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Characterization of some b-chromatic edge critical graphs

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Abstract

A b-coloring is a proper coloring of the vertices of a graph such that each color class has a vertex that is adjacent to a vertex of every other color. The b-chromatic number $b(G)$ of a graph G is the largest k such that G admits a b-coloring with k colors. A graph G is b-chromatic edge critical if for any edge e of G, the b-chromatic number of $G - e$ is less than the b-chromatic number of G. We call these graphs b-chromatic edge critical or just edge b-critical. We show that any edge b-critical graph G satisfies $b(G) = \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G, and we characterize edge b-critical P_4 sparse graphs and edge b-critical quasi-line graphs.

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex-set V and edge-set E. A coloring of the vertices of G is a mapping $c : V \to \{1, 2, \ldots\}$. For every vertex $v \in V$ the integer $c(v)$ is called the color of v. A coloring is *proper* if any two adjacent vertices have different colors. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G admits a proper coloring using k colors.

A *b-coloring* of a graph G by k colors is a proper coloring of the vertices of G such that in each color class there exists a vertex having neighbors in all the other $k - 1$ color classes. We call any such vertex a *b-vertex*. The concept of b-coloring was introduced in [6, 7]. The b-chromatic number $b(G)$ of a graph G is the largest integer such that G admits a b-coloring with k colors.

If e is an edge of a graph $G = (V, E)$, then $G - e$ is the subgraph of G that results after removing from G the edge e. Note that the end vertices of e are not removed from G. A graph G is called *edge b-critical* if $b(G - e) < b(G)$, for every edge e in G.

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We finish this section with some definitions and notation which are used throughout the paper. For the other necessary definitions and notation, we follow that of Berge [1]. Consider a graph $G = (V, E)$. For any $A \subset V$, let $G[A]$ be the subgraph of G induced by A. For any vertex v of G, the *neighborhood* of v is the set $N_G(v) = \{u \in V(G) \mid (u, v) \in E\}$ (or $N(v)$ if there is no confusion). Let $\omega(G)$ denote the size of a maximum clique of G. If G and H are two vertex-disjoint graphs, the *union* of G and H is the graph $G + H$ whose vertex-set is $V(G) \cup V(H)$ and edge-set is $E(G) \cup E(H)$. For an integer $p \geq 2$, the union of p copies of a graph G is denoted pG. The *join* of graphs G and H is the graph denoted $G \vee H$ obtained from $G + H$ by adding all edges between G and H. Given a collection $\mathcal{H}$ of graphs, a graph G is called $\mathcal{H}$*-free* if G does not have an induced subgraph that is isomorphic to any member of $\mathcal{H}$. In case $\mathcal{H}$ has only one member H we say that G is H-free. We let P_k denote the path with k vertices, and K_k denote the complete graph with k vertices. The *corona* of a graph G is obtained by attaching a pendant edge at each vertex of G.

In this paper, we prove that if G is edge b-critical graph with maximum degree $\Delta(G)$, then $b(G) = \Delta(G) + 1$. We give a characterization of edge b-critical P_4 sparse graphs and edge b-critical quasi-line graphs.

2 Preliminary results

In this section, we show that any edge b-critical graph G satisfies $b(G) = \Delta(G) + 1$. In [2], Faik proved that for any graph G, the removal of an edge can decrease the b-chromatic number by at most one.

Proposition 2.1 *[2] Let G be a graph and e an edge of G; then $b(G-e) \geq b(G)-1$.*

Definition 2.2 *A graph G is said to be an edge b-critical graph if for any edge e, $b(G - e) = b(G) - 1$.*

Let $G = (V, E)$ be a graph and let $G' = (V', E)$ be a graph obtained from G by removing all isolated vertices. Since $E(G) = E(G')$, G is edge b-critical if and only if G' is edge b-critical. So we may suppose that none of the graphs in this paper contain isolated vertices.

By S we denote the set of b-vertices such that any two b-vertices have different colors. If $|S| = b(G)$ then we say that the set S is a *b-system* of G. An *independent set* of vertices in a graph is a set of mutually non-adjacent vertices. Now we begin our study with the following theorem:

Theorem 2.3 *Let $G = (V, E)$ be an edge b-critical graph, and let c be a b-coloring of G with $b(G)$ colors. Then:*

i) Any two b-vertices of c have different colors.
ii) The b-system S of G is unique.
iii) $V \setminus S$ is an independent set.

iv) $\forall x \in V \setminus S$, $d_G(x) \leq |S| - 2$.

Proof. Let G be an edge b-critical graph.

i) Assume that there exist two b-vertices x, y of the same color. Now we prove that all neighbors of x and y are b-vertices. If there exists a non b-vertex $u \in N(x) \cup N(y)$, then $b(G - xu) \geq b(G)$ or $b(G - yu) \geq b(G)$, a contradiction. In this case, since all colors (other than the color of x, y) appear in both $N(x)$ and $N(y)$, for any vertex $z \in N(x)$, $b(G - xz) \geq b(G)$, a contradiction.

ii) This is a direct consequence of (*i*).

iii) By (*ii*), $V \setminus S$ contains no b-vertex. If $V \setminus S$ contains two adjacent vertices u and v, then $b(G - uv) \geq b(G)$, a contradiction.

vi) Let u be a vertex $V \setminus S$. Vertex u is adjacent to at most $|S| - 2$ vertices of S, for otherwise, u would be a b-vertex. ■

We now establish the next result.

Theorem 2.4 *If G is an edge b-critical graph, then $b(G) = \Delta(G) + 1$.*

Proof. Let $b(G) = k$. For a given graph G, it may be easily noted that $k \leq \Delta(G)+1$. So, to prove that $k = \Delta(G) + 1$, it suffices to show that for any vertex x of an edge b-critical graph G, $d_G(x) \leq k - 1$. Assume that there exists a vertex y such that $d_G(y) > k - 1$. If y is a b-vertex, then $N(y)$ must contain two vertices u, v with the same color. Then one of u, v, say u, is not a b-vertex, for otherwise, we have a contradiction to Theorem 2.3. But in this case, $b(G - yu) \geq b(G)$, a contradiction. If y is not a b-vertex, then Theorem 2.3 implies that $d_G(y) \leq k - 2$, a contradiction to the assumption that $d_G(y) > k - 1$. So $k = \Delta(G) + 1$. ■

Observation 2.5 *If $G \neq K_n$ is a graph with $b(G) = \omega(G)$, then G is not edge b-critical.*

Proof. Let $G \neq K_n$ be a graph with $b(G) = \omega(G)$, and let K be a maximum clique of G. Let $E' = \{xy \in E(G) : x \in K,\ y \in V \setminus K\} \cup \{xy \in E(G(V \setminus K))\}$. Since G is without isolated vertices, $E' \neq \emptyset$. Thus for any edge e of E', we have $b(G - e) \geq \omega(G) = b(G)$. ■

A graph is called a *split graph* if its vertex set can be partitioned into a clique and an independent set. It is easy to see that if G is a split graph, then $b(G) = \omega(G)$. So the following is an immediate result.

Observation 2.6 *If $G \neq K_n$ is a split graph, then G is not an edge b-critical graph.*

3 Edge b-critical P_4-sparse graphs

In [4] Hoàng introduced the class of P_4-sparse graphs as the graphs for which every set of five vertices induces at most one P_4.

A *spider* is a graph whose vertex set can be partitioned into sets S, K and R such that

(a) S is a stable, K is a clique and $|S| = |K| \geq 2$.
(b) Every vertex in R is adjacent to all the vertices in K and adjacent to no vertex in S.
(c) There exists a bijection $f : S \to K$ such that either
(c.1) for all vertices $x \in S$, $N(x) \cap K = \{f(x)\}$,
or else,
(c.2) for all vertices $x \in S$, $N(x) \cap K = K \setminus \{f(x)\}$.

If the condition of case (c.1) holds, then the spider G is called a *thin spider*, whereas if the condition of case (c.2) holds then G is a *thick spider*. Note that the complement of a thin spider is a thick spider and vice versa. A spider graph with $|S| = |K| = 2$ and $|R| \leq 1$ is simultaneously thin and thick. We shall denote a spider by (S, K, R) or (S, K) if R is empty.

The P_4-sparse graphs have been characterized independently by Hoàng [4], and Jamison and Olariu [3].

Theorem 3.1 *[4, 3] If G is P_4-sparse graph then G or $\overline{G}$ is disconnected, or G is a spider.*

From Theorem 3.1, we can deduce the following observation.

Observation 3.2 *Let G be a disconnected P_4-sparse graph. Then any connected component of G is either a spider graph or the join of two graphs.*

Theorem 3.3 *Let G be a spider graph. Then G is edge b-critical if and only if G is a complete graph.*

Proof. It is obvious that the complete graph is edge b-critical. Let G be an edge b-critical spider graph with vertex set $R \cup K \cup S$. We shall show that $R \cup S$ contains no b-vertex and all of K are b-vertices. It is straightforward to show that the degree of every vertex of $R \cup S$ is at most $|R| + |K| - 1$. If x is a b-vertex, then Theorem 2.4 implies that $d_G(x) = \Delta(G) \geq |K| + |R|$. Thus $R \cup S$ contains no b-vertex.

Let $k = |K|$. Suppose that K contains a non b-vertex y. Then for every vertex $x \in S$ which is adjacent to y, we have $b(G - xy) \geq b(G)$, a contradiction. So all of K are b-vertices. Let $E' = \{e : e \in E(G[R])\} \cup \{yu : y \in K, u \in S \cup R\}$. For every edge e' of E', $b(G - e') \geq b(G)$, a contradiction. So G is a complete graph of order k. ■

The following observation is immediate.

Observation 3.4 *Let G be a disconnected edge b-critical P_4-sparse graph and let G_i be a connected component of G. If G_i is spider (S, K, R) then:*
i) $S \cup R$ contains no b-vertex and all of K are b-vertices.
ii) In every b-coloring of G with $b(G)$ colors, no color in S appears in R.

Definition 3.5 *Let G_i be a connected graph with vertex-set $S^i \cup K^i \cup R^i$ which satisfies the following conditions:*
1. K^i is a clique and R^i, S^i are independent sets.
2. $K^i \neq \emptyset$, $S^i \cup R^i \neq \emptyset$.
3. G_i is a thin spider (S^i, K^i, R^i) or the join of two graphs $G[R^i]$ and $G[K^i]$ (different from a clique). Note that if $R^i = \emptyset$, then G_i is a thin spider (S^i, K^i).
For any integer $p \geq 2$, let $G = \bigcup_{i=1}^{p} G_i$ be the union of p connected components such that, in addition to the three previous conditions, S^i, K^i and R^i satisfy the following condition.

$$\text{If } S^i = \emptyset, \text{ then } |R^i| = \left| \bigcup_{j=1\ (j\neq i)}^{p} K^j \right| \text{ else } |R^i| = \left| \bigcup_{j=1\ (j\neq i)}^{p} K^j \right| - 1.$$

Let $\mathcal{G}$ be a collection of graphs defined as above. Then we remark that any graph G of $\mathcal{G}$ satisfies $\Delta(G) = \Delta(G_i) = \sum_{i=1}^{p} |K^i| - 1, \quad i \in \{1, \dots, p\}$.

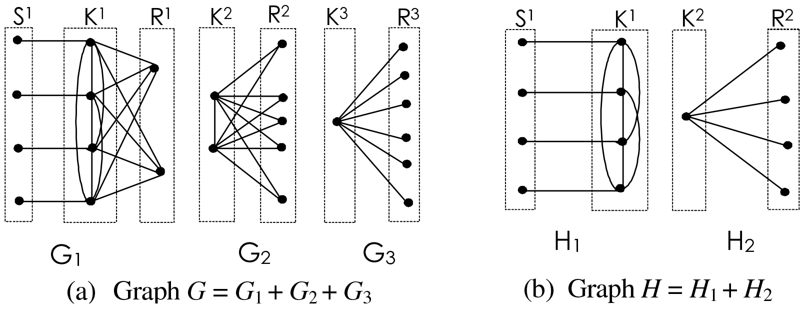

Figure 1: Two graphs in family $\mathcal{G}$

Observation 3.6 *Let G be a graph of $\mathcal{G}$. Then $b(G) = \Delta(G) + 1$*

Proof. Let G be a graph of $\mathcal{G}$. A b-coloring of a graph G with $\Delta(G) + 1$ colors is obtained by coloring each vertex of $\bigcup_{i=1}^{p} K^i$ with a different color and, for every stable R^i, coloring each of $|R^i|$ vertices with a different color such that this color does not appear in K^i. Finally, coloring each vertex of S^i with the same color such that this color does not appear in $(K^i \cup R^i)$. ■

It is straightforward to verify the following result.

Observation 3.7 *Let G be a graph of $\mathcal{G}$. Then G is an edge b-critical graph.*

Lemma 3.8 *Let G be a disconnected edge b-critical P_4-sparse graph. Then all b-vertices in any connected component of G form a clique.*

Proof. Let G_i be a connected component of G. By Observation 3.2, we can distinguish between two cases. If G_i is a spider then, by Observation 3.4, all b-vertices of G_i form a clique. If G_i is the join of two graphs $G[A]$ and $G[B]$, then in every coloring of G_i, no color can appear in both A and B. Suppose that G_i contains two vertices $x, u \in A$ of the same color. Theorem 2.3 implies that one of x, u, say u, is a non b-vertex. It follows that, for any vertex z of B, $b(G - uz) \geq b(G)$, a contradiction. Thus any two vertices of $G[A] \vee G[B]$ have different colors. So all b-vertices of G_i form a clique. ∎

The previous lemma remains true if G is a connected graph. So a direct consequence of this lemma is the following.

Observation 3.9 *$G = G_1 \vee G_2$ is edge b-critical if and only if G is a complete graph.*

The following lemma was proved by Hoàng and Kouider [5].

Lemma 3.10 *[5] Let $p \geq 1$ be an integer. Let G' and K_p be two vertex-disjoint graphs where K_p is a clique on p vertices, and let $G = G' + K_p$. Then we have $b(G) = max\{b(G'), p\}$.*

Now we give a characterization of the edge b-critical P_4-sparse graphs.

Theorem 3.11 *Let $G = (V, E)$ be a P_4-sparse graph. Then G is edge b-critical if and only if G is a complete graph or $G \in \mathcal{G}$.*

Proof. Let $G = (V, E)$ be a P_4-sparse graph. It is obvious that the complete graph is edge b-critical. Also, by Observation 3.7, any graph of $\mathcal{G}$ is edge b-critical. Let us now prove the necessary condition. By Theorem 3.1, we can distinguish between three cases:

Case 1: *$\overline{G}$ is not connected.* Then G is the join of two graphs G_1 and G_2. Thus, by Observation 3.9, G is a complete graph.

Case 2: *G is not connected.* Then G is the union of at least two components. Let G_i be a connected component of G.

Claim 1: G_i is not a clique.

Proof. Suppose that G_i is a clique on p vertices. Since G is not connected, $G = G' + G_i$, where G' is an induced subgraph of G. Since G contains no isolated vertex, it follows that $E(G') \neq \emptyset$ and $|V(G_i)| \geq 2$. By Lemma 3.10, $b(G) = b(G')$ or p. If $b(G) = p$ then $b(G - e) \geq b(G)$, for every edge e in G' or else $b(G - e) \geq b(G)$, for every edge e in G_i, a contradiction. So Claim 1 holds.

Claim 2: $\Delta(G_i) = \Delta(G)$.

Proof. Otherwise, Theorem 2.4 implies that G_i contains no b-vertex. Since G_i is without isolated vertices, it follows that $b(G - e) \geq b(G)$, for every edge e in G_i, a contradiction. So Claim 2 holds.

Claim 3: G_i is a split graph with vertex-set $K^i \cup S^i$, where K^i is a clique and S^i, $|S^i| \geq 2$, is a stable. Furthermore, all vertices of K^i have the same number of neighbors in S^i.

Proof. Let G_i be a connected component of G. Let K^i be a set of all b-vertices of G in G_i. Lemma 3.8 implies that K^i is a clique. By Theorem 2.3 and Claim 1, $V(G_i) \setminus K^i$ is an independent non-empty set. Let $S^i = V(G_i) \setminus K^i$. Suppose that S^i contains only one vertex u. Since G_i has no isolated vertex, u is adjacent to some vertex x of K^i. It follows that $\Delta(G_i) = |K^i|$. If there exists a vertex $y \neq x$ of K^i which is not adjacent to u, then $d_G(y) = |K^i| - 1 = \Delta(G_i) - 1$. By Claim 2, $d_G(y) = \Delta(G) - 1$, a contradiction to Theorem 2.4. Then G_i is a clique, which contradicts the Claim 1. This implies that S^i contains at least two vertices. Since all of K^i are b-vertices, it follows that all vertices of K^i have the same number of neighbors in S^i. So Claim 3 holds.

Claim 4: G_i cannot be a thick spider.

Proof. Let G_i be a spider (S, K, R). If $|S| = |K| = 2$, we can consider G_i as a thin spider. So assume that $|S| \geq 3$ and suppose that G_i is a thick spider. By Claim 3, $S \cup R = S^i$ and $K = K^i$. Then any two vertices of S have different colors. Assume that S contains two vertices u, v of the same color. Since $|K| = |S| \geq 3$, there exists a b-vertex $x \in K$ which is adjacent to u, v. This implies that $b(G - xu) \geq b(G)$ or $b(G - xv) \geq b(G)$, a contradiction. So all vertices of S have a same color. Since x is not adjacent to all vertices of S, it follows that it is adjacent to a some vertex of R such that its color appears in S. This contradicts the Observation 3.4. So Claim 4 holds.

The corona of the graph H, denoted by $H \circ K_1$, is the graph obtained from H by attaching a pendant edge at each vertex of H. If H is a complete graph, then $H \circ K_1$ is called the clique corona of H. The star $K_{1,p}$ is the tree of order $p+1$ in which p vertices are of degree 1 and one vertex is of degree p. We note that the clique corona is a thin spider (S, K), and the star $K_{1,p}$ is the join of two graphs $K_1 \vee S_p$, where S_p is a stable of order p and K_1 is a clique of order 1.

Claim 5: If G_i is a spider (K, S), then $G = K_p \circ K_1 + K_{1,p}$, where $p = |K|$ and $|V(G)| = 3p + 1$ (see Figure 1(b)).

Proof. Let G_i be a spider (K, S). Then Claims 3 and 4 imply that $K = K^i$, $S = S^i$, and G_i is a clique corona $K_p \circ K_1$ where $p = |K|$. Moreover, Observation 3.4 implies that G_i contains $|K|$ b-vertices. Since $\Delta(G_i) = |K|$, Theorem 2.4 and Claim 2 imply that $b(G) = |K| + 1$. Thus $G = G_i + G'$ where G' is an induced subgraph of G which contains one b-vertex y. Theorem 2.3 and Claim 1 imply that $V(G') \setminus \{y\}$ is an independent non-empty set. Since G contains no isolated vertex, all vertices of $V(G') \setminus \{y\}$ are adjacent to y. This implies that G' is a star of order $|K| + 1$. So $G = K_p \circ K_1 + K_{1,p}$, where $3\,|K| + 1 = |V(G)|$. So Claim 5 holds.

Using Observation 3.2 and previous claims we can deduce that G_i is either the join of two graphs $G[K^i]$ and $G[S^i]$ where K^i and S^i are, respectively, a clique and a stable set which satisfy $\Delta(G_i) = |K^i| + |S^i| - 1$, or it is a thin spider (S, K, R) with $\Delta(G_i) = |K| + |R|$ where $S \cup R = S^i$ and $K = K^i$, or a thin spider (S, K) where $S = S^i$, $K = K^i$. On the other hand, Lemma 3.8 and Theorem 2.3 imply that $\bigcup_{i=1}^{k} K^i$ is a b-system of G. Thus $b(G) = \sum_{i=1}^{k} |K^i|$. Theorem 2.4 and Claim 2 imply that $\sum_{i=1}^{k} |K^i| = \Delta(G) + 1 = \Delta(G_i) + 1$. Consequently $G \in \mathcal{G}$.

Case 3: *G is a spider. Then Theorem 3.3 implies that G is a complete graph.*

This completes the proof of Theorem 3.11. ■

4 Edge b-critical quasi-line graphs

A *quasi-line* graph is a graph in which the neighborhood of any vertex can be covered by two cliques. The class of quasi-line graphs is a proper superclass of line graphs and a proper subclass of claw-free graphs. In what follows, we characterize edge b-critical quasi-line graphs.

We first introduce a graph H_0 which plays an important role in this section. Let K_n be a clique on n vertices and let xy be an edge of K_n. The graph H_0 is obtained from the clique K_n by removing an edge xy. Add two extra vertices u, v, put an edge between x and u, and between y and v. It is easy to check that H_0 is edge b-critical and $b(H_0) = b(K_n) = n$. We now begin this section with two lemmas needed to prove the next results.

Lemma 4.1 *Let G be an edge b-critical quasi-line graph. Then every b-vertex of G may be adjacent to at most two non b-vertices.*

Proof. Let $G = (V, E)$ be an edge b-critical graph and let S be a b-system of G. Let x be any vertex of S. By Theorem 2.3, $V \setminus S$ is an independent set. If x is adjacent to 3 vertices of $V \setminus S$, then G contains a $K_{1,3}$ as an induced subgraph, a contradiction. ■

Lemma 4.2 *If $G \neq K_n$, H_0, is an edge b-critical quasi-line graph, then every b-vertex is adjacent to at least one non b-vertex.*

Proof. Let $G \neq K_n, H_0$ be an edge b-critical quasi-line graph and let x be a b-vertex for some b-coloring c of G with $b(G)$ colors. Then $N(x)$ is the union of two cliques A and B. Suppose that all neighbors of x are b-vertices. If $N(x)$ is a clique, then by Theorem 2.4, $b(G) = \Delta(G)+1 = \omega(G)$. Since G has no isolated vertices, Observation 2.5 implies that $G = K_n$, a contradiction. So $N(x)$ is not a clique. In this case, there exist two b-vertices $x_1 \in A$, $x_2 \in B$ such that x_1 is not adjacent to x_2. This implies that x_1 (respectively, x_2) is adjacent to a some vertex u (respectively, v) of color

$c(x_2)$ (respectively, $c(x_1)$). By Theorem 2.3, u and v are non b-vertices. Since all of $N(x)$ are b-vertices, it follows that u and v are not adjacent to x. So we claim that

$$x_i \ (i = 1, 2) \text{ is adjacent exactly to one non } b\text{-vertex.}$$

Assume that x_1 is adjacent to another non b-vertex $u_1 \neq u$. By Theorem 2.3, u_1 is not adjacent to u. So, since u_1 is not adjacent to x, $\{x, x_1, u, u_1\}$ form a $K_{1,3}$, a contradiction. Likewise x_2 is adjacent exactly to one non b-vertex v. The claim is proved. Then x_i is adjacent to all b-vertices except x_j, $i \neq j$, $i = 1, 2$, $j = 1, 2$.

On the other hand, every b-vertex of $N(x)$ other than x_1, x_2 is adjacent to all b-vertices. Indeed, suppose that there exists a b-vertex $x_3 \in A$ which is not adjacent to a b-vertex $x_4 \in B$. So x_3 is adjacent to a non b-vertex w of color $c(x_4)$. By the previous claim, x_1, x_2 are not adjacent to w. Then $\{x_1, x_3, w, x_2\}$ form a $K_{1,3}$, a contradiction.

This implies that $G = H_0$, a contradiction. ■

Let $\mathcal{F} = \{F_1, \ldots, F_7\}$ be the set of graphs depicted in Figure 2.

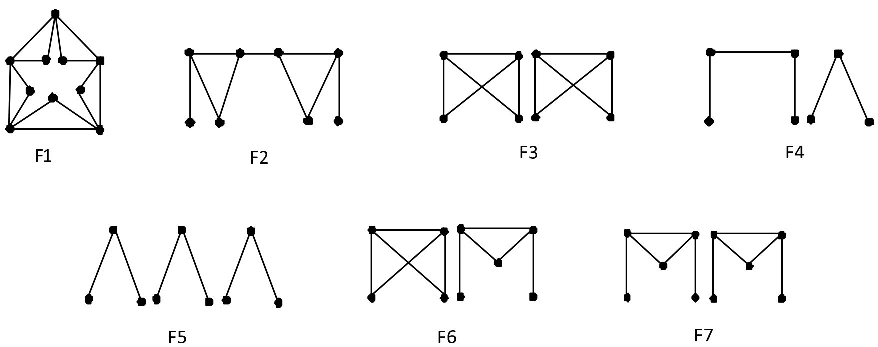

Figure 2 : Class $\mathcal{F} = \{F_1, \ldots, F_7\}$

Lemma 4.3 *Let $G = (V, E)$ be an edge b-critical quasi-line graph different from F_i, $i = 4, 5, 6, 7$, and let S be a b-system of G. Then any two vertices of $V \setminus S$ have different colors.*

Proof. Let $G = (V, E)$ be an edge b-critical quasi-line graph different to F_i, $i = 4, 5, 6, 7$. Let S be a b-system of G, and let c be a b-coloring of G with $b(G)$ colors. There are two cases.

Case 1: $b(G) \geq 4$.

Suppose that there exist two vertices $u, v \in V \backslash S$ of the same color. Then we claim that

$$N(u) \cap N(v) = \emptyset.$$

Theorem 2.3 implies that u and v have no common neighbor in $V \backslash S$. Let x be a vertex of S. If $x \in N(v) \cap N(u)$ then $b(G - xu) \geq b(G)$ or $b(G - xv) \geq b(G)$, a contradiction. Therefore u and v have no common neighbor in S. Thus $N(u) \cap N(v) = \emptyset$.

Let c be a b-coloring of G with $b(G)$ colors. Since G contains no isolated vertex, it follows that $N(u) \cap S \neq \emptyset$ and $N(v) \cap S \neq \emptyset$. Let x, y be two vertices of S such that u is adjacent to x, and v is adjacent to y. By Theorem 2.3, $c(x) \neq c(y)$. Let z be a vertex of S of color $c(u)$. Then z is not adjacent to x and y, for otherwise $b(G-xu) \geq b(G)$ or $b(G-yv) \geq b(G)$. Let u' and v' be two vertices of $N(z)$ of colors $c(x)$ and $c(y)$ respectively. Vertices u' and v' are two nonadjacent vertices belonging to $V \setminus S$, otherwise we would have a contradiction to Theorem 2.3. Since $b(G) \geq 4$, $I = S \setminus \{x, y, z\}$ is non-empty. So by Lemma 4.1, z is adjacent to all vertices of I.

If x is not adjacent to y, then x has exactly one neighbor in $V \setminus S$ other than u of color $c(y)$. For otherwise G contains a $K_{1,3}$ as an induced subgraph, a contradiction. Likewise y has exactly one neighbor in $V \setminus S$ other than v of color $c(x)$.

In this case, x and y are adjacent to all vertices of I. It follows that x, y and z have the same neighborhood in I. Consequently, for every vertex $w \in I$, $\{w, x, y, z\}$ form a $K_{1,3}$, a contradiction.

If x is adjacent to y, then one of x, y has exactly two neighbors in $V \setminus S$. Assume that this is not true. By Lemmas 4.1 and 4.2, x (respectively, y) has exactly one neighbor u (respectively, v) in $V \setminus S$. Since $b(G) \geq 4$, vertices x, y and z are adjacent to all of I. This implies that, for any vertex $w \in I$, w is non adjacent to u' and v'. Then, for any vertex $w \in I$, $\{z, w, u', v'\}$ form a $K_{1,3}$, a contradiction. So one of x, y, say x, has exactly two neighbors u and u_1 in $V \setminus S$.

If y is non adjacent to u_1, then $\{x, y, u_1, u\}$ form a $K_{1,3}$, a contradiction; otherwise we distinguish two subcases with respect to the size of S.

If $b(G) = 4$, then I contains exactly one vertex w of color $c(u_1)$. Vertex w cannot be adjacent to any vertex of $\{x, y, u, u_1, v\}$, for otherwise $b(G - xu_1) \geq b(G)$ or $b(G - yu_1) \geq b(G)$ or $b(G - wu) \geq b(G)$ or $b(G - wv) \geq b(G)$. If w is adjacent to $u\prime$ and $v\prime$ or one of $u\prime$, $v\prime$, then $G = F_6$ or F_7 (see Figure 2),or else G contains a $K_{1,3}$ as an induced subgraph; a contradiction.

If $b(G) \geq 5$, then I contains a vertex $w' \neq w$ which is adjacent to x and y, for otherwise x (or y) is adjacent to three non b-vertices of colors $c(u), c(w)$ and $c(w')$, a contradiction to Lemma 4.1. In this case, w' cannot be adjacent to u' and v'. This implies that $\{w', z, u', v'\}$ form a $K_{1,3}$, a contradiction.

Case 2: $b(G) \leq 3$.

Let x and y, respectively, be two b-vertices of colors 1 and 2. Suppose that x (respectively, y) is adjacent to a non b-vertex u (respectively, v) of color 3. It is obvious that u is not adjacent to y, and v is not adjacent to x. Let $z \in S$ be a b-vertex of color 3. Vertex z is not adjacent to x, y. Then z is adjacent to two non b-vertices u' and v' of colors 1 and 2, respectively. Vertex x is not adjacent to v', for otherwise v' is a b-vertex, which contradicts Theorem 2.3. If x is not adjacent to y, then it is adjacent to a vertex u_1 of color 2. By Theorem 2.3, $\{u, u_1, u'\}$ is an independent set. By a symmetric argument, y is adjacent to a some vertex v_1 of color 1, and $\{v, v_1, v'\}$ is an independent set. This implies that $G = F_5$, a contradiction. If x is adjacent to y, then $G = F_4$, a contradiction.

Finally, it is easy to see that if G is edge b-crtical with $b(G) = 2$, then $G = K_2$. Thus every vertex of G is a b-vertex, i.e. $V \setminus S = \emptyset$. ■

Lemma 4.4 *If $G \neq K_n, H_0$ is an edge b-critical quasi-line graph, then $b(G) \leq 5$.*

Proof. It is easy to check that $b(F_4) = b(F_5) = 3 \leq 5$ and $b(F_6) = b(F_7) = 4 \leq 5$. Let $G \neq K_n, H_0, F_i$, $i = 4, 5, 6, 7$, be an edge b-critical quasi-line graph, and let x be a b-vertex for some b-coloring c of G with $b(G)$ colors. To prove that $b(G) \leq 5$, it suffices to show that, for every b-vertex x, $d_G(x) \leq 4$. Since G is a quasi-line graph, it is the union of two cliques A and B, ($|A| \geq |B|$). By Lemmas 4.1 and 4.2, $N(x)$ contains r non b-vertices, $1 \leq r \leq 2$. Suppose that there exists a b-vertex x such that $d_G(x) > 4$. Then $|A| \geq 3$.

Case 1: $r = 2$

Let $u, v \in N(x)$ be two non b-vertices. Suppose that u and v are in A. By Theorem 2.3, u is not adjacent to v, a contradiction to the assumption that A is a clique. Similarly u, v are not both in B. So we may suppose that $u \in A$ and $v \in B$. Let y be a b-vertex of color $c(u)$. Vertex $y \notin B$, for otherwise $b(G - xu) \geq b(G)$. Then y is adjacent to a vertex u_1 of color $c(x)$. By Theorem 2.3, u_1 is not a b-vertex. Vertex y is not adjacent to any vertex of A, for otherwise, if there exists a vertex $w \in A, w \neq u$ such that y is adjacent to w, then $b(G - uw) \geq b(G)$, a contradiction. In this case, since $|A| \geq 3$, y is adjacent to at least two vertices u_2, u_3, of which colors appear in A. By Theorem 2.3, u_2 and u_3 are non b-vertices. Moreover, $u_2, u_3 \notin B$, for otherwise, $b(G - xu_2) \geq b(G)$ or $b(G - xu_1) \geq b(G)$. By Theorem 2.3, $\{y, u_1, u_2, u_3\}$ is an independent set. This implies that G contains a $K_{1,3}$, a contradiction.

Case 2: $r = 1$

Let $u \in N(x)$ be a non b-vertex. By **Case 1**, since $|A| \geq 3$, vertex u belongs to B and $1 \leq |B| \leq 2$. It follows that all vertices of A are b-vertices. Vertex u cannot be adjacent to all of A, for otherwise, u would be a b-vertex. Thus, there exists a b-vertex of A, say x_1, which is non-adjacent to u. This implies that x_1 is adjacent to a vertex y of color $c(u)$. By Lemma 4.3, y is a b-vertex. Vertex y is not adjacent to x, for otherwise, $b(G - xu) \geq b(G)$. Then y is adjacent to a vertex u_1 of color $c(x)$. By Theorem 2.3, u_1 is a non b-vertex. Obviously, u_1 is not adjacent to any vertex of $N(x)$. So we claim that

$$y \text{ is adjacent to all vertices of } A$$

Suppose that there exists a vertex $x_i \in A$, $i \neq 1$, such that y is not adjacent to x_i. Then y is adjacent to a some vertex u_2 of color $c(x_i)$. By Theorem 2.3, u_2 is a non b-vertex. Also, vertex u_2 is not adjacent to any vertex of A, for otherwise, if there exists a vertex x_j of $A \setminus \{x_i\}$ such that x_j is adjacent to u_2, then $b(G - x_i x_j) \geq b(G)$. By Theorem 2.3, u_2 is not adjacent to u_1. This implies that $\{x_1, y, u_1, u_2\}$ form a $K_{1,3}$, a contradiction. Thus y is adjacent to all of A. So, the claim is proved.

If $B = \{u\}$ then $|A| \geq 4$ and $G = H_0$, a contradiction. If $B = \{u, x'\}$ where x' is a b-vertex, then we claim that

$$x' \text{ is adjacent to all vertices of } A, \ |A| \geq 3$$

Firstly, x' is adjacent to at most one non b-vertex other than u, for otherwise G contains a $K_{1,3}$. Assume that there exists a vertex $x_i \in A$ non adjacent to x'. Then

x' is adjacent to exactly one non b-vertex u_3 of color $c(x_i)$. This implies that x' is adjacent to all of $A\setminus\{x_i\}$. It follows that vertex u_3 is not adjacent to any vertex of A. If there exists a vertex $x_j \neq x_i$ of A which is adjacent to u_3, then $b(G-x_ju_3) \geq b(G)$, a contradiction. Since all vertices of A are adjacent to y, u is not adjacent to any vertex of A. In this case, for any vertex $x_k \neq x_i$ of A, the set $\{x_k, x', u, u_3\}$ forms a $K_{1,3}$, a contradiction. Then x' is adjacent to all of A.

Vertex y is not adjacent to x', for otherwise $b(G-ux') \geq b(G)$. Since y is a b-vertex of color $c(u)$, y is adjacent to a some non b-vertex u_4 of color $c(x')$. Since x' is adjacent to all of A, u_4 is non adjacent to any vertex of A. By Theorem 2.3, u_1 is not adjacent to u_4. This implies that $\{x_1, y, u_1, u_4\}$ form a $K_{1,3}$, contradiction. ■

We proceed now to give a characterization of edge b-critical quasi-line graphs

Theorem 4.5 *Let G be a quasi-line graph. Then G is edge b-critical if and only if $G = K_n, H_0$ or $G \in \mathcal{F}$ (see Figure 2).*

Proof. The 'if' part is easy to check by examining the graphs in Figure 2, K_n and H_0. Let us now prove the 'only if' part. Let G be an edge b-critical quasi-line graph. If $b(G) \geq 6$, then by Lemma 4.4, $G = K_n$ or H_0. Now, we are considering the case where $b(G) \leq 5$. Let x be a b-vertex for some b-coloring c of G with $b(G)$ colors. Since G is a quasi-line, $N(x) = A \cup B$ where A and B, ($|A| \geq |B|$), are two cliques. By Lemmas 4.1 and 4.2, $N(x)$ contains one or two non b-vertices. It is easy to show that the theorem holds for $G = F_i, i = 4,5,6,7$. Now suppose that $G \neq F_i, i = 4,5,6,7$. By Theorem 2.3 and Lemma 4.3, all b-vertices (non b-vertices) have different colors. We can distinguish between two cases:

Case 1: $N(x)$ contains a single non b-vertex.

Let $u \in N(x)$ be a non b-vertex, and let $\{x_i : 1 \leq i \leq 3\}$ denote the set of all b-vertices of $N(x)$.

Case 1.1: $b(G) = 5$. We distinguish among three cases.

a) $A = \{x_1, x_2\}$ and $B = \{x_3, u\}$. Vertex u cannot be adjacent to all of A, for otherwise u would be a b-vertex. So there exists a vertex of A, say x_1, which is not adjacent to u. Then x_1 is adjacent to a some vertex $y \notin N(x)$ of color $c(u)$. By Lemma 4.3, y is a b-vertex. Since y is not adjacent to x, then y is adjacent to some vertex u_1 of color $c(x)$. Vertex u_1 is a non b-vertex, for otherwise we would have a contradiction to Theorem 2.3. Also, vertex u_1 is not adjacent to any vertex of A, for otherwise $b(G-x_1u_1) \geq b(G)$ or $b(G-x_2u_1) \geq b(G)$. We claim that

$$y \text{ is adjacent to } x_2.$$

Suppose the contrary. Then y is adjacent to a some vertex u_2 of color $c(x_2)$. By Theorem 2.3, u_2 is a non b-vertex. Also u_2 is not adjacent to x_1, for otherwise $b(G-x_1u_2) \geq b(G)$. This implies that $\{x_1, y, u_1, u_2\}$ form a $K_{1,3}$, a contradiction. So y is adjacent to x_2.

On the other hand, y is not adjacent to x_3, for otherwise $b(G-x_3u) \geq b(G)$. It follows that y is adjacent to a vertex u_3 of color $c(x_3)$. By Theorem 2.3, u_1 is not adjacent to u_3. This implies that vertex u_3 is adjacent to all of A, for otherwise

G contains a $K_{1,3}$. It follows that x_3 is not adjacent to any vertex of A. So x_3 is adjacent to two non b-vertices u_4 and u_5 of colors $c(x_1)$ and $c(x_2)$. In this case, $\{x_3, u, u_4, u_5\}$ form a $K_{1,3}$, a contradiction. This case cannot occur.

b) $A = \{x_1, x_2, x_3\}$ and $B = \{u\}$. Since u is a non b-vertex, there exists a vertex of A, say x_1, which is not adjacent to u. This implies that x_1 is adjacent to a some vertex y of color $c(u)$. So, by the previous **Case 1.1 (a)**, y is a b-vertex which is adjacent to x_2. By a symmetric argument, y is adjacent to x_3. Also, there exists some non b-vertex u_1 of color $c(x)$ that is adjacent to y and not to any vertex of $N(x)$. This implies that $G = H_0$ with $|V(H_0)| = 7$.

c) $A = \{x_1, x_2, u\}$ and $B = \{x_3\}$. Vertex u cannot be adjacent to x_3, for otherwise, u would be a b-vertex. So x_3 is adjacent to some vertex y of color $c(u)$. In the same way, we can show that y is a b-vertex that is adjacent to a some non b-vertex u_1 of color $c(x)$, and u_1 is not adjacent to any vertex of $N(x)$. On the other hand, y is not adjacent to x_1 and x_2, for otherwise, $b(G - x_1u) \geq b(G)$ or $b(G - x_2u) \geq b(G)$. It follows that y is adjacent to two vertices u_3 and u_4 of colors $c(x_1)$ and $c(x_2)$. By Theorem 2.3, u_2 and u_3 are not b-vertices, and by Theorem 2.3, $\{u_1, u_2, u_3\}$ is an independent set. This implies that G contains a $K_{1,3}$, a contradiction. This case cannot occur.

Case 1.2: $b(G) = 4$. We distinguish among two cases.

a) $A = \{x_1, x_2\}$ and $B = \{u\}$. Vertex u cannot be adjacent to all of A, for otherwise u would be a b-vertex. So there exists a vertex of A, say x_1, which is not adjacent to u. This implies that x_1 is adjacent to a some vertex y of color $c(u)$. In the same way, we can show that y is a b-vertex that is adjacent to some non b-vertex u_1 of color $c(x)$, and u_1 is not adjacent to any vertex of $N(x)$. Also, by the previous claim we can show that y is adjacent to x_2. This implies that $G = H_0$ with $|V(H_0)| = 6$.

b) $A = \{x_1, u\}$ and $B = \{x_2\}$. Vertex u cannot be adjacent to x_2, for otherwise u would be a b-vertex. So x_2 is adjacent to some vertex y of color $c(u)$. Vertex y is a b-vertex that is adjacent some non b-vertex u_1 of color $c(x)$. Clearly, u_1 is not adjacent to any vertex of $N(x)$. Also, y is not adjacent to x_1, for otherwise $b(G - x_1u) \geq b(G)$. So y is adjacent to some non b-vertex u_2 of color $c(x_1)$. Vertex u_2 is adjacent to x_2, for otherwise $\{x_2, y, u_1, u_2\}$ form a $K_{1,3}$. Since x_1 is a b-vertex, x_1 is adjacent to some non b-vertex u_3 of color $c(x_2)$. This implies that $G = F_2$.

Case 1.3: $b(G) = 3$.

We may suppose that $A = \{x_1\}$ and $B = \{u\}$. Vertex u cannot be adjacent to x_1, for otherwise u would be a b-vertex. So x_1 is adjacent to some b-vertex y of color $c(u)$. Since x is not adjacent to y, we have y adjacent to some vertex u_1 of color $c(x)$. It is easy to check that u_1 is not adjacent to any vertex of $N(x)$. This implies that $G = P_5 = H_0$ with $|V(H_0)| = 5$.

Case 2: $N(x)$ contains two non b-vertices.

Let $u, v \in N(x)$ be two non b-vertices, and let $\{x_i : 1 \leq i \leq 2\}$ denote the set of all b-vertices of $N(x)$. By Lemma 4.4, **Case 1**, u, v cannot both belong to A or B. So we may suppose that $u \in A$ and $v \in B$.

Case 2.1: $b(G) = 5$. We distinguish among two cases.

a) $A = \{x_1, u\}$ and $B = \{x_2, v\}$. Let y_1 and y_2 be two vertices of colors $c(u)$ and

$c(v)$, respectively. By Lemma 4.3, y_1 and y_2 are b-vertices. Vertex y_1 is not adjacent to x_1, for otherwise $b(G - x_1u) \geq b(G)$. Then y_1 is not adjacent to any vertex of A. So y_1 is adjacent to some vertex u_1 of color $c(x_1)$. By Theorem 2.3, u_1 cannot be adjacent to u. By a symmetric argument, y_2 is not adjacent to any vertex of B. Also y_2 is adjacent to some vertex u_2 of color $c(x_2)$, and u_2 cannot be adjacent to v. So we claim that

$$y_1, y_2 \text{ are adjacent to the same vertex } u_3 \text{ of color } c(x).$$

Since y_1 is not adjacent to x, y_1 is adjacent to a some vertex u' of color $c(x)$. Likewise y_2 is adjacent to a some vertex u'' of color $c(x)$. By Theorem 2.3, u', u'' are non b-vertices. So by Lemma 4.3, $u' = u''$. Let $u_3 = u' = u''$. So the claim is proved.

Clearly, vertex u_3 cannot be adjacent to any vertex of $N(x)$. If y_1 is adjacent to v, then $\{v, y_1, u_1, u_3\}$ form a $K_{1,3}$, a contradiction. Similarly, y_2 cannot be adjacent to u. By a symmetric argument, y_1 is not adjacent to u_2, and y_2 is not adjacent to u_1. Since y_1 is a b-vertex, it is adjacent to x_2, otherwise we would have a contradiction to Lemma 4.3. Similarly, y_2 is adjacent to x_1. If u_1 is not adjacent to x_2, then $\{x_2, y_1, u_1, u_3\}$ form a $K_{1,3}$, a contradiction. Likewise u_2 is adjacent to x_1. Vertex y_1 is adjacent to y_2, otherwise we would have a contradiction to Lemma 4.3. This implies that $G = F_1$.

b) $A = \{x_1, x_2, u\}$ and $B = \{v\}$. Let y be a b-vertex of color $c(u)$. Similarly to the previous case, we can show that y cannot be adjacent to $A \cup \{x\}$. Then y is adjacent to three non b-vertices of colors $c(x)$, $c(x_1)$ and $c(x_2)$. By Theorem 2.3, $N[y]$ contains a $K_{1,3}$, a contradiction. This case cannot occur.

Case 2.2: $b(G) = 4$.

$A = \{x_1, u\}$ and $B = \{v\}$. Let y_1 and y_2 be two b-vertices of colors $c(u)$ and $c(v)$ respectively. Vertex y_1 cannot be adjacent to x_1, for otherwise $b(G - x_1u) \geq b(G)$. So y_1 is adjacent to some vertex u_1 of color $c(x_1)$. Clearly, u_1 is a non b-vertex that is non-adjacent to any vertex of $N(x)$. Since y_1 and y_2 are non-adjacent to x, by a similar argument to the previous cases, we can show that there exists a non b-vertex u_2 of color $c(x)$ which is adjacent in both to y_1 and y_2. Obviously, u_2 cannot be adjacent to $N(x)$. If y_1 is adjacent to v, then $\{v, y_1, u_1, u_2\}$ form a $K_{1,3}$, a contradiction. So y_1 is adjacent to y_2, for otherwise we would have a contradiction to Lemma 4.3. If y_2 is adjacent to u_1, then x_1 is adjacent to v. This implies that $G = F_3$. Otherwise y_2 is adjacent to x_1. So $G = F_2$.

Case 2.3: $b(G) = 3$.

$A = \{u\}$ and $B = \{v\}$. Let y_1 and y_2 be two b-vertices of colors $c(u)$ and $c(v)$ respectively. If y_1 and y_2 have the same neighbor, say x', then x' is a b-vertex of color $c(x)$, a contradiction to Theorem 2.3. It follows that y_1 (respectively, y_2) is adjacent to a some non b-vertex u_1 (respectively, u_2) of color $c(x)$, a contradiction to Lemma 4.3. This case cannot occur.

Finally, it is easy to check that if G is edge b-critical with $b(G) = 2$, then $G = K_2$. ∎

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