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# A characterization of locating-domination edge critical graphs

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**Abstract**

A locating-dominating set $D$ of a graph $G = (V(G), E(G))$ is a set $D \subseteq V(G)$ such that every vertex of $V(G) - D$ is adjacent to a vertex of $D$ and for every pair of distinct vertices $u, v$ in $V(G) - D$, $N(u) \cap D \neq N(v) \cap D$. The minimum cardinality of a locating-dominating set is denoted by $\gamma_L(G)$. A graph G is said to be *a locating domination edge removal critical graph*, or just $\gamma_L^+$-ER-critical graph*, if* $\gamma_L(G-e) > \gamma_L(G)$ for all $e \in E(G)$. The purpose of this paper is to characterize the class of $\gamma_L^+$-ER-critical graphs.

## 1 Introduction

In a graph $G = (V(G), E(G))$, the *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$, the closed *neighborhood is* $N_G[v] = N[v] = N(v) \cup \{v\}$ and the *degree of* $v$ is the size of its open neighborhood. A vertex of degree one is called a *pendant vertex* (or a *leaf*) and its neighbor is called a *support vertex*. We denote by $S(G)$ (respectively, $L(G)$) the set of support vertices (respectively, leaves) of $G$ and by $L_v$ the set of leaves adjacent to a support vertex $v$. A support vertex $v$ is *strong (respectively, weak)* if $|L_v| \geq 2$ (respectively, $|L_v| = 1$). An edge incident with a leaf is called a pendant edge. The subgraph induced in $G$ by a subset of vertices $S$ is denoted $G[S]$. A subset $S$ is an independent set if no edge exists between any two vertices of $G[S]$.

A subset $D$ of vertices of $V(G)$ is a *locating-dominating set* (LDS) of $G$ if every vertex in $V(G)-D$ is adjacent to a vertex in $D$ and every pair of distinct vertices $u$ and $v$ of $V(G)-D$ satisfies $N(u)\cap D \neq N(v)\cap D$. The *locating-domination number* $\gamma_L(G)$ is the minimum cardinality of a locating-dominating set. Locating domination was introduced by Slater [4, 5]. For any parameter $\mu(G)$ associated with a graph property $\mathcal{P}$, we refer to a set of vertices with Property $\mathcal{P}$ and cardinality $\mu(G)$ as a $\mu(G)$-set.

For many parameters of graphs, the study of critical, minimal or maximal graphs under vertex removal, edge removal or edge addition is classical (see [2, 3]). When we remove an edge from a graph $G$ the locating-domination number can increase, decrease or remain unchanged.

For instance, if $G$ is a $P_5$, then $\gamma_L(G) = 2$ and $\gamma_L(G-e) = 3$ for all $e \in E$. If $G$ is a clique $K_4$, then $\gamma_L(G) = 3$ and $\gamma_L(G-e) = 2$ for all $e \in E$. If $G$ is a star $K_{1,p}, p \geq 2$, then $\gamma_L(G) = \gamma_L(G-e) = p$ for all $e \in E$.

A graph G is said to be *a locating domination edge removal critical graph*, or just $\gamma_L^+$-ER-critical graph, *if* $\gamma_L(G-e) > \gamma_L(G)$ for all $e \in E(G)$.

The purpose of this paper is to characterize the class of $\gamma_L^+$-$ER$-critical graphs.

## 2 Preliminary results

The following results will be of use throughout the paper.

**Observation 1** *[1] For every graph $G$, the set $S(G)$ of all support vertices is contained in some $\gamma_L(G)$-set and for each $v \in S(G)$, every $\gamma_L(G)$-set contains at least $|L_v|$ vertices in $\{v\} \cup L_v$.*

**Proposition 2** *If $D$ is a $\gamma_L(G)$-set of a $\gamma_L^+$-ER-critical graph $G = (V, E)$, then $D$ and $V - D$ are independent sets.*

**Proof.** If an edge $e$ exists in $G[D]$ (respectively, in $G[V-D]$), then $D$ is also an LDS of $G-e$. Thus $\gamma_L(G-e) \leq \gamma_L(G)$, which contradicts the fact that $G$ is a $\gamma_L^+$-ER-critical graph. ■

**Proposition 3** *If $G$ is a $\gamma_L^+$-ER-critical graph, then for every edge $e$ of $G$, $\gamma_L(G-e) = \gamma_L(G)+1$.*

**Proof.** Let $D$ be a $\gamma_L(G)$-set of $G$ and $e = xy \in E$. Since $G$ is a $\gamma_L^+$-ER-critical graph, $\gamma_L(G-e) \geq \gamma_L(G)+1 = |D|+1$. By Proposition 2, exactly one vertex of $x$ and $y$ is in $D$. Assume $x \in D$. Then $D \cup \{y\}$ is a LDS of $G-e$, so $|D \cup \{y\}| = |D|+1 \geq \gamma_L(G-e) \geq |D|+1$, from which it follows that $\gamma_L(G-e) = \gamma_L(G)+1$. ■

**Proposition 4** *If $G$ is a $\gamma_L^+$-$ER$-critical graph, then every support vertex is weak.*

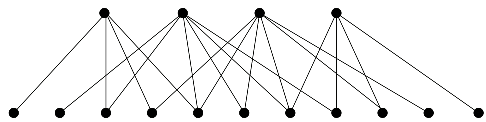
**Proof.** Suppose to the contrary that there exists a strong support vertex $v$ in $G$ and let $e = vx$ be a pendant edge attached to $v$. By Observation 1, there exists a $\gamma_L(G)$-set $D$ of $G$ which contains $v$ and $x$, so $D$ is not an independent set, which contradicts Proposition 2. ■

**Definition 5** *Let $H = (X, Y, E)$ be a connected bipartite graph such that: for every $w$ in $Y$ and for every nonempty subset $X' \subseteq N(w)$ there exists a unique $w' \in Y$ such that $N(w') = X'$.*

Let $\mathcal{H}$ be the set of all such graphs.

Examples:

$P_2 \in \mathcal{H}$



Graph of $\mathcal{H}$

**Remark 1** *Let $H = (X, Y, E)$ be a graph in $\mathcal{H}$ different from $P_2$. By Observation 1, $X$ is the unique $\gamma_L(H)$-set of $H$.*

## 3 Characterization

**Lemma 6** *If $G \in \mathcal{H}$, then $G$ is a $\gamma_L^+$-ER-critical graph.*

**Proof.** It is obvious that $P_2$ is a $\gamma_L^+$-ER-critical graph. Let $G = (X, Y, E)$ be a graph in $\mathcal{H}$ different from $P_2$. Delete any edge $e = vw$ with $v \in X$ and $w \in Y$. Then $X \cup \{w\}$ is a LDS of $G - e$. Hence $\gamma_L(G - e) \leq |X \cup \{w\}| = |X| + 1$. If $e$ is a non-pendant edge, then by Observation 1 and since $X$ is not a LDS of $G - e$, it follows that $\gamma_L(G - e) > |X|$ . If $e$ is a pendant edge, then also by Observation 1 and since $X - \{v\}$ is not a LDS of $G - w$, we have $\gamma_L(G - e) > |X - \{v\}| + 1 = |X|$. In each case, $\gamma_L(G - e) = \gamma_L(G) + 1$, so $G$ is a $\gamma_L^+$-ER-critical graph. ■

**Lemma 7** *If $G = (V, E)$ is a connected $\gamma_L^+$-ER-critical graph of order $n \geq 3$, then $S(G)$ is the unique $\gamma_L(G)$-set of $G$.*

**Proof.** Suppose that $G$ is a $\gamma_L^+$-ER-critical graph. Let $D$ be a $\gamma_L(G)$-set of $G$ which contains all supports. By Proposition 2, $D$ and $V - D$ are independent sets, and thus $G$ is a connected bipartite graph. Now we prove that every vertex of $D$ is a support vertex. Let $v \in D$. Then for every vertex $w \in N(v) \subseteq V - D$ with $N(w) = \{v, v_1, \ldots, v_k\} \subseteq D, k \geq 1$, there exists a unique vertex $w^{k-1} \in N(v)$ with $N(w^{k-1}) = \{v, v_1, \ldots, v_{k-1}\}$, for otherwise $D$ is a LDS of $G - e$ with $e = wv_k$, which contradicts the fact that $G$ is $\gamma_L^+$-ER-critical. We repeat this process for $w^l \in N(v)$ with $N(w^l) = \{v, v_1, \ldots, v_l\}$ where $1 \leq l \leq k - 1$. Consequently, there exists $w^0 \in N(v)$ with $N(w^0) = \{v\}$. Hence, every vertex of $D$ is a support, therefore $D = S(G)$. Assume that $G$ has a second $\gamma_L(G)$-set $D' \neq D$. By Observation 1 and Proposition 4, there exists a pendant vertex $w$ in $D'$ attached to a weak support vertex $v$. By Proposition 2, $v \notin D'$ and each vertex of $N(v)$ is in $D'$. Since $G$ is connected and $n \geq 3$, $|N(v)| \geq 2$, so $|D'| = |S(G)| \geq |N(v)| + |S(G) - \{v\}| \geq |S(G)| + 1$, a contradiction. ■

We obtain our characterization of the class of $\gamma_L^+$-ER-critical graphs as a consequence of Lemmas 6 and 7.

**Theorem 8** *A nontrivial connected graph $G = (V, E)$ is a $\gamma_L^+$-ER-critical graph if and only if $G \in \mathcal{H}$*

**Proof.** The "if " part follows from Lemma 6, so let us prove the "only if" part. If $G = P_2$ then $G \in \mathcal{H}$. Let $G$ be a connected $\gamma_L^+$-ER-critical graph of order $n \geq 3$. By Proposition 2 and Lemma 7, $G$ is a bipartite graph $H$ with bipartition $X = S(G)$ and $Y = V - S(G)$, and $S(G)$ is the unique $\gamma_L(G)$-set of $G$. Now, it remains to show that: for every $w$ in $Y$ and for every nonempty subset $X' \subseteq N(w)$ there exists an unique $w' \in Y$ such that $N(w') = X'$.
For this, let $w \in Y$, $N(w) = \{v_1, \ldots, v_k\}, k \geq 1$ and $X' \subseteq N(w)$. We consider the following cases.

**Case 1.** $|X'| = k$. If $k = 1$, then $X' = N(w) = \{v_1\}$ and $w$ is a pendant vertex. By Proposition 4, $v_1$ is a weak support vertex, so $w$ is the unique pendant vertex in $Y$ such that $N(w) = X'$. If $k \neq 1$, then since $X$ is a $\gamma_L(G)$-set of $G$, $w$ is the unique vertex in $Y$ such that $N(w) = X'$.

**Case 2.** $1 \leq |X'| \leq k - 1$. Let $X' = \{x_1, \ldots, x_l\} \subseteq N(w)$ with $l \leq k - 1$. With a similar manner as in the proof of Lemma 7, if $l = k - 1$, then there exists a unique vertex $w^l \in Y$ with $N(w^l) = \{x_1, \ldots, x_l\}$, for otherwise $D$ is a LDS of $G - e$ with $e = wv$ and $v \in N(w) - N(w^l)$, which contradicts the criticality of $G$. We repeat this process for $w^j \in Y$ with $N(w^j) = \{x_1, \ldots, x_j\}$ where $l + 1 \leq j \leq k - 2$. Consequently, there exists $w^l \in Y$ with $N(w^l) = X'$. ■

Notice that a disconnected graph $G$ is $\gamma_L^+$-ER-critical graph if and only if each component of $G$ is $\gamma_L^+$-ER-critical graph. So we have the following result.

**Corollary 1** *A nonempty graph $G = (V, E)$ is $\gamma_L^+$-ER-critical graph if and only if $G$ is the union of independent sets and graphs of $\mathcal{H}$.*

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