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Global offensive alliance numbers in graphs with emphasis on trees

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Abstract

For a graph $G = (V, E)$, a non-empty set $S \subseteq V$ is a *global offensive alliance* if for every $v \in V - S$, at least half of the vertices from the closed neighborhood of v are in S. A set $S \subseteq V$ is a *global strong offensive alliance* if for each vertex $v \in V - S$, a strict majority of the vertices of the closed neighborhood of v are in S. The cardinality of a minimum global (strong) offensive alliance of a graph G is called the global (strong) offensive alliance number of G. We determine bounds on the global offensive alliance and the global strong offensive alliance numbers of a graph, and characterize the trees achieving two of these lower bounds.

1 Introduction

In real life, an alliance is a group of entities that unite for a common cause. The applications of alliances are widespread from social and business associations to national defense coalitions. Motivated by these varied applications, Hedetniemi, Hedetniemi, and Kristiansen [13] introduced several types of alliances in graphs, including the global (strong) offensive alliances that we consider in this paper. Their introductory paper generated much interest in the topic, and consequently, several different alliances have been studied; for examples see [6, 10, 11, 14, 15]. In particular, offensive alliances were studied in [5, 7, 16, 17].

To formally define alliances, we need some additional definitions. In general, we follow [12]. In a graph $G = (V(G), E(G)) = (V, E)$ of order $n(G)$, or simply n when the graph G is clear from the context, the *neighborhood* of a vertex $v \in V$ is $N_G(v) = N(v) = \{u \in V \mid uv \in E\}$. If S is a subset of vertices, its neighborhood is $N_G(S) = N(S) = \cup_{v\in S} N(v)$. The *closed neighborhoods* of v and S are $N_G[v] = N[v] = N(v) \cup \{v\}$ and $N_G[S] = N[S] = N(S) \cup S$, respectively. A set $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has at least a neighbor in S, that is, $N[S] = V$. The *domination number* $\gamma(G)$ is the cardinality of a minimum dominating set of G.

A set $S \subseteq V$ is a *global offensive alliance* of G if for every $v \in V - S$, $|N[v] \cap S| \geq |N[v] - S|$ and is a *global strong offensive alliance* of G if for every $v \in V - S$, $|N[v] \cap S| > |N[v] - S|$. Every graph has a global (strong) offensive alliance, since V is such a set. We abbreviate global offensive alliance as *goa* and global strong offensive alliance as *gsoa*. The *global offensive alliance number* $\gamma_o(G)$ (respectively, *global strong offensive alliance number* $\gamma_{\hat{o}}(G)$) is the cardinality of a minimum goa (respectively, gsoa) of G. If S is a goa (respectively, gsoa) of G and $|S| = \gamma_o(G)$ (respectively, $|S| = \gamma_{\hat{o}}(G)$), then we say that S is a $\gamma_o(G)$-set (respectively, $\gamma_{\hat{o}}(G)$-set). Note that a goa (respectively, gsoa) is a dominating set of G. Hence, for any graph G, we have $\gamma(G) \leq \gamma_o(G) \leq \gamma_{\hat{o}}(G)$. In [4] Cami, Balakrishnan, Deo and Dutton have proved that finding optimal global offensive alliance is an NP-complete problem. So it seems natural to find bounds on $\gamma_o(G)$ and $\gamma_{\hat{o}}(G)$ for an arbitrary graph G.

Many applications of alliances, including the coalition of nations for defense purposes, were stated in [13]. Considering this application for a global offensive alliance, it is reasonable to think of that each vertex in $V - S$ is vulnerable to possible attack by the vertices in S (assuming strength in numbers). On the other hand, since an attack by a global offensive alliance S on the vertices of $V - S$ can result in no worse than a "tie", the vertices in S can "successfully" attack $V - S$.

In Section 3, we determine bounds on the global offensive alliance and global strong alliance numbers of graphs. In particular, we show that for trees T, $\gamma_{\hat{o}}(T) \geq \gamma_o(T) + 1$. It is known that the global strong alliance number of a tree T is bounded below by the independence number $\beta_0(T)$. In Section 4, we characterize the extremal trees attaining $\gamma_{\hat{o}}(T) = \beta_0(T)$, and in Section 5 we characterize the trees achieving $\gamma_{\hat{o}}(T) = \gamma_o(T) + 1$. To achieve this, we begin with additional terminology and useful known results in Section 2.

2 Terminology and Background

Several results play an important role in our investigations. Before we list them, we give some more terminology.

A vertex of degree one is called an *endvertex* or a *leaf*, and its neighbor is called a *support vertex*. We also denote the set of leaves of a graph G by $L(G)$ and the set of support vertices by $S(G)$. Let $|L(G)| = \ell$ and $|S(G)| = s$. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$. The *subdivision graph* of a graph G is that graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw. If a tree T is a subdivision graph of a nontrivial tree T', then we say that T is a *subdivided tree*, and the $n(T') - 1$ new vertices resulting from the subdivision of the edges of T' are called *subdivision vertices.* Note that a subdivided tree has order at least three and at least one subdivision vertex. The *corona graph* $G \circ K_1$ of a graph G is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added. Let P_n denote the path graph of order n.

A subset $S \subseteq V$ is a 2-*dominating set* if every vertex of $V - S$ has at least two neighbors in S, and S is independent if no edge of G has its two endvertices in S. The 2-*domination number* $\gamma_2(G)$ is the cardinality of a minimum 2-dominating set. A *vertex cover* in a graph G is a set of vertices that covers all edges of G. The minimum cardinality of a vertex cover in a graph G is called the *covering number* of G and is denoted by $\alpha_0(G)$.

We are now ready to list some known results that will be useful here.

Theorem 1 (Fink and Jacobson [8]) *If T is a tree of order n, then $\gamma_2(T) \geq (n+1)/2$, with equality if and only if $T = P_1$ or T is a subdivided tree.*

Theorem 2 (Volkmann [18]) *If T is a tree of order n, then $\gamma_2(T) = \lceil (n+1)/2 \rceil$ if and only if*

i) T is the trivial graph or T is a subdivided tree, or

ii) T can be formed from two trees T_1 and T_2 which satisfy conditions (i) by adding an edge between a vertex of T_1 and a vertex of T_2.

Theorem 3 (Volkmann [18]) *For every nontrivial tree T, $\gamma_2(T) \geq \gamma(T) + 1$ with equality if and only if T is a subdivided star, the corona of a star, or a subdivided double star.*

Let $i(G)$ denote the independent domination number of a graph G.

Theorem 4 (Hansberg and Volkmann [9]) *For every nontrivial tree T, $\gamma_2(T) \geq i(T) + 1$ with equality if and only if $\gamma_2(T) = \gamma(T) + 1$ or $T = S_{2,2}$.*

Theorem 5 (Blidia, Chellali and Favaron [1]) *For any tree T, $\beta_0(T) \leq \gamma_2(T)$.*

Theorem 6 (Chellali and Haynes [6]) *For any tree T, $\gamma_o(T) \leq \beta_0(T) \leq \gamma_{\hat{o}}(T)$.*

Observation 7 (Hedetniemi, Hedetniemi and Kristiansen [13]) *Let G be a graph.*

(i) If $\delta(G) \geq 1$, then $\gamma_o(G) \leq \alpha_0(G)$.

(ii) If $\delta(G) \geq 2$, then $\gamma_{\hat{o}}(G) \leq \alpha_0(G)$.

3 Bounds

We begin with some observations.

Observation 8 *For any graph G, the endvertices of G are contained in every $\gamma_{\hat{o}}(G)$-set.*

Observation 9 *For any graph G, $\gamma_{\hat{o}}(G) \geq \gamma_2(G)$.*

Proof. If S is any $\gamma_{\hat{o}}(G)$-set, then every vertex of $V - S$ has at least two neighbors in S. Thus S is a 2-dominating set of G and so $\gamma_2(G) \leq |S| = \gamma_{\hat{o}}(G)$. ■

We note that the difference $\gamma_{\hat{o}}(G) - \gamma_2(G)$ may be strict and arbitrarily large even for trees. To see this, consider the tree T_k formed by $k \geq 2$ copies of the double star $S_{2,2}$ by adding a new vertex adjacent to exactly one support vertex of each double star. Then $\gamma_2(T_k) = 4k + 1$, while $\gamma_{\hat{o}}(T_k) = 5k$.

Theorem 10 *If G is an r-partite graph, then*

$$\alpha_0(G) \leq \frac{(r-1)(n-\ell)+s}{r}.$$

Proof. Let $A = L(G) \cup S(G)$. If $V(G) = A$, then $S(G)$ is a covering set and thus

$$\alpha_0(G) \leq s(G) = \frac{(r-1)(n-\ell)+s}{r}.$$

In the remaining case that $|A| < |V(G)|$, let $V_1, V_2, \ldots, V_r$ be a partition of the r-partite graph $G - A$ such that $|V_1| \geq |V_2| \geq \ldots \geq |V_r|$, where $V_i = \emptyset$ is possible for $i \geq 2$. Then $V(G) - (L(G) \cup V_1)$ is a covering set of G. Since

$$|V_1| \geq \frac{|V_1| + |V_2| + \ldots + |V_r|}{r} = \frac{n - |A|}{r},$$

we obtain

$$\begin{aligned}\alpha_0(G) &\leq n - \ell - |V_1| \\ &\leq n - \ell - \frac{n-\ell-s}{r} \\ &= \frac{(r-1)(n-\ell)+s}{r},\end{aligned}$$

and the proof is complete. ■

For the corona $H \circ K_1$, where H is an arbitrary r-partite graph, we have equality in the inequality of Theorem 10, and therefore it is best possible.

Using Gallai's identity $\alpha_0(G)+\beta_0(G) = n$, Theorem 10 leads to the next corollary.

Corollary 11 *If G is an r-partite graph, then*

$$\beta_0(G) \geq \left\lceil \frac{n + (r-1)\ell - s}{r} \right\rceil .$$

The case $r = 2$ in Corollary 11 leads to the next bound.

Corollary 12 (Blidia, Chellali, Favaron, Meddah [2]) *If G is a bipartite graph, then*

$$\beta_0(G) \geq \left\lceil \frac{n + \ell - s}{2} \right\rceil .$$

Next we characterize the r-partite graphs with equality in the inequality of Corollary 11.

Theorem 13 *Let G be an r-partite graph, and let $H = G - (L(G) \cup S(G))$. Then $\beta_0(G) = \left\lceil \frac{n(G)+(r-1)\ell(G)-s(G)}{r} \right\rceil$ if and only if $\beta_0(H) = \left\lceil \frac{n(H)}{r} \right\rceil$.*

Proof. Every $\beta_0(G)$-set containing $L(G)$ contains a maximum independent set of H. Thus $\beta_0(G) = \ell(G) + \beta_0(H)$. Using the fact that $n(H) = n(G) - \ell(G) - s(G)$, we obtain

$$\begin{aligned} \beta_0(H) - \frac{n(H)}{r} &= \beta_0(G) - \ell(G) - \frac{n(G) - \ell(G) - s(G)}{r} \\ &= \beta_0(G) - \frac{n(G) + (r-1)\ell(G) - s(G)}{r} \end{aligned}$$

and therefore

$$\beta_0(H) - \left\lceil \frac{n(H)}{r} \right\rceil = \beta_0(G) - \left\lceil \frac{n(G) + (r-1)\ell(G) - s(G)}{r} \right\rceil .$$

The last identity immediately leads to the desired characterization, and the proof is complete. ■

Using König's result that $\alpha_0(G)$ is equal to the matching number in a bipartite graph G, Theorem 13 implies the next result for $r = 2$.

Corollary 14 (Blidia, Favaron, Lounes [3]) *Let G be a bipartite graph, and let $H = G - (L(G) \cup S(G))$. Then $\beta_0(G) = \frac{n(G)+\ell(G)-s(G)}{2}$ if and only if H contains a perfect matching.*

By the way, if $n+\ell-s$ is odd for a bipartite graph G, then $\beta_0(G) = \lceil \frac{n+\ell-s}{2} \rceil$ if and only if H contains an almost perfect matching.

Using Observation 7 and Theorem 10 for $r = 2$, we arrive at the next result by Chellali.

Corollary 15 (Chellali [5]) *For every bipartite graph G without isolated vertices, we have*

$$\gamma_o(G) \leq \frac{n-\ell+s}{2}.$$

By Theorem 3 and Observation 9, we have:

Theorem 16 *For every nontrivial tree T, $\gamma_{\hat{o}}(T) \geq \gamma(T)+1$ with equality if and only if T is a subdivided star, a corona of a star, or a subdivided double star.*

Since $\gamma_{\hat{o}}(S_{2,2}) = 5 > i(S_{2,2}) + 1 = 4$ and by Theorem 4, we also obtain:

Theorem 17 *For every nontrivial tree T, $\gamma_{\hat{o}}(T) \geq i(T)+1$ with equality if and only if $\gamma_{\hat{o}}(T) = \gamma(T) + 1$.*

Theorem 18 *For every graph G with order $n \geq 3$, s support vertices, and ℓ leaves, $\gamma_{\hat{o}}(G) \geq \gamma_o(G) + \ell - s$.*

Proof. Let D be a $\gamma_{\hat{o}}(G)$-set. Then $L(G) \subseteq D$. Thus $S(G) \cup D - L(G)$ is a goa of G, and so $\gamma_o(G) \leq |S(G) \cup D - L(G)| = s + \gamma_{\hat{o}}(G) - \ell$. ∎

Combining Theorem 5 and Observations 7 and 9, we obtain the following inequality chain.

Corollary 19 *For every nontrivial tree T, we have*

$$\gamma(T) \leq \gamma_o(T) \leq \alpha_0(T) \leq \beta_0(T) \leq \gamma_2(T) \leq \gamma_{\hat{o}}(T).$$

Theorem 20 *For every nontrivial tree T, $\gamma_{\hat{o}}(T) \geq \gamma_o(T) + 1$.*

Proof. If $\gamma_{\hat{o}}(T) = \gamma_o(T)$, then by Corollary 19, we have $\gamma_o(T) = \beta_0(T) = \gamma_2(T) = \gamma_{\hat{o}}(T)$ and since by Corollary 15, $\gamma_o(T) \leq n/2 \leq \beta_0(T)$, it follows that $\gamma_2(T) = n/2$, a contradiction to Theorem 1. Therefore $\gamma_{\hat{o}}(T) \geq \gamma_o(T) + 1$. ∎

Note that Theorem 20 cannot be deduced from Theorem 18 since for a subdivided star SS_p with $p \geq 2$ leaves, we have $\ell = s$, but $\gamma_{\hat{o}}(T) = p + 1 = \gamma_o(T) + 1$.

4 Trees T with $\gamma_{\hat{o}}(T) = \beta_0(T)$

In this section, we characterize the trees T with $\beta_0(T) = \gamma_{\hat{o}}(T)$. By Corollary 19, if a tree T satisfies $\beta_0(T) = \gamma_{\hat{o}}(T)$, then $\beta_0(T) = \gamma_2(T) = \gamma_{\hat{o}}(T)$. Blidia, Chellali, and Favaron [1] gave a constructive characterization of trees T with $\beta_0(T) = \gamma_2(T)$. Let L_v denote the set of leaves adjacent to a vertex v. Let $\mathcal{G}$ be the family of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees, where T_1 is a star $K_{1,t}$ $(t \geqslant 2)$ of center vertex w, $T = T_k$, and, if $k \geq 2$, T_{i+1} is obtained recursively from T_i by one of the three operations defined below. When an operation is performed on a tree T_i we let $A(T_{i+1}) = A(T_i) \cup L_x$, where $A(T_1) = L_w$.

- **Operation** $\mathcal{O}_1$: Add a star $K_{1,p}$, $p \geqslant 1$, centered at a vertex x and join x by an edge to a leaf y of T_i.
- **Operation** $\mathcal{O}_2$: Add a star $K_{1,p}$, $p \geqslant 1$, centered at a vertex x and join x by an edge to a non-leaf vertex y of $A(T_i)$.
- **Operation** $\mathcal{O}_3$: Add a star $K_{1,p}$, $p \geqslant 2$, centered at a vertex x and join x by an edge to a vertex y of $V(T_i) - A(T_i)$.

Theorem 21 (Blidia, Chellali and Favaron [1]) *Let T be a tree. Then the following statements are equivalent:*

a) $\gamma_2(T) = \beta_0(T)$,

b) $T = K_1$ *or* $T \in \mathcal{G}$,

c) *T has a unique $\gamma_2(T)$-set that also is a unique $\beta_0(T)$-set. ($A(T)$ is such a set if T is nontrivial).*

Let us define $\mathcal{G}_1$ as the subfamily of $\mathcal{F}$ consisting of trees constructed from T_1 by recursively applying Operations $\mathcal{O}_1, \mathcal{O}_2$, or $\mathcal{O}_4$, where Operation $\mathcal{O}_4$ is defined as follow:

- **Operation** $\mathcal{O}_4$: Add a star $K_{1,p}$, $p \geqslant 3$, centered at a vertex x and join x by an edge to a vertex y of $V(T_i) - A(T_i)$ with the condition $|N[y] \cap A(T_i)| \geq |N[y] \cap (V(T_i) - A(T_i))| + 2$.

Theorem 22 *Let T be a tree. Then $\gamma_{\hat{o}}(T) = \beta_0(T)$ if and only if $T = K_1$ or $T \in \mathcal{G}_1$.*

Proof. If $T = K_1$, then $\gamma_{\hat{o}}(T) = \beta_0(T)$. If $T \in \mathcal{G}_1$, then by Theorem 21, $A(T)$ is a unique $\gamma_2(T)$-set that also is a unique $\beta_0(T)$-set. Thus $L(T) \subseteq A(T)$. It is easy to check from the way T is constructed that $A(T)$ is a gsoa of T. Thus, using Observation 9, we obtain $\beta_0(T) = \gamma_2(T) \leq \gamma_{\hat{o}}(T) \leq |A(T)| = \gamma_2(T) = \beta_0(T)$. Hence $\gamma_{\hat{o}}(T) = \beta_0(T)$.

To show the converse, we use induction on the order n of T. If $n = 1$, then $T = K_1$. Since $\gamma_{\hat{o}}(P_2) > \beta_0(P_2)$, let $n \geq 3$. If diam(T)= 2, then $T = K_{1,t}(t \geq 2)$

which belongs to $\mathcal{G}_1$ and $A(T) = L(T)$ is the unique $\gamma_{\hat{o}}(T)$-set. Assume that any tree T' of order $3 \le n' < n$ with $\gamma_{\hat{o}}(T') = \beta_0(T')$ is in $\mathcal{G}_1$ and that $A(T')$ (as defined in the construction) is the unique $\gamma_{\hat{o}}(T')$-set.

Let T be a tree of order n and $\mathrm{diam}(T) \ge 3$ such that $\gamma_{\hat{o}}(T) = \beta_0(T)$. Let u be a support vertex of T such that $T' = T - (L_u \cup \{u\})$ is a tree (this is possible because $\mathrm{diam}(T) \ge 3$), and let v the unique non-leaf neighbor of u in T'. Then T' has order at least three for otherwise $\gamma_{\hat{o}}(T) = |L_u|+2 > \beta_0(T) = |L_u|+1$. It is a simple exercise to see that $\beta_0(T') = \beta_0(T) - |L_u|$. If S is any $\gamma_{\hat{o}}(T)$-set, then $L_u \subseteq S$ and, without loss of generality, $u \notin S$ (else replace it by v in S). It follows that $S \cap V(T')$ is a gsoa of T', and so $\gamma_{\hat{o}}(T') \le |S| - |L_u| = \gamma_{\hat{o}}(T) - |L_u|$. Thus $\gamma_{\hat{o}}(T') \le \beta_0(T')$ and equality holds by Corollary 19. By the inductive hypothesis, T' is in $\mathcal{G}_1$ with a unique $\gamma_{\hat{o}}(T')$-set $A(T')$, implying that $S \cap V(T') = A(T')$. Then $A(T') \cup L_u$ is a $\gamma_{\hat{o}}(T)$-set. If $v \in A(T')$, then $T \in \mathcal{G}_1$ and is obtained from T' by using either Operation $\mathcal{O}_1$ or $\mathcal{O}_2$ depending on whether or not v is a leaf of T', respectively. Assume that $v \notin A(T')$. Then since as seen above $u \notin S$ and $A(T') = S \cap V(T')$, $|N[v] \cap S| \ge |N[v] \cap (V(T) - S)| + 1$ implies that $|N[v] \cap A(T')| \ge |N[v] \cap (V(T') - A(T'))| + 2$. Similarly, since neither u nor v is in S and $|N[u] \cap S| \ge |N[u] \cap (V(T) - S)| + 1 \ge 3$, it follows that the subgraph induced by $u \cup L_u$ is a star $K_{1,p}$ with $p \ge 3$. Thus T is obtained from T' by using Operation $\mathcal{O}_4$, and so $T \in \mathcal{G}_1$.

We claim that $A(T)$ is the unique $\gamma_{\hat{o}}(T)$-set. To show this, we assume to the contrary that $D \ne A(T)$ is a $\gamma_{\hat{o}}(T)$-set. Since every gsoa is also a 2-dominating set and $\beta_0(T) = \gamma_2(T) = \gamma_{\hat{o}}(T)$, it follows that D is a $\gamma_2(T)$-set different from $A(T)$, a contradiction to Theorem 21. ■

5 Trees T with $\gamma_{\hat{o}}(T) = \gamma_o(T) + 1$

Next we give a characterization of the trees T achieving the lower bound of Theorem 20.

Theorem 23 *For a nontrivial tree T, $\gamma_{\hat{o}}(T) = \gamma_o(T) + 1$ if and only if one of the following is true:*

i) T is a subdivided tree,

ii) T is obtained from trees T_1 and T_2 that are subdivisions of trees T_1' and T_2', respectively, by adding an edge between a vertex u_1 of T_1' and a vertex u_2 of T_2', where at least one of u_1 and u_2 is a leaf or an isolate of $T_1 \cup T_2$.

Proof. Let T be a tree of order $n \ge 2$ with $\gamma_{\hat{o}}(T) = \gamma_o(T) + 1$. Clearly if $n = 2$, then $T = P_2$, and $ii)$ of the theorem holds where T_i' for $i \in \{1, 2\}$ is the trivial graph. Thus assume that $n \ge 3$. Theorem 18 implies that $s \le \ell \le s + 1$. If $\ell = s + 1$, then Observation 9, Theorem 1, and Corollary 15 imply that $(n+1)/2 \le \gamma_2(T) \le \gamma_{\hat{o}}(T) = \gamma_o(T) + 1 \le (n - \ell + s)/2 + 1 = (n + 1)/2$. Therefore we have equality throughout this chain. In particular, $\gamma_2(T) = (n + 1)/2$. By Theorem 1, T is a subdivided tree. Since $\ell = s + 1$, a support vertex of T must be adjacent to two leaves. But since T

is a subdivision of a nontrivial tree T', this is only possible if $T' = P_2$ and hence, $T = P_3$.

Assume now that $\ell = s$. Using again Observation 9, Theorem 1 and Corollary 15, we obtain $(n+1)/2 \le \gamma_2(T) \le (n+2)/2$. Since $\gamma_2(T)$ is an integer, we have that $\gamma_2(T) = \lceil (n+1)/2 \rceil$. If n is odd, then $\gamma_2(T) = (n+1)/2$ and so by Theorem 1, T is a subdivided tree. We note that every subdivided tree has odd order. Thus, if n is even, then by Theorem 2, T is obtained from two trees T_1 and T_2 joined by an edge, where each of T_1 and T_2 is either the trivial graph or a subdivided tree.

For $i \in \{1, 2\}$, let T_i be the subdivision of T'_i, and let $A_i = A(T_i)$ denote the set of subdivision vertices of T_i. (Note that if T_i is the trivial graph, then T'_i is the trivial graph.) We note that $n(T) = n(T_1) + n(T_2) = n(T'_1) + |A_1| + n(T'_2) + |A_2| = n(T'_1)+n(T'_1)-1+n(T'_2)+n(T'_2)-1 = 2n(T'_1)+2n(T'_2)-2$. If $u_i \in V(T'_i)$ for $i \in \{1, 2\}$, and at least one of u_1 and u_2 is a leaf or an isolate in $T_1 \cup T_2$, then $ii)$ of the theorem holds. In all other cases, it is a simple exercise to show that $A_1 \cup A_2$ is a goa of T. Then $\gamma_o(T) \le |A_1 \cup A_2| \le |A_1|+|A_2| = n(T'_1)-1+n(T'_2)-1 = n(T'_1)+n(T'_2)-2 < n/2$, a contradiction.

For the converse, let T be a subdivided tree of order n, and let A denote the set of subdivision vertices of T. Then $|A| = n - |A| - 1$. Also $V(T) - A$ is a gsoa, and so $\gamma_2(T) \le \gamma_{\hat{o}}(T) \le |V(T) - A| = (n + 1)/2 = \gamma_2(T)$. Thus $\gamma_2(T) = \gamma_{\hat{o}}(T)$. If D is a $\gamma_o(T)$-set, then for each $a \in A$, we have $a \in D$ or its two neighbors are in D. Thus $\gamma_o(T) \ge |A|$, and since A is a goa of T, we obtain $\gamma_o(T) = |A|$. It follows that $\gamma_o(T) = |A| = |V(T) - A| - 1 = \gamma_2(T) - 1 = \gamma_{\hat{o}}(T) - 1$ implying that $\gamma_o(T) + 1 = \gamma_{\hat{o}}(T)$.

Now let T be obtained from trees T_1 and T_2 that are subdivisions of trees T'_1 and T'_2, respectively, by adding an edge between a vertex u_1 of T'_1 and a vertex u_2 of T'_2, where at least one of u_1 and u_2 is a leaf or an isolate of $T_1 \cup T_2$. Without loss of generality, let u_2 be a leaf or an isolate in $T_1 \cup T_2$. If both u_1 and u_2 are isolates in $T_1 \cup T_2$, then $T = P_2$ and $1 = \gamma_o(T) = \gamma_{\hat{o}}(T) - 1$. Assume then that u_1 is not an isolate. Then $A_1 \ne \emptyset$.

By Theorem 2 and Corollary 19, $\lceil (n + 1)/2 \rceil = \gamma_2(T) \le \gamma_{\hat{o}}(T)$. Since $V(T)-A(T)$ is a gsoa of T, we have $\gamma_{\hat{o}}(T) \le |V(T) - A(T)| = |A(T)| + 2 = \lceil (n + 1)/2 \rceil$. Hence, $\gamma_{\hat{o}}(T) = |A(T)| + 2$.

Let D be a $\gamma_o(T)$-set. Then since u_2 is either an isolate or a leaf in T_2, we have that u_1 or u_2 is in D. (We note that $u_1 \notin A(T)$). Moreover, for each $a \in A$, $a \in D$ or its two neighbors are in D. Thus $|D| \ge |A| + 1$. Since $A(T) \cup \{u_2\}$ is a goa of T, it follows that $\gamma_o(T) = |A| + 1$ and $\gamma_{\hat{o}}(T) = |A| + 2 = \gamma_o(T) + 1$. ■

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