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Equitable $L(2,1)$-labelings of Sierpiński graphs

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Abstract

An $L(2,1)$-*labeling* of a graph G is *equitable* if the number of elements in any two color classes differ by at most one. The *equitable* $L(2,1)$-*labeling number* $\lambda_e(G)$ of G is the smallest integer k such that G has an equitable $L(2,1)$-labeling. *Sierpiński graphs* $S(n,k)$ generalize the Tower of Hanoi graphs — the graph $S(n,3)$ is isomorphic to the graph of the Tower of Hanoi with n disks. In this paper, we show that for any $n \geq 2$ and any $k \geq 2$, $\lambda_e(S(n,k)) = 2k$.

1 Introduction

An $L(2,1)$-*labeling* of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x)-f(y)| \geq 2$ if $d(x,y)=1$ and $|f(x)-f(y)| \geq 1$ if $d(x,y)=2$, where $d(x,y)$ denotes the distance between x and y in G. The $L(2,1)$-*labeling number* $\lambda(G)$ of G is the smallest number k such that G has an $L(2,1)$-labeling with $\max\{f(v) : v \in V(G)\} = k$.

The $L(2,1)$-labeling concept grew from the problem of assigning frequencies to radio transmitters at various nodes in a territory. To avoid interferences, transmitters that are close must receive frequencies that are sufficiently apart. Since frequencies are quantized in practice, there is no loss of generality in assuming that they admit integer values. Motivated by this problem, Hale [11] first proposed the problem with the objective of minimizing the span of frequencies. Later Griggs and Yeh [10] proposed the above defined $L(2,1)$-labeling. Currently, many papers [6, 8, 9, 17, 22] focusing on the $L(2,1)$-labeling and its generalizations are compiled in two recent surveys [2, 24].

In 1973, Meyer [19] introduced the notion of equitable (vertex) coloring of graphs and conjectured that the equitable chromatic number of a connected graph G, which is neither a complete graph nor an odd cycle, is at most $\Delta(G)$. It is well-known [4] that if a graph G has an edge k-coloring, then G has an equitable edge k-coloring. Fu

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[7] first investigated the equitable total coloring of graphs. Later Wang [23] proved the equitable total coloring conjecture of graphs with maximum degree 3. An $L(2,1)$-labeling of a graph G is *equitable* if the number of elements in any color classes differ by at most one. The *equitable* $L(2,1)$*-labeling number* $\lambda_e(G)$ of G is the smallest integer k such that G has an equitable $L(2,1)$-labeling.

In 1997, the graphs $S(n,3)$ were generalized by Klavžar and Milutinović [15] to the *Sierpiński graphs* $S(n,k)$ for $k \geq 3$. The motivation for this generalization came from topological studies of Lipscomb's space [18, 20], where it is shown that this space is a generalization of the Sierpiński triangular curve (Sierpiński gasket). As it turned out, the graphs $S(n,k)$ possess many appealing properties and were studied from different points of view, as for instance several coding [3, 8, 16] and several metric properties [21]. Klavžar et al. [16] proved that the Sierpiński graphs $S(n,k)$ have perfect domination sets [1], i.e., are 100% efficient. For some recent results on the Sierpiński graphs see [6, 12–14].

In Section 2 we introduce the Sierpiński graphs of interest and describe preliminary lemmas. In the last section, we give the equitable $L(2,1)$-labeling numbers of Sierpiński graphs for any $n \geq 2$ and any $k \geq 2$, and also give a new proof of $L(2,1)$-labeling numbers of $S(n,k)$ (in [8], they proved that $\lambda(S(n,k)) = 2k$ for any $n \geq 2$ and any $k \geq 3$).

2 Preliminaries

In this section we first introduce the Sierpiński graphs. Sierpiński graphs $S(n,k)$ are defined for $n \geq 1$ and $k \geq 1$ as follows.

The vertex set of $S(n,k)$ consists all n-tuples of integers $1,2,\dots,k$, that is, $V(S(n,k)) = \{1,2,\dots,k\}^n$ ($|V(S(n,k))| = k^n$). We will write $(u_1,u_2,\dots,u_n)$, $u_r \in \{1,2,\dots,k\}$, $r \in \{1,\dots,n\}$ for the vertices of $S(n,k)$. Two different vertices $(u_1,u_2,\dots,u_n)$, $u_r \in \{1,2,\dots,k\}$, $r \in \{1,\dots,n\}$ and $(v_1,v_2,\dots,v_n)$, $v_r \in \{1,2,\dots,k\}$, $r \in \{1,\dots,n\}$ are adjacent if and only if there exists an $h \in \{1,\dots,n\}$ such that

(a) $u_t = v_t$, for $t = 1,\dots,h-1$;

(b)$u_h \neq v_h$; and

(c)$u_t = v_h$ and $v_t = u_h$ for $t = h+1,\dots,n$.

In the rest of the paper we will write $(u_1u_2\dots u_n)$ for $(u_1,u_2,\dots,u_n)$ or even shorter, $u_1u_2\dots u_n$. The Sierpiński graphs $S(3,4)$ and $S(2,5)$, together with the corresponding vertex labelings, are shown in Fig. 1.

The vertices $(1\dots1)$, $(2\dots2)$, $\dots$, $(k\dots k)$ are called the *extreme vertices* of $S(n,k)$; the other vertices will be called *inner vertices* of $S(n,k)$. For $i = 1,2,\dots,k$, let $S_i(n,k)$ be the subgraph of $S(n,k)$ induced by the vertex set $V_i' = \{(ij_1\dots j_{n-1}) \mid j_r \in \{1,\dots,k\}, r \in \{1,\dots,n-1\}\}$. Clearly $S_i(n,k)$ is isomorphic to $S(n-1,k)$. Consequently, for $k \geq 2$, $S(n,k)$ contains k copies of the graph $S(n-1,k)$ and k^{n-1} copies of the complete graph $S(1,k) = K_k$.

The graph $S(n,k)$ can be constructed inductively from $S(n-1,k)$ as follows (cf. Fig. 1 and Fig. 2):

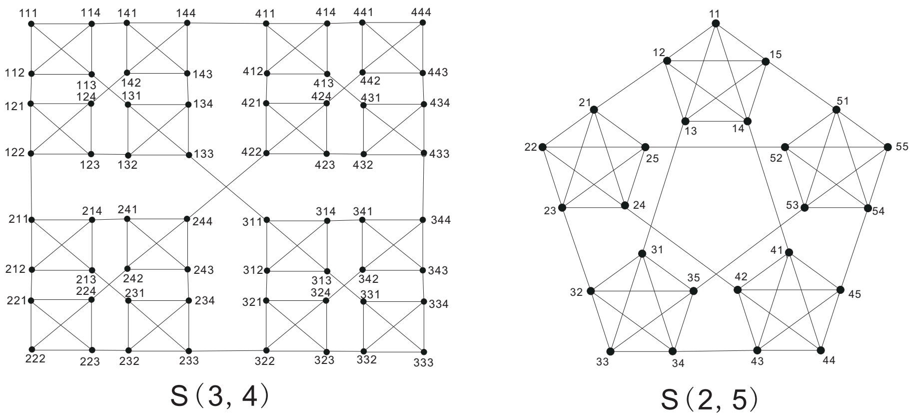

Fig. 1. Sierpiński graphs $S(3,4)$ and $S(2,5)$

Take k copies $S_1(n,k), S_2(n,k), \ldots, S_k(n,k)$ of $S(n-1,k)$, where for $i = 1, 2, \ldots, k$, we have

$$V(S_i(n,k)) = \{(ij_1 \ldots j_{n-1}) \mid (j_1 \ldots j_{n-1}) \in V(S(n-1,k))\}.$$

For any i and any j with $i \neq j$, we add an edge between the extreme vertex $(ijj \ldots j)$ of $S_i(n,k)$ and the extreme vertex $(jii \ldots i)$ of $S_j(n,k)$. Note that no edge incident with the vertex $(ii \ldots i)$ of $S_i(n,k)$ is added.

Lemma 2.1. [8] *For $n = 2$ and any $k \geq 3$, $\lambda(S(2,k)) \geq 2k$.*

Lemma 2.2. [17] *For the complete graph K_k, $\lambda(K_k) = 2k - 2$.*

3 Equitable $L(2,1)$-labelings

In this section we give optimal equitable $L(2,1)$-labeling numbers of Sierpiński graphs and the optimal $L(2,1)$-labeling numbers of $S(n,k)$ by a new proof.

A 2-distance coloring [5] of a graph G is a function φ from the vertex set of graph G to the set of positive integers such that no two vertices at distance less than or equal to 2 are assigned the same color. The 2-*distance chromatic number* of a graph G, denoted by χ_{2d}, is the smallest integer k for which G admits a 2-distance coloring with k colors.

Theorem 3.1. *For any $n \geq 2$ and any $k \geq 2$, there exists a partition of $V(S(n,k))$, say, $V_1, V_2, \ldots, V_{k+1}$, such that for any two distinct vertices u and v in V_i, $d(u,v) > 2$ for $i = 1, \ldots, k+1$.*

Proof. In order to prove the theorem, it is enough to show that $\chi_{2d}(S(n,k)) = k+1$ for any $n \geq 2$ and any $k \geq 2$. For any $n \geq 2$, $S(n,2)$ is a path of length $2^n - 1$. We color the vertices of $S(n,2)$ by the circular coloring using colors $1,2,3$ on this path. It is easy to see that $\chi_{2d}(S(n,2)) = 3$. Now we consider the case for any $n \geq 2$ and any $k \geq 3$.

First we show that $\chi_{2d}(S(n,k)) \geq k+1$. Note that $S(2,k)$ is isomorphic to a subgraph of $S(n,k)$, for any $n \geq 2, k \geq 3$. Thus in order to show that $\chi_{2d}(S(n,k)) \geq k+1$, we only need to show that $\chi_{2d}(S(2,k)) \geq k+1$. The graph $S(2,k)$ consists of k complete subgraphs on k vertices induced by the vertex sets $V_i^* = \{ij \mid j = 1,2,\ldots,k\}$ for $i \in \{1,2,\ldots k\}$. The vertex $ij \in V_i^*$ is adjacent to the vertex $ji \in V_j^*$ for $i \neq j$. The vertex $(12) \in V_1^*$ is adjacent to the vertex $(21) \in V_2^*$, hence the distance between the vertex (21) and a vertex of V_1^* is at most 2 in $S(2,k)$. It holds that $\chi_{2d}(S(n,k)) \geq k+1$.

Now we need to show that $\chi_{2d}(S(n,k)) \leq k+1$. In order to prove $\chi_{2d}(S(n,k)) \leq k+1$, we shall form a 2-distance coloring φ_n of $S(n,k)$ that uses $k+1$ colors by induction on n. Suppose that $n=2$ and $k \geq 3$. We form a coloring φ_2 of $S(2,k)$ that uses $k+1$ colors as follows:

$$\varphi_2(ij) = j\ , \quad \text{if } i \neq j, \ \text{ for } \quad i,j \in \{1,2,\ldots,k\},$$

$$\varphi_2(ii) = k+1\ , \quad \text{for} \quad i \in \{1,2,\ldots,k\}.$$

It is easy to see that φ_2 is a 2-distance coloring of $S(n,k)$ for $n=2$ and any $k \geq 3$ by the definition of 2-distance coloring (For the case $k=4$, $S(2,4)$ is shown in Fig. 2.1). Suppose $n=3$ and any $k \geq 3$. We form a coloring φ_3 of $S(3,k)$ that uses $k+1$ colors as follows:

$$\varphi_3(ijl) = i, \quad \text{if } j = l \quad \text{for} \quad i,j,l \in \{1,2,\ldots,k\},$$

$$\varphi_3(ijl) = k+1, \ \text{if } i = l \neq j \ \text{ for } \ i,j,l \in \{1,2,\ldots,k\},$$

$$\varphi_3(ijl) = l, \quad \text{other cases}, \quad \text{for } \ i,j,l \in \{1,2,\ldots,k\}.$$

Similarly, φ_3 is a 2-distance coloring for $n=3$ and any $k \geq 3$ by the definition of 2-distance coloring (As an example, for $k=4$, $S(3,4)$ is shown in Fig. 2.2).

Suppose that the result holds for $n-1$ (for any $n-1 \geq 2$ and any $k \geq 3$), i.e., there exists a 2-distance coloring φ_{n-1} of $S(n-1,k)$ that uses $k+1$ colors. Let $u = (ij_1 \ldots j_{n-1}) \in V(S_i(n,k))$, $v = (jj_1 \ldots j_{n-1}) \in V(S_j(n,k))$ and $u^{(i)} = (j_1 \ldots j_{n-1}), v^{(j)} = (j_1 \ldots j_{n-1}) \in V(S(n-1,k))$, $i,j,j_r \in \{1,\ldots,k\}$, $r \in \{1,\ldots,n-1\}$. Now we consider n (for any $n > 3$ and any $k \geq 3$). We form a coloring φ_n of $S(n,k)$ that uses $k+1$ colors as follows:

$$\begin{gathered}\varphi_n(u) = \varphi_{n-1}^i(u), \quad \text{if } u = (ij_1 \ldots j_{n-1}) \in V(S_i(n,k)), \\ \text{for } \ i, j_r \in \{1,\ldots,k\}, \ r \in \{1,\ldots,n-1\}, \\ \text{where } \varphi_{n-1}^i(u) \text{ is obtained from } \varphi_{n-1}(u^{(i)}) \text{ using permuting of colors} \\ (1)\ldots(i-1)(i\ k+1)(i+1)\ldots(k), \quad i \in \{1,2,\ldots,k\}.\end{gathered}$$

Note that φ_n is a coloring of $S(n,k)$, φ_{n-1}^i is a coloring of $S_i(n,k)$, and φ_{n-1} is a coloring of $S(n-1,k)$. It is easy to see that, for any $u \in S_i(3,k)$, $\varphi_3(u) = \varphi_2^i(u)$ and $\varphi_2^i(u)$ is obtained from $\varphi_2(u^{(i)})$ using permuting of colors $(1)\ldots(i-1)(i\ k+1)(i+1)\ldots(k)$ for $i \in \{1,2,\ldots,k\}$. Now, we shall show that the coloring φ_n is a 2-distance coloring that uses $k+1$ colors in $S(n,k)$ for $n \geq 3$.

Case 1 Suppose that n is odd.

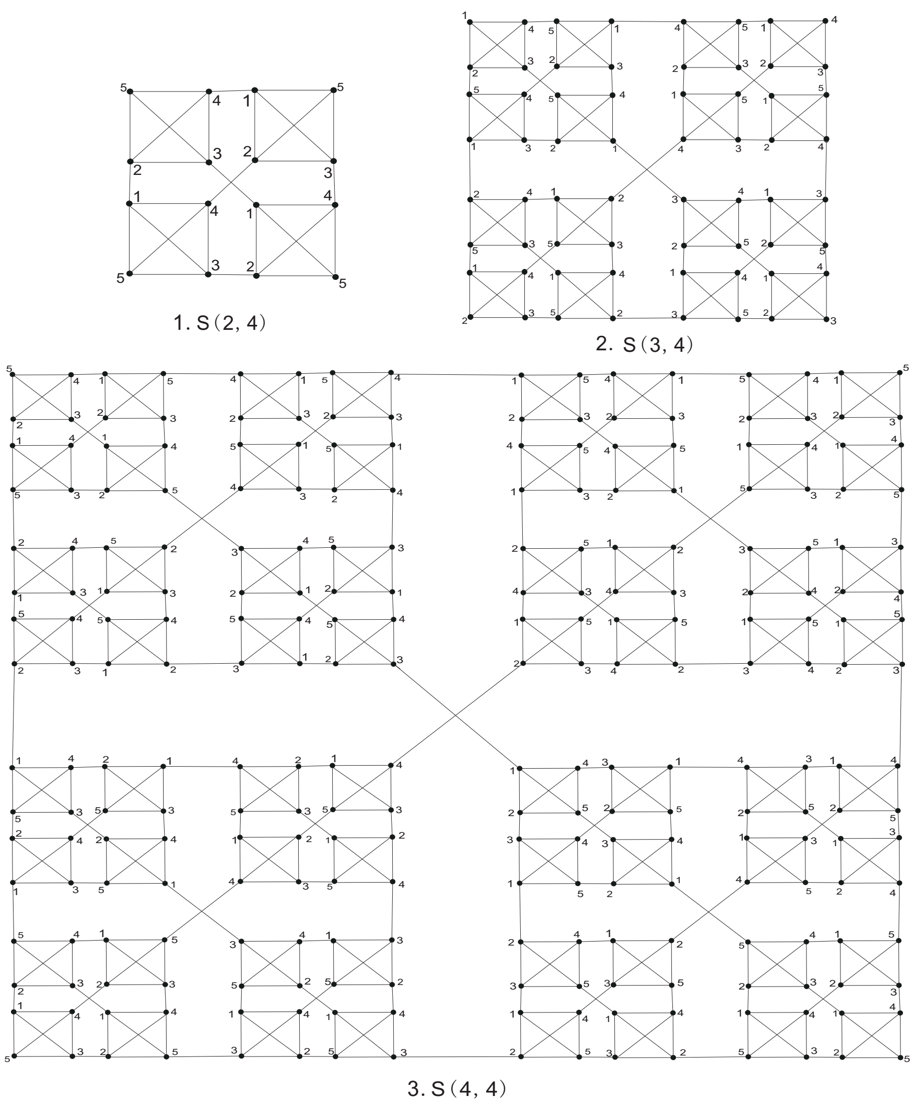

Fig. 2. 2-distance colorings of Sierpiński graphs $S(2,4)$, $S(3,4)$ and $S(4,4)$

We must first give the following results for $i, j \in \{1, 2, \ldots, k\}$,

$$\varphi_{n-1}(i, i, \ldots, i, j) = j, \text{ if } (i, i, \ldots, i, j) \in V(S_i(n-1, k)), \text{ for } i \neq j,$$

$$\varphi_{n-1}(i, i, \ldots, i, j) = k+1, \quad \text{if } (i, i, \ldots, i, j) \in V(S_i(n-1, k)), \text{ for } i = j.$$

Case 1.1 Suppose that $d(u, v) = 1$ for any $u, v \in V(S(n, k))$.

Then u is adjacent to v in Sierpiński graphs $S(n, k)$. Suppose that $u, v \in V(S_i(n, k))$ for $i \in \{1, \ldots, k\}$. Clearly, $\varphi_n(u) = \varphi^i_{n-1}(u) \neq \varphi^i_{n-1}(v) = \varphi_n(v)$. Suppose that $u \in V(S_i(n, k))$ and $v \in V(S_j(n, k))$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$. Since u is adjacent to v, u is $(ij \ldots j)$ and v is $(ji \ldots i)$ by the structure of $S(n, k)$. Since $\varphi_{n-1}(u^{(i)}) = k+1 = \varphi_{n-1}(v^{(j)})$ by the definition of φ_{n-1}, $\varphi^i_{n-1}(u) = i$ and $\varphi^j_{n-1}(v) = j$ for $i, j \in \{1, 2, \ldots, k\}$ and $i \neq j$. Thus $\varphi_n(u) = \varphi^{(i)}_{n-1}(u) \neq \varphi^{(j)}_{n-1}(v) = \varphi_n(v)$.

Case 1.2 Suppose that $d(u, v) = 2$ for $u, v \in V(S(n, k))$.

Suppose that $u, v \in V(S_i(n, k))$ for $i \in \{1, \ldots, k\}$. By the induction hypothesis, $\varphi_n(u) = \varphi^i_{n-1}(u) \neq \varphi^i_{n-1}(v) = \varphi_n(v)$. Suppose that $u \in V(S_i(n, k))$ and $v \in V(S_j(n, k))$ for $i, j \in \{1, 2, \ldots, k\}$ and $i \neq j$. By the structure of $S(n, k)$, we distinguish two cases.

Case 1.2.1 Suppose that $u = ij \ldots j$ and $v = ji \ldots il$ for $i \neq j$, $l \neq i$ and $i, j, l \in \{1, 2, \ldots, k\}$.

Since $\varphi_{n-1}(u^{(i)}) = k+1$ and $\varphi_{n-1}(v^{(j)}) = l$ by the definition of φ_{n-1}, either $\varphi^i_{n-1}(ij \ldots j) = i$, $\varphi^j_{n-1}(ji \ldots il) = l$ for $i \neq j$, $l \neq i$, $l \neq j$ and $i, j, l \in \{1, 2, \ldots, k\}$; or $\varphi^i_{n-1}(ij \ldots j) = i$, $\varphi^j_{n-1}(ji \ldots il) = k+1$ for $i \neq j$, $l \neq i$, $l = j$ and $i, j, l \in \{1, 2, \ldots, k\}$. Thus $\varphi_n(u) = \varphi^i_{n-1}(u) \neq \varphi^j_{n-1}(v) = \varphi_n(v)$.

Case 1.2.2 Suppose that $u = ij \ldots jl$ and $v = ji \ldots i$ for $i \neq j$, $l \neq j$ and $i, j, l \in \{1, 2, \ldots, k\}$.

Since $\varphi_{n-1}(u^{(i)}) = l$ and $\varphi_{n-1}(v^{(j)}) = k+1$ by the definition of φ_{n-1}, either $\varphi^i_{n-1}(ij \ldots jl) = l$, $\varphi^j_{n-1}(ji \ldots i) = j$ for $i \neq j$, $l \neq i$, $l \neq j$ and $i, j, l \in \{1, 2, \ldots, k\}$; or $\varphi^i_{n-1}(ij \ldots jl) = k+1$, $\varphi^j_{n-1}(ji \ldots i) = j$ for $i \neq j$, $l \neq j$, $l = i$ and $i, j, l \in \{1, 2, \ldots, k\}$. Thus $\varphi_n(u) = \varphi^i_{n-1}(u) \neq \varphi^j_{n-1}(v) = \varphi_n(v)$.

Case 2 Suppose that n is even.

We must first give the following results for $i, j \in \{1, 2, \ldots, k\}$,

$$\varphi_{n-1}(i, i, \ldots, i, j) = j, \quad \text{if } (i, i, \ldots, i, j) \in V(S_i(n, k)) \ .$$

Case 2.1 Suppose that $d(u, v) = 1$ for any $u, v \in V(S(n, k))$.

Suppose that $u, v \in V(S_i(n, k))$ for $i \in \{1, \ldots, k\}$. Clearly, $\varphi_n(u) = \varphi^i_{n-1}(u) \neq \varphi^i_{n-1}(v) = \varphi_n(v)$. Suppose that $u \in V(S_i(n, k))$ and $v \in V(S_j(n, k))$ for $i, j \in \{1, 2, \ldots, k\}$ and $i \neq j$. Since u is adjacent v in $S(n, k)$, u is $(ij \ldots j)$ and v is $(ji \ldots i)$ by the structure of $S(n, k)$. Since $\varphi_{n-1}(u^{(i)}) = j$ and $\varphi_{n-1}(v^{(j)}) = i$ by the definition of φ_{n-1}, $\varphi^i_{n-1}(ij \ldots j) = j$ and $\varphi^j_{n-1}(ji \ldots i) = i$ for $i, j \in \{1, 2, \ldots, k\}$ and $i \neq j$. Thus $\varphi_n(u) = \varphi^i_{n-1}(u) \neq \varphi^j_{n-1}(v) = \varphi_n(v)$.

Case 2.2 Suppose that $d(u, v) = 2$ for any $u, v \in V(S(n, k))$.

Suppose that $u, v \in V(S_i(n,k))$ for $i \in \{1,\ldots,k\}$. By the induction hypothesis, $\varphi_n(u) = \varphi^i_{n-1}(u) \neq \varphi^i_{n-1}(v) = \varphi_n(v)$. Suppose that $u \in V(S_i(n,k))$ and $v \in V(S_j(n,k))$ for $i, j \in \{1,2,\ldots,k\}$ and $i \neq j$. Similarly, we distinguish two cases.

Case 2.2.1 Suppose that $u = ij\ldots j$ and $v = ji\ldots il$ for $i \neq j$, $l \neq i$ and $i, j, l \in \{1,2,\ldots,k\}$.

Since $\varphi_{n-1}(u^{(i)}) = j$ and $\varphi_{n-1}(v^{(j)}) = l$ by the definition of φ_{n-1}, either $\varphi^i_{n-1}(ij\ldots j) = j$, $\varphi^j_{n-1}(ji\ldots il) = l$ for $i \neq j$, $l \neq i$, $l \neq j$ and $i, j, l \in \{1,2,\ldots,k\}$; or $\varphi^i_{n-1}(ij\ldots j) = j$, $\varphi^j_{n-1}(ji\ldots il) = k+1$ for $i \neq j$, $l \neq i$, $l = j$ and $i, j, l \in \{1,2,\ldots,k\}$. Thus $\varphi_n(u) = \varphi^i_{n-1}(u) \neq \varphi^j_{n-1}(v) = \varphi_n(v)$.

Case 2.2.2 Suppose that $u = ij\ldots jl$ and $v = ji\ldots i$ for $i \neq j$, $l \neq j$ and $i, j, l \in \{1,2,\ldots,k\}$.

Since $\varphi_{n-1}(u^{(i)}) = l$ and $\varphi_{n-1}(v^{(j)}) = i$ by the definition of φ_{n-1}, either $\varphi^i_{n-1}(ij\ldots jl) = l$, $\varphi^j_{n-1}(ji\ldots i) = i$ for $i \neq j$, $l \neq j$, $l \neq i$, and $i, j, l \in \{1,2,\ldots,k\}$; or $\varphi^i_{n-1}(ij\ldots jl) = k+1$, $\varphi^j_{n-1}(ji\ldots i) = i$ for $i \neq j$, $l \neq j$, $l = i$ and $i, j, l \in \{1,2,\ldots,k\}$. Thus $\varphi_n(u) = \varphi^i_{n-1}(u) \neq \varphi^j_{n-1}(v) = \varphi_n(v)$.

By the principle of induction, $\chi_{2d}(S(n,k)) \leq k+1$, and so $\chi_{2d}(S(n,k)) = k+1$ (As an example for the case $k = 4$ and $n = 4$, $S(4,4)$ is shown in Fig. 2.3). Let $\varphi_n^{-1}(i) = V_i$ be the set of vertices assigned color i in $S(n,k)$ for $i \in \{1,2,\ldots,k+1\}$. The theorem is proved. □

Theorem 3.2. *Let φ_n be the 2-distance coloring of $S(n,k)$, $n \geq 2$, $k \geq 3$, as given in Theorem 3.1. Then,*

(i) If n is odd, then $|\varphi_n^{-1}(i)| = \frac{k^n-k}{k+1} + 1$ for $i \in \{1,2,\ldots,k\}$ and $|\varphi_n^{-1}(k+1)| = \frac{k^n-k}{k+1}$.

(ii) If n is even, then $|\varphi_n^{-1}(i)| = \frac{k^n-1}{k+1}$ for $i \in \{1,2,\ldots,k\}$ and $|\varphi_n^{-1}(k+1)| = \frac{k^n-1}{k+1} + 1$.

Proof. By induction on n. Suppose that $n = 2$ and any $k \geq 3$. By the definition of φ_2 in Theorem 3.1, $|\varphi_2^{-1}(i)| = k-1$ for $i \in \{1,2,\ldots,k\}$ and $|\varphi_2^{-1}(k+1)| = k$. (As an example for $S(2,4)$, we have $|\varphi_2^{-1}(i)| = 3$ for $i = 1,2,3,4$ and $|\varphi_2^{-1}(5)| = 4$, which is shown in Fig. 2.1). Suppose that $n = 3$ and any $k \geq 3$. Similarly, we have $|\varphi_3^{-1}(i)| = k^2 - k + 1$ for $i \in \{1,2,\ldots,k\}$ and $|\varphi_3^{-1}(k+1)| = k^2 - k$. (For $S(3,4)$, we have $|\varphi_3^{-1}(i)| = 13$ for $i = 1,2,3,4$ and $|\varphi_3^{-1}(5)| = 12$, which is shown in Fig. 2.2).

Suppose that the results hold for $n-1$ ($n-1 \geq 2$ and any $k \geq 3$), i.e.,

(i) if $n-1$ is odd, then $|\varphi_{n-1}^{-1}(i)| = \frac{k^{n-1}-k}{k+1} + 1$ for $i \in \{1,2,\ldots,k\}$ and $|\varphi_{n-1}^{-1}(k+1)| = \frac{k^{n-1}-k}{k+1}$;

(ii) if $n-1$ is even, then $|\varphi_{n-1}^{-1}(i)| = \frac{k^{n-1}-1}{k+1}$ for $i \in \{1,2,\ldots,k\}$ and $|\varphi_{n-1}^{-1}(k+1)| = \frac{k^{n-1}-1}{k+1} + 1$.

Suppose that n is odd. By Theorem 3.1, φ_{n-1}^{i} is obtained from φ_{n-1} using permuting of colors $(1)\dots(i-1)(i\ k+1)(i+1)\dots(k)$ for $i \in \{1,2,\dots,k\}$. Thus for every $S_i(n,k)$, $|\varphi_n^{-1}(i)| = |\varphi_{n-1}^{-1}(k+1)| = \frac{k^{n-1}-1}{k+1} + 1$ for $i \in \{1,2,\dots,k\}$ and $|\varphi_n^{-1}(j)| = \frac{k^{n-1}-1}{k+1}$ for $j \in \{1,2,\dots,k+1\}$ and $i \neq j$. It follows that for $S(n,k)$

$$|\varphi_n^{-1}(k+1)| = k(\frac{k^{n-1}-1}{k+1}) = \frac{k^n-k}{k+1}$$

and

$$|\varphi_n^{-1}(i)| = (k-1)\frac{k^{n-1}-1}{k+1} + (\frac{k^{n-1}-1}{k+1}+1) = \frac{k^n-k}{k+1} + 1 \ for\ i \in \{1,2,\dots,k\}.$$

Now suppose that n is even. Since $n-1$ is odd, by the induction hypothesis, $|\varphi_{n-1}^{-1}(i)| = \frac{k^{n-1}-k}{k+1} + 1$ for $i \in \{1,2,\dots,k\}$ and $|\varphi_{n-1}^{-1}(k+1)| = \frac{k^{n-1}-k}{k+1}$. (Note that φ_{n-1}^{i} is obtained from φ_{n-1} using permuting of colors $(1)\dots(i-1)(i\ k+1)(i+1)\dots(k)$ for $i \in \{1,2,\dots,k\}$). For every $S_i(n,k)$, $|\varphi_n^{-1}(i)| = \frac{k^{n-1}-k}{k+1}$ for $i \in \{1,2,\dots,k\}$ and $|\varphi_n^{-1}(j)| = \frac{k^{n-1}-k}{k+1} + 1$ for $j \in \{1,2,\dots,k+1\}$ and $j \neq i$. Thus for $S(n,k)$

$$|\varphi_n^{-1}(k+1)| = k(\frac{k^{n-1}-k}{k+1}+1) = \frac{k^n-1}{k+1} + 1$$

and

$$|\varphi_n^{-1}(i)| = (k-1)(\frac{k^{n-1}-k}{k+1}+1) + \frac{k^{n-1}-k}{k+1} = \frac{k^n-1}{k+1} \quad for \quad i \in \{1,2,\dots,k\}.$$

□

Theorem 3.3. *For any $n \geq 2$ and any $k \geq 2$, $\lambda_e(S(n,k)) = 2k$.*

Proof. First we show that $\lambda(S(n,k)) = 2k$ for any $n \geq 2$ and any $k \geq 2$.

Suppose that $n > 2$ and $k = 2$. Since $\lambda_{2d}(S(n,2)) = 3$ by Theorem 3.1, we form a labeling ℓ of $S(n,2)$ as follows:

$$\ell(u) = 0 \qquad \text{if } u \in \varphi_n^{-1}(1),$$
$$\ell(u) = 2 \qquad \text{if } u \in \varphi_n^{-1}(2),$$
$$\ell(u) = 4 \qquad \text{if } u \in \varphi_n^{-1}(3).$$

It is clear that ℓ is an $L(2,1)$-labeling and $\lambda(S(n,2)) \leq 4$. It is easy to see that $\lambda(S(n,2)) \geq 4$. Thus it holds for any $n > 2$ and $k = 2$.

Suppose that $n \geq 2$ and $k \geq 3$. By Theorem 3.1, there exists a 2-distance coloring φ_n of $S(n,k)$ that uses $k+1$ colors, then the vertex set of $S(n,k)$ can be partitioned into $k+1$ sets, i.e., $\varphi_n^{-1}(1), \varphi_n^{-1}(2), \dots, \varphi_n^{-1}(k+1)$. We form a labeling ℓ of $S(n,k)$ for $n \geq 2$ and $k \geq 3$ as follows:

$$\ell(u) = 2(i-1), \quad \text{if } u \in \varphi_n^{-1}(i) \text{ for } i \in \{1,2,\dots,k+1\}.$$

It is easy to see that ℓ is an $L(2,1)$-labeling and $\lambda(S(n,k)) \leq 2k$. Note that $S(n-1,k)$ is isometric to a subgraph of $S(n,k)$ for any $n \geq 2$ and any $k \geq 3$. In order to prove that $\lambda(S(n,k)) \geq 2k$, we only need to show that $\lambda(S(2,k)) \geq 2k$. By Lemma 2.1, $\lambda(S(2,k)) \geq 2k$. Thus $\lambda(S(n,k)) \geq 2k$, and so $\lambda(S(n,k)) = 2k$.

Now we show that the $L(2,1)$-labeling ℓ of $S(n,k)$ is equitable for any $n \geq 2$ and any $k \geq 2$. For the case $n > 2$ and $k = 2$. Since $\lambda(S(n,2)) = 4$ and $S(n,2)$ is the path on 2^n vertices by Theorem 3.1,

$$\text{if } n \text{ is odd, then } |\varphi_n^{-1}(1)| = |\varphi_n^{-1}(2)| = |\varphi_n^{-1}(3)| + 1;$$

$$\text{if } n \text{ is even, then } |\varphi_n^{-1}(1)| = |\varphi_n^{-1}(2)| + 1 = |\varphi_n^{-1}(3)| + 1.$$

For any $n \geq 2$ and any $k \geq 3$, by Theorem 3.2, it is easy to see that

$$||\varphi_n^{-1}(i)| - |\varphi_n^{-1}(j)|| = 0 \ \ and \ \ ||\varphi_n^{-1}(i)| - |\varphi_n^{-1}(k+1)|| = 1, \ i,j \in \{1,2,\ldots,k\}.$$

By the definition of equitable $L(2,1)$-labeling, $\lambda_e(S(n,k)) = 2k$ for any $n \geq 2$ and any $k \geq 2$.

□

Remark 3.1. *Indeed, (i) for any $n \geq 2$ and $k = 1$, $S(n,1)$ is isomorphic to K_1; (ii) for $n = 2$ and $k = 2$, $S(2,2)$ is a path of 3, it is clear that $\chi_{2d}(S(2,2)) = \lambda(S(2,2)) = \lambda_e(S(2,2)) = 3$; (iii) for $n = 1$ and any $k \geq 2$, $S(1,k) = K_k$. Since $\lambda_{2d}(K_k) = \chi(K_k) = k$ and $\lambda(S(1,k)) = \lambda(K_k) = 2k-2$ by Lemma 2.2, $|\varphi_n^{-1}(i)| = |\varphi_n^{-1}(j)| = 1$ for $i,j \in \{1,\ldots,k\}$. Thus $\lambda_e(S(1,k)) = 2k-2$ for any $k \geq 2$.*

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