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A neighborhood and degree condition for panconnectivity

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Abstract

Let G be a 2-connected graph of order n with $x, y \in V(G)$. For $u, v \in V(G)$, let $P_i[u, v]$ denote the path with i vertices which connects u and v. In this paper, we prove that if $n \geq 5$ and $|N_G(u) \cup N_G(v)| + d_G(w) \geq n+1$ for every triple of independent vertices u, v, w of G, then there exists a $P_i[x, y]$ in G for $5 \leq i \leq n$, or G belongs to one of three exceptional classes. This implies a positive answer to a conjecture by Wei and Zhu [*Graphs Combin.* 14 (1998), 263–274].

1 Introduction

In this paper, we consider only finite graphs without loops or multiple edges. For standard graph-theoretic terminology not explained in this paper, we refer the reader to [1]. We denote by $N_G(x)$ the neighborhood of a vertex x in a graph G. For a subgraph H of G and a vertex $x \in V(G)$, we also let $N_H(x) = N_G(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$. If there is no danger of confusion, we often identify a subgraph H of a graph G with its vertex set $V(H)$. For $X \subseteq V(G)$, $N_G(X)$ denotes the set of vertices in G which are adjacent to some vertex in X, and $G[X]$ denotes the subgraph induced by X in G. For $A, B \subseteq V(G)$, we denote by $E(A, B)$ the set of edges of $E(G)$ connecting a vertex of A and a vertex of B. For $u, v \in V(G)$, let $P_i[u, v]$ denote the path with i vertices which connects u and v. A graph G is called m-panconnected if for any $u, v \in V(G)$ and for any $m \le i \le |V(G)|$, there exists a $P_i[u, v]$.

In [2], Wei and Zhu proved the following theorem.

Theorem 1 (Wei and Zhu [2])**.** *Let G be a 2-connected graph of order n. Suppose that $|N_G(u) \cup N_G(v)| + d_G(w) \ge n + 1$ for every triple of independent vertices u, v, w of G. Then for any two vertices x, y of G such that $\{x, y\}$ is not a cut set of G, there exists a $P_i[x, y]$ in G for $7 \le i \le n$.*

In addition, Wei and Zhu also conjectured that Theorem 1 holds in case $i = 6$, and constructed an example which shows that i cannot be reduced to 5. In this paper, we establish this conjecture and determine all the exceptional classes which make i to be reduced to 5. We will show the exceptional classes of Theorem 2 at the end of this section.

Theorem 2. *Let G be a 2-connected graph of order $n \ge 5$. Suppose that $|N_G(u) \cup N_G(v)| + d_G(w) \ge n + 1$ for every triple of independent vertices u, v, w of G. Then for any vertices $x, y \in V(G)$, there exists a $P_i[x, y]$ in G for $5 \le i \le n$, or G belongs to one of three exceptional classes $\mathcal{G}_i$ $(1 \le i \le 3)$.*

By observing the exceptional classes of Theorem 2, we can obtain the following result.

Theorem 3. *Let G be a 2-connected graph of order $n \ge 6$. Suppose that $|N_G(u) \cup N_G(v)| + d_G(w) \ge n + 1$ for every triple of independent vertices u, v, w of G. Then for any vertices $x, y \in V(G)$, there exists a $P_i[x, y]$ in G for $6 \le i \le n$, or G belongs to an exceptional class $\mathcal{G}_1$.*

Notice that Theorems 2 and 3 yield results concerning panconnectivity.

Exceptional classes of Theorem 2

Let $G(x, y)$ be a 2-connected graph with $x, y \in V(G(x, y))$. For $1 \le i \le 3$, let $\mathcal{G}_i$ be a set of graphs whose elements $G_i = G_i(x, y)$ satisfy the following properties:

(1) $\mathcal{G}_1$: $\{x, y\}$ is a cut set of G_1, and $G_1 \backslash \{x, y\}$ consists of two components C_1, C_2 such that C_1 and C_2 are complete.

(2) $\mathcal{G}_2$: For convenience, let $v_1 = x$ and $v_2 = y$. For each $i = 1, 2$, let $A_i = N_{G_2}(v_i) \backslash \{v_{3-i}\}$, and $B_i = N_{G_2}(A_i) \backslash (A_1 \cup A_2 \cup \{v_1, v_2\})$. Then $A_1 \cap A_2 = \emptyset$, $B_1 \cap B_2 = \emptyset$, $E(A_1, A_2) = \emptyset$, $E(B_1, B_2) \ne \emptyset$, $V(G_2) = A_1 \dot\cup A_2 \dot\cup B_1 \dot\cup B_2 \dot\cup \{v_1, v_2\}$, and $v_1 v_2 \in E(G_2)$. For each $i = 1, 2$, $B_i \ne \emptyset$, $G[A_i \cup B_i]$ is complete, and all the vertices of A_i are joined with v_i.

(3) $\mathcal{G}_3$: Let $A = N_{G_3}(x) \cup N_{G_3}(y)$ and $W = V(G_3) \backslash (A \dot\cup \{x, y\})$. Then $A = N_{G_3}(x) \cap N_{G_3}(y)$, A consists of at most two components, the order of the component of A is at most two, W is complete, $N_W(u) \ne \emptyset$ and $N_W(u) \cap N_W(v) = \emptyset$ hold for any $u, v \in A$, and $xy \in E(G_3)$.

2 Proof of Theorem 2

Suppose that G satisfies the assumption of Theorem 2, and let $x, y \in V(G)$. In case $n = 5$, we can easily obtain the conclusion. Therefore we may assume $n \geq 6$. Suppose that G belongs to none of the exceptional classes $\mathcal{G}_i$ $(i = 1, 2, 3)$.

By the assumption of Theorem 2, we can easily obtain the following fact.

Fact 1. *For any triple of independent vertices u, v, w of G, the following hold:*

(i) $d_G(u) \geq 4.$;

(ii) $|N_G(u) \cap (N_G(v) \cup N_G(w))| \geq 4.$

By Fact 1, we obtain the following fact.

Fact 2. (i) *For $u \in V(G)$ such that $d_G(u) \leq 3$, $G[V(G) \backslash (N_G(u) \cup \{u\})]$ is complete.*

(ii) *For any triple of independent vertices u, v, w of G such that $d_G(u) = 4$, $N_G(v) \cup N_G(w) = V(G) \backslash \{u, v, w\}$.*

Fact 3. *$\{x, y\}$ is not a 2-cut set of G.*

Proof. Suppose that $\{x, y\}$ is a 2-cut set of G. By Fact 1 (ii), we can easily see that $G \backslash \{x, y\}$ consists of two components, each of which is complete. Hence G belongs to the exceptional class $\mathcal{G}_1$, a contradiction. $\square$

Let $v_1 = x$ and $v_2 = y$, and let $\mathcal{P}_m$ be the set of a path $P_m[v_1, v_2]$ in G. By Theorem 1 and Fact 3, we see that there exists a $P_i[v_1, v_2]$ in G for $7 \leq i \leq n$. Therefore it suffices to show that $\mathcal{P}_5 \neq \emptyset$ and $\mathcal{P}_6 \neq \emptyset$. Suppose that $\mathcal{P}_5$ or $\mathcal{P}_6$ is an empty set.

For $i = 1, 2$, let $A_i := N_G(v_i) \backslash \{v_{3-i}\}$, $B_i := N_G(A_i) \backslash (A_1 \cup A_2 \cup \{v_1, v_2\})$, $A := A_1 \cup A_2$ and $B := B_1 \cup B_2$.

Lemma 2.1. *The following hold:*

(i) *$|A_1| \geq 2$ and $|A_2| \geq 2$;*

(ii) *if $\mathcal{P}_3 = \emptyset$, then $\mathcal{P}_4 \neq \emptyset$ and $\mathcal{P}_5 \neq \emptyset$.*

Proof. (i) Suppose that $|A_1| = 1$ or $|A_2| = 1$. By symmetry, we may assume $|A_1| = 1$. Let $y_1 \in A_1$. By Fact 2 (i), $G[V(G) \backslash \{v_1, v_2, y_1\}]$ is a complete graph of order at least 3. Since G is 2-connected, there exist two distinct vertices $u_1, u_2 \in V(G) \backslash \{v_1, v_2, y_1\}$ such that $u_1y_1, u_2v_2 \in E(G)$. Therefore $\mathcal{P}_5 \neq \emptyset$ and $\mathcal{P}_6 \neq \emptyset$, a contradiction. Therefore the statement (i) holds.

(ii) First, suppose that $\mathcal{P}_4 \neq \emptyset$ and $\mathcal{P}_3 = \mathcal{P}_5 = \emptyset$. Let $y_1 \in A_1$ and $y_2 \in A_2$ such that $y_1y_2 \in E(G)$. By the statement (i), there exist $y_1' \in A_1 \backslash \{y_1\}$ and $y_2' \in$

$A_2\backslash\{y_2\}$. Since $\mathcal{P}_3 = \mathcal{P}_5 = \emptyset$, $\{y_1, y_1', v_2\}$ and $\{y_2, y_2', v_1\}$ are independent sets, and $(N_G(y_1) \cup N_G(y_1')) \cap (N_G(y_2) \cup N_G(y_2')) = N_G(v_1) \cap N_G(v_2) = \emptyset$. Therefore we obtain $|N_G(y_1) \cup N_G(y_1')| + |N_G(y_2) \cup N_G(y_2')| + d_G(v_1) + d_G(v_2) \leq n+n = 2n$, a contradiction.

Next, suppose that $\mathcal{P}_3 = \mathcal{P}_4 = \emptyset$. We shall show that G belongs to the exceptional class $\mathcal{G}_2$, that is, (1) $A_1 \cap A_2 = \emptyset$ and $E(A_1, A_2) = \emptyset$; (2) for $i = 1, 2$, $B_i \neq \emptyset$, $B_1 \cap B_2 = \emptyset$ and $G[A_i \cup B_i]$ is complete; (3) $V(G) = \{v_1, v_2\} \cup A \cup B$ and $E(B_1, B_2) \neq \emptyset$; and (4) $v_1v_2 \in E(G)$. We will prove the properties (1)–(4).

(1) Since $\mathcal{P}_3 = \emptyset$, $A_1 \cap A_2 = \emptyset$. Since $\mathcal{P}_4 = \emptyset$, $E(A_1, A_2) = \emptyset$.

(2) If there exist $u_1, u_1' \in A_1$ such that $u_1u_1' \notin E(G)$, then $\{u_1, u_1', v_2\}$ is an independent set. This contradicts Fact 1(ii). Therefore, by symmetry of A_1 and A_2, $G[A_1]$ and $G[A_2]$ are complete. By Fact 3, $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$. Since $\mathcal{P}_5$ or $\mathcal{P}_6$ is an empty set and $|A_1|, |A_2| \geq 2$, it follows that $B_1 \cap B_2 = \emptyset$. Therefore, by Fact 1 (ii), $G[A_1 \cup B_1]$ and $G[A_2 \cup B_2]$ are complete.

(3) Suppose that there exists $w \in V(G\backslash(\{v_1, v_2\} \cup A \cup B))$, and let $y_2 \in A_2$. Then $\{v_1, y_2, w\}$ is an independent set. This contradicts Fact 1 (ii). Hence $V(G) = \{v_1, v_2\} \cup A \cup B$. Therefore, by Fact 3, $E(B_1, B_2) \neq \emptyset$.

(4) If $v_1v_2 \notin E(G)$, then for $z_1 \in B_1$, $\{v_1, v_2, z_1\}$ is an independent set. This contradicts Fact 1 (ii). Hence $v_1v_2 \in E(G)$. Thus, G belongs to the exceptional class $\mathcal{G}_2$, a contradiction. ☐

Lemma 2.2. *If $\mathcal{P}_5 = \emptyset$, then the following hold.*

(i) *The independence number of $G[A]$ is at most two.*

(ii) *$G[A]$ is a connected graph of order at least three.*

Proof. (i) Note that $x \in N_G(v_1)$ and $y, z \in N_G(v_2)$ are not independent, since $(N_G(y) \cup N_G(z)) \cap N_G(x) \subseteq \{v_1, v_2\}$. Suppose that $x, y, z \in N_G(v_2)$ are independent. By Lemma 2.1 (ii), there exists $u \in N_G(v_1) \cap N_G(v_2)$. Then, by the above note, we see that $y, z \notin N_C(v_1)$, and may assume that $yu, zu \in E(G)$. Then $N_G(v_1) \cap (N_G(y) \cup N_G(z)) \subseteq \{u, v_2\}$, since $\mathcal{P}_5 = \emptyset$, a contradiction. By symmetry of v_1 and v_2, we obtain the statement (i).

(ii) By the statement (i), the number of the components of $G[A]$ is at most two. Suppose that the number of components of $G[A]$ is two. Let W_1, W_2 be the components of $G[A]$, and $W_3 = G\backslash(W_1 \cup W_2 \cup \{v_1, v_2\})$. By the statement (i), W_1 and W_2 are complete. By Fact 3, $W_3 \neq \emptyset$, $N_G(W_1) \cap W_3 \neq \emptyset$ and $N_G(W_2) \cap W_3 \neq \emptyset$ hold. By Lemma 2.1 (ii), there exists $w_0 \in N_G(v_1) \cap N_G(v_2)$. By the symmetry of v_1 and v_2, we may assume that $w_0 \in W_2$ and $W_1 \cap N_G(v_1) \neq \emptyset$. Then $|W_2| \leq 2$ since $\mathcal{P}_5 = \emptyset$. Suppose that $W_1 \cap N_G(v_2) = \emptyset$. Since $d_G(v_2) \leq 3$, $W_1 \cup W_3$ is complete by Fact 2 (i). If there exists $w' \in W_2$ such that $w' \notin N_G(v_2)$, then $W_1 \subseteq N_G(w')$ holds by Fact 2 (i). This contradicts the fact that W_1 and W_2 are components of $G[A]$. Therefore $W_2 \subseteq N_G(v_2)$ holds. These imply $\mathcal{P}_5 \neq \emptyset$, a contradiction. Therefore we may assume that $W_1 \cap N_G(v_2) \neq \emptyset$. Then $|W_1| \leq 2$ since $\mathcal{P}_5 = \emptyset$.

First, suppose that there exist $w_1, w_2 \in A \cup W_3$ such that $\{v_1, w_1, w_2\}$ is an independent set. By Fact 1 (i), we may assume that $w_1 \in W_3$. By Fact 1 (ii), $4 \leq |(N_G(w_1) \cup N_G(w_2)) \cap N_G(v_1)| = |(N_{A\cup\{v_2\}}(w_1) \cup N_{A\cup\{v_2\}}(w_2)) \cap N_{A\cup\{v_2\}}(v_1)| \leq 4$. This implies $A \backslash \{w_2\} \subseteq (N_G(w_1) \cup N_G(w_2)) \cap N_G(v_1)$. Suppose that $w_2 \in W_j$ $(j = 1, 2)$. By the above inequality, $|W_1| = |W_2| = 2$ holds. Then $W_{3-j} \subseteq N_G(w_1) \cap N_G(v_1)$ since $N_{W_{3-j}}(w_2) = \emptyset$. Since $N_{W_{3-j}}(v_2) \neq \emptyset$, we have $\mathcal{P}_5 \neq \emptyset$, a contradiction. Therefore $w_2 \in W_3$. Then $A \subseteq N_G(v_1) \cap (N_G(w_1) \cup N_G(w_2))$ holds. By the symmetry of w_1 and w_2, we may assume that $w_0 \in N_G(w_2)$. Then $A \backslash \{w_0\} \cap N_G(w_2) = \emptyset$ holds since $\mathcal{P}_5 = \emptyset$. This implies $A \backslash \{w_0\} \subseteq N_G(w_1)$. Then $\mathcal{P}_5 \neq \emptyset$ holds since $A \subseteq N_G(v_1)$ and $W_1 \cap N_G(v_2) \neq \emptyset$, a contradiction.

Next, suppose that there exist no $w_1, w_2 \in A \cup W_3$ such that $\{v_1, w_1, w_2\}$ is an independent set. Then W_3 is complete, and if there exists $w \in W_i$ $(i = 1, 2)$ such that $w \notin N_G(v_1)$, then $W_3 \subseteq N_G(w)$ and $W_{3-i} \subset N_G(v_1)$. We now show that G belongs to the exceptional class $\mathcal{G}_3$, that is, (1) $A \subseteq N_G(v_i)$ for $i = 1, 2$, (2) $N_{W_3}(w) \neq \emptyset$ for any $w \in A$ and $N_{W_3}(w) \cap N_{W_3}(w') = \emptyset$ for any distinct $w, w' \in A$, and (3) $v_1v_2 \in E(G)$.

(1) Assume that there exists $w_3 \in A$ such that $w_3 \notin N_G(v_1)$. Then note that $w_3 \in N_G(v_2)$ holds, and moreover, $A \backslash \{w_3\} \subseteq N_G(v_1)$, because $N_{W_i}(v_1) \neq \emptyset$ holds for any $i = 1, 2$. Since $N_G(W_i) \cap W_3 \neq \emptyset$ holds for $i = 1, 2$, we obtain $\mathcal{P}_5 \neq \emptyset$, a contradiction. Therefore $A \subseteq N_G(v_1)$ holds, and so $A \subseteq N_G(v_2)$ by the symmetry of v_1 and v_2.

(2) For some $j = 1, 2$ and $w_4 \in W_j$, suppose that $N_{W_3}(w_4) = \emptyset$. Then since $d_G(w_4) \leq 3$, it follows from Fact 2 (i) that $W_{3-j} \cup W_3$ is complete.

Since $N_G(W_j \backslash \{w_4\}) \cap W_3 \neq \emptyset$, $\mathcal{P}_5 \neq \emptyset$ holds, a contradiction. Therefore $N_{W_3}(w) \neq \emptyset$ holds for any $w \in A$, and moreover, since $\mathcal{P}_5 = \emptyset$, $N_{W_3}(w) \cap N_{W_3}(w') = \emptyset$ holds for any distinct $w, w' \in A$.

(3) Suppose that $v_1v_2 \notin E(G)$. Then, for $w \in W_3$, $\{w, v_1, v_2\}$ is an independent set, and $4 \leq |(N_G(v_1) \cup N_G(v_2)) \cap N_G(w)|$ holds by Fact 1 (ii). This implies that $A \subseteq N_G(w)$, a contradiction. Therefore $v_1v_2 \in E(G)$ holds. Therefore G belongs to the exceptional class $\mathcal{G}_3$, a contradiction.

Therefore $G[A]$ is connected. Suppose that $|A| = 2$. As in the proof of Fact 3, we see that $H := G \backslash (A \cup \{v_1, v_2\})$ is complete and $v_1v_2 \in E(G)$. By Lemma 2.1 (i), $A \subseteq N_G(v_1) \cap N_G(v_2)$. Since $\mathcal{P}_5 = \emptyset$, $N_H(a) \cap N_H(b) = \emptyset$ for $\{a, b\} = A$. Therefore G belongs to the exceptional class $\mathcal{G}_3$, a contradiction. □

Lemma 2.3. $\mathcal{P}_4 \neq \emptyset$ *and* $\mathcal{P}_6 = \emptyset$.

Proof. It suffices to show that $\mathcal{P}_4 \neq \emptyset$ and $\mathcal{P}_5 \neq \emptyset$, since $\mathcal{P}_5 = \emptyset$ or $\mathcal{P}_6 = \emptyset$. By Lemma 2.1 (ii), we may assume that $\mathcal{P}_3 \neq \emptyset$. Let $P := v_1v_3v_2 \in \mathcal{P}_3$.

Suppose that $\mathcal{P}_5 = \emptyset$. Then there exists no path of length three in A such that one of the endvertices is v_3. This implies that $A \backslash \{v_3\} \subseteq N_G(v_3)$ by Lemma 2.2 (ii), and $A \backslash \{v_3\}$ is an independent set. By Lemma 2.2 (i), $G[A]$ is a path of order three, and v_3 is not the endvertex. Let $a_1, a_2 \in A \backslash \{v_3\}$. By Lemma 2.1 (i), we may assume that $a_i \in N_G(v_i)$ for $i = 1, 2$. Then $v_1a_1v_3a_2v_2 \in \mathcal{P}_5$, a contradiction. Therefore

$\mathcal{P}_5 \neq \emptyset$, and so $\mathcal{P}_6 = \emptyset$.

Suppose that $\mathcal{P}_4 = \emptyset$. Assume that $N_{G\backslash P}(v_1) = N_{G\backslash P}(v_2) = \{y\}$. Then $d_G(v_1) \leq 3$. Let $D = G\backslash(P \cup \{y\})$. By Fact 2 (i), $G[V(D)]$ is complete. Since G is 2-connected, we see that $N_D(v_3) \neq \emptyset$, $N_D(y) \neq \emptyset$ and $|N_D(v_3) \cup N_D(y)| \geq 2$. Then $\mathcal{P}_6 \neq \emptyset$, a contradiction. Therefore, there exist $y_1 \in N_{G\backslash P}(v_1)$ and $y_2 \in N_{G\backslash P}(v_2)$ such that $y_1 \neq y_2$. Since $\mathcal{P}_4 = \emptyset$, $\{v_3, y_1, y_2\}$ is an independent set. By Fact 1(ii), $|N_G(v_3) \cap (N_G(y_1) \cup N_G(y_2))| \geq 4$ and so there exist $y_3, y_4 \in N_G(v_3) \cap (N_G(y_1) \cup N_G(y_2))\backslash\{v_1, v_2\}$ such that $y_3 \neq y_4$. Since $\mathcal{P}_4 = \emptyset$, $y_3v_1, y_4v_1 \notin E(G)$ holds. Since $\mathcal{P}_6 = \emptyset$, $y_3y_4 \notin E(G)$ holds. Therefore $\{v_1, y_3, y_4\}$ is an independent set. Since $\mathcal{P}_6 = \emptyset$, $(N_G(y_1) \cup N_G(y_2)) \cap (N_G(y_3) \cup N_G(y_4)) = \emptyset$. Since $\mathcal{P}_4 = \emptyset$, $N_G(v_1) \cap N_G(v_3) \subseteq \{v_2\}$. Then we obtain $(|N_G(y_3) \cup N_G(y_4)| + d_G(v_1)) + (|N_G(y_1) \cup N_G(y_2)| + d_G(v_3)) \leq n+1+n$, a contradiction. □

Lemma 2.4. *There exists no $v \in V(G\backslash P)$ such that $N_P(v) = \{v_1, v_2\}$ for any $P \in \mathcal{P}_4$.*

Proof. Suppose that there exist $P = v_1v_4v_3v_2 \in \mathcal{P}_4$ and $v \in V(G\backslash P)$ such that $N_P(v) = \{v_1, v_2\}$. By Fact 3, we can take a path $Q = vy_1 \cdots y_k v'$, where $y_i \in V(G\backslash P)$ and $v' \in \{v_3, v_4\}$. Choose such a path Q as short as possible. By the symmetry of v_3 and v_4, we may assume $v' = v_3$. In case $k = 1, 2$, we can easily find a $P_6[v_1, v_2]$, a contradiction. Therefore we may assume that $k \geq 3$. Since $N_P(v) = \{v_1, v_2\}$ and by the minimality of $|V(P)|$, $\{v, y_2, v_4\}$ is an independent set, and furthermore, $N_P(v_4) \cap (N_P(v) \cup N_P(y_2)) \subseteq \{v_1, v_2\}$, and $N_{G\backslash P}(v_4) \cap N_{G\backslash P}(v) = \emptyset$. Therefore, by Fact 1 (ii), we have $|N_{G\backslash P}(v_4) \cap N_{G\backslash P}(y_2)| \geq 2$. Take $w_1 \in N_{G\backslash P}(v_4) \cap N_{G\backslash P}(y_2)$ with $w_1 \neq y_3$. By the minimality of $|V(P)|$, we obtain $k = 3$ and $y_3, w_1 \notin N_G(v)$. Since $\mathcal{P}_6 = \emptyset$, $y_3w_1 \notin E(G)$. Hence $\{v, y_3, w_1\}$ is an independent set. Again, by the minimality of $|V(P)|$, $N_G(v) \cap (N_G(y_3) \cup N_G(w_1)) \subseteq \{v_1, v_2\}$, which contradicts Fact 1 (ii). □

Lemma 2.5. $|\{v \in V(G\backslash P)|\ |N_P(v)| \geq 2\}| \leq 1$ *holds for any $P \in \mathcal{P}_4$.*

Proof. Suppose that there exists $P = v_1v_4v_3v_2 \in \mathcal{P}_4$ such that

$$|\{v \in V(G\backslash P)|\ |N_P(v)| \geq 2\}| \geq 2,$$

say $x_1, x_2 \in \{v \in V(G\backslash P) \mid |N_P(v)| \geq 2\}$.

At first, we shall show that $|N_P(x_1) \cup N_P(x_2)| \geq 3$ holds. Assume $|N_P(x_1) \cup N_P(x_2)| = 2$, and $N_P(x_1) \cup N_P(x_2) = \{v_i, v_j\}$ $(i < j)$. By Lemma 2.4, we may assume that $\{v_i, v_j\} = \{v_2, v_3\}$, $\{v_i, v_j\} = \{v_2, v_4\}$ or $\{v_i, v_j\} = \{v_3, v_4\}$. If $x_1x_2 \notin E(G)$, since $\mathcal{P}_6 = \emptyset$, $\{x_1, x_2, v_{j+1}\}$ is an independent set, and $(N_G(x_1) \cup N_G(x_2)) \cap N_G(v_{j+1}) \subseteq \{v_i, v_j\}$, a contradiction. Therefore $x_1x_2 \in E(G)$ holds. Since $\mathcal{P}_6 = \emptyset$, we may assume that $i = 2$ and $j = 4$, and hence $v_1v_3 \notin E(G)$. Then $\{x_1, v_1, v_3\}$ is an independent set, and $(N_G(v_1) \cup N_G(v_3)) \cap N_G(x_1) \subseteq \{v_2, v_4\}$, a contradiction. Therefore $|N_P(x_1) \cup N_P(x_2)| \geq 3$ holds.

By considering a reverse orientation of P, we may assume that $v_1, v_3, v_4 \in N_P(x_1) \cup N_P(x_2)$ or $v_1, v_2, v_3 \in N_P(x_1) \cup N_P(x_2)$. Since $\mathcal{P}_6 = \emptyset$ and by Lemma 2.4, we may assume that $v_1, v_3 \in N_P(x_1)$, by symmetry of x_1 and x_2. Then (i) $v_3, v_4 \in N_P(x_2)$, (ii) $v_4, v_1 \in N_P(x_2)$, or (iii) $v_2, v_3 \in N_P(x_2)$ and $v_4 \notin N_P(x_2)$ holds. Let $D = G \backslash (P \cup \{x_1, x_2\})$. If (i) holds, then $\{x_1, v_2, v_4\}$ is an independent set and $N_D(x_1) \cap (N_D(v_2) \cup N_D(v_4)) = \emptyset$ holds, which implies $N_G(x_1) \cap (N_G(v_2) \cup N_G(v_4)) \subseteq \{v_1, v_3\}$, a contradiction. If (ii) holds, then $\{x_1, x_2, v_2\}$ is an independent set and $N_D(x_1) \cap (N_D(x_2) \cup N_D(v_2)) = \emptyset$ holds, a contradiction. If (iii) holds, then $\{x_1, x_2, v_4\}$ is an independent set and $N_D(v_4) \cap (N_D(x_1) \cup N_D(x_2)) = \emptyset$ holds, a contradiction. □

By Lemma 2.5, for any $P = v_1 v_4 v_3 v_2 \in \mathcal{P}_4$, there exists a path $Q = v_i y_1 \cdots y_k v_j$ ($k \geq 2$, $i \neq j$) such that $y_i \in V(G \backslash P)$ ($1 \leq i \leq k$) and $d_P(y_1) = 1$. Take P and Q so that (i) Q is as short as possible, and (ii) $\{v_i, v_j\} = \{v_1, v_2\}$ if possible, subject to (i).

Lemma 2.6. $k = 2$.

Proof. By the minimality of $|V(Q)|$, $y_1 y_3 \notin E(G)$ holds. Note that $d_P(y_1) = 1$.

First, suppose that $k = 3$. Since $\mathcal{P}_6 = \emptyset$, considering a reverse orientation of P, we may assume that (i) $Q = v_1 y_1 y_2 y_3 v_4$, (ii) $Q = v_1 y_1 y_2 y_3 v_2$, (iii) $Q = v_1 y_3 y_2 y_1 v_4$, or (iv) $Q = v_4 y_1 y_2 y_3 v_3$ holds. Suppose that (i) holds. Since $\mathcal{P}_6 = \emptyset$, we have $v_2 v_4 \notin E(G)$. Therefore $\{v_2, v_4, y_1\}$ is an independent set. By the minimality of $|V(Q)|$, $(N_G(v_2) \cup N_G(v_4)) \cap N_G(y_1) \subseteq \{v_1\}$, a contradiction. Suppose that at least one of (ii)–(iv) holds. If (ii) holds, let $a = v_3$; otherwise, let $a = v_2$. Since $\mathcal{P}_6 = \emptyset$, $a y_3 \notin E(G)$. Therefore $\{y_1, y_3, a\}$ is an independent set. By the minimality of $|V(Q)|$, $(N_G(y_1) \cup N_G(y_3)) \cap N_G(a) \subseteq V(P) \backslash \{a\}$, a contradiction.

Next, suppose that $k \geq 4$. Let $N_P(y_1) = \{v_i\}$. By the minimality of $|V(Q)|$, $\{y_1, y_3, v_p\}$ and $\{y_1, y_3, v_q\}$ are independent sets for $v_p, v_q \in V(P) \backslash \{v_i\}$, and $(N_P(y_1) \cup N_P(y_3)) \cap (N_P(v_p) \cup N_P(v_q)) \subseteq \{v_i\}$ holds. Therefore we can take $w_1 \in N_{G \backslash P}(y_3) \cap N_{G \backslash P}(v_p)$ and $w_2 \in N_{G \backslash P}(y_3) \cap N_{G \backslash P}(v_q)$ such that $w_1 \neq w_2$ by Fact 1 (ii). Then $v_p w_1 y_3 w_2 v_q$ is shorter than Q, a contradiction. □

Lemma 2.7. $N_P(y_1) \cup N_P(y_2) = \{v_1, v_2\}$.

Proof. Assume not. Since $\mathcal{P}_6 = \emptyset$, we may assume that Q connects v_1 and v_3. Let $Q' = v_1 a b v_3$ ($\{a, b\} = \{y_1, y_2\}$). Since $\mathcal{P}_6 = \emptyset$, $\{a, v_2, v_4\}$ is an independent set and $N_P(a) \cap (N_P(v_4) \cup N_P(v_2)) \subseteq \{v_1, v_3\}$. This implies that there exist $w_1, w_2 \in N_{G \backslash P}(a) \cap N_{G \backslash P}(v_2)$ such that $w_1 \neq w_2$ by Fact 1 (ii). By Lemma 2.5, we may assume $d_P(w_1) = 1$. Since $\mathcal{P}_6 = \emptyset$, $d_P(a) = 1$ holds. Therefore, the existence of a path $v_1 a w_1 v_2$ contradicts the choice of Q. □

By Lemmas 2.4 and 2.7, we may assume that $N_P(y_1) = \{v_1\}$, $N_P(y_2) = \{v_2\}$ and so $Q = v_1 y_1 y_2 v_2$.

By Fact 3, there exists a path $R = y_i z_1 \cdots z_l v_j$ ($1 \leq i \leq 2$, $3 \leq j \leq 4$), where $z_h \in V(G \backslash (P \cup Q))$ ($1 \leq h \leq l$). Take such a path R as short as possible. By the symmetry of y_1 and y_2, we may assume that $i = 1$.

First, suppose that $l = 1$. Since $\mathcal{P}_6 = \emptyset$, $v_4z_1 \notin E(G)$. This implies $v_3z_1 \in E(G)$ and $v_2v_4 \notin E(G)$. Suppose that $v_2z_1 \in E(G)$. Then $P' = v_1y_1z_1v_2 \in \mathcal{P}_4$, $d_{P'}(v_3) \geq 2$ and $d_{P'}(y_2) \geq 2$ hold. This contradicts Lemma 2.5. Therefore $v_2z_1 \notin E(G)$ holds. Then $\{v_2, v_4, z_1\}$ is an independent set and $N_G(z_1) \cap (N_G(v_2) \cup N_G(v_4)) \subseteq \{v_1, v_3\}$, a contradiction.

Next, suppose that $l = 2$. Since $\mathcal{P}_6 = \emptyset$, $y_2z_2, z_2v_3 \notin E(G)$. This implies $z_2v_4 \in E(G)$. By Lemma 2.7, we have $y_2v_3 \notin E(G)$. Therefore $\{y_2, z_2, v_3\}$ is an independent set. By the minimality of $|V(Q)|$ and since $\mathcal{P}_6 = \emptyset$, we have $N_G(v_3) \cap (N_G(y_2) \cup N_G(z_2)) \subseteq \{v_1, v_2, v_4\}$, a contradiction.

Finally, suppose that $l \geq 3$. By Lemma 2.7 and the minimality of $|V(R)|$, $\{y_1, z_2, v_3\}$ and $\{y_1, z_2, v_4\}$ are independent sets, and $N_G(y_1) \cap (N_G(v_3) \cup N_G(v_4)) \subseteq \{v_1\}$ holds. Let $D = V(P) \cup V(Q) \cup \{z_1\}$. Then $N_D(z_2) \cap (N_D(v_3) \cup N_D(v_4)) \subseteq \{v_2\}$. These imply $|N_{G\backslash D}(z_2) \cap N_{G\backslash D}(v_3)| \geq 2$ and $|N_{G\backslash D}(z_2) \cap N_{G\backslash D}(v_4)| \geq 2$ by Fact 1 (ii). Therefore we can take $w_1 \in N_{G\backslash D}(z_2) \cap N_{G\backslash D}(v_3)$ and $w_2 \in N_{G\backslash D}(z_2) \cap N_{G\backslash D}(v_4)$ such that $w_1 \neq w_2$. Since $\mathcal{P}_6 = \emptyset$ and by the minimality of $|V(R)|$, we can see $\{w_1, w_2, y_1\}$ is an independent set. Hence $N_G(y_1) \cap (N_G(w_1) \cup N_G(w_2)) \subseteq \{v_1, v_2\}$, a contradiction.

References

[1] J.A. Bondy, *Basic Graph Theory: Paths and Circuits* in: Handbook of Combinatorics, Vol. I, eds. R. Graham, M. Grötschel and L. Lovász (Elsevier, Amsterdam), 1995, pp. 5–110.

[2] B. Wei and Y. Zhu, On the panconnectivity of graphs with large degrees and neighborhood unions, *Graphs Combin.* **14** (1998), 263–274.

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