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Symmetrically inequivalent partitions of a square array: part II

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Abstract

An $n \times n \times p$ *proper array* is a three-dimensional array composed of directed cubes that obeys certain constraints. Because of these constraints, the $n \times n \times p$ proper arrays may be classified via a schema in which each $n \times n \times p$ proper array is associated with a particular $n \times n$ planar face. By representing each connected component present in the $n \times n$ planar face with a distinct letter, and the position of each outward pointing connector by a circle, an $n \times n$ array of circled letters is formed. This $n \times n$ array of circled letters is the *word representation* associated with the $n \times n \times p$ proper array. The main result of this paper involves the enumeration of all $n \times n$ word representations modulo symmetry, where the symmetry is derived from the group $D_4 = C_4 \times C_2$ acting on the set of word representations. This enumeration is achieved by forming a linear combination of six exponential generating functions, each of which is derived from a particular symmetry operation.

1 Introduction

This paper is a continuation of the author's previous work on three-dimensional proper arrays [2, 3, 4]. In [3], the objects studied were $n \times n \times p$ proper arrays, where $n \times n \times p$ proper arrays are three-dimensional configurations of directed cubes which are enumerated by the transition matrix $\boldsymbol{M_{n\times n}}$. The main theorem of [3] involved the construction of an exponential generating function for the number of symmetrically inequivalent partitions of an $n \times n$ square. This exponential generating function provided a lower bound for the basis size of $\boldsymbol{M_{n\times n}}$. The $n \times n$ partitions of the square were defined to be the $n \times n$ *letter representations* associated with the $n \times n \times p$ proper array. Two $n \times n$ letter representations were said to be *symmetrically equivalent* if they mapped to one another via a symmetry operation of the $n \times n$

square. For more details about proper arrays and the transition matrix, the reader is referred to [3] or [4].

The goal of this paper is provide an upper bound for the basis size of $\boldsymbol{M_{n\times n}}$. To accomplish this goal, we work with the set of $n \times n$ circled letter arrays (see Section 2 or [4]). Call the set of $n \times n$ circled letter arrays $\mathcal{P}_n$. We should note that $\mathcal{P}_n$ contains the states of the $n \times n \times p$ proper arrays. That is, $\mathcal{P}_n$ contains all the $n \times n$ *word representations* associated with $n \times n \times p$ proper arrays [4]. In order to find an upper bound for the basis size of $\boldsymbol{M_{n\times n}}$, we must enumerate all the symmetrically inequivalent element of $\mathcal{P}_n$, where two elements of $\mathcal{P}_n$ are *symmetrically equivalent* if and only if the symmetry group of the square, $D_4 = C_4 \times C_2$, maps one into the other. Figure 1.1 provides an illustration of eight symmetrically equivalent $3 \times 3 \times 3$ proper arrays, along with their eight equivalent 3×3 word representations. We enumerate the symmetrically inequivalent elements of $\mathcal{P}_n$ by the exponential generating function provided in Theorem 2.1. Note that this exponential generating function is itself a linear combination of six exponential generating functions. An auxiliary result, given in Appendix B, provides a generating function for the $n \times n$ word representations fixed by the two diagonal reflection maps of D_4.

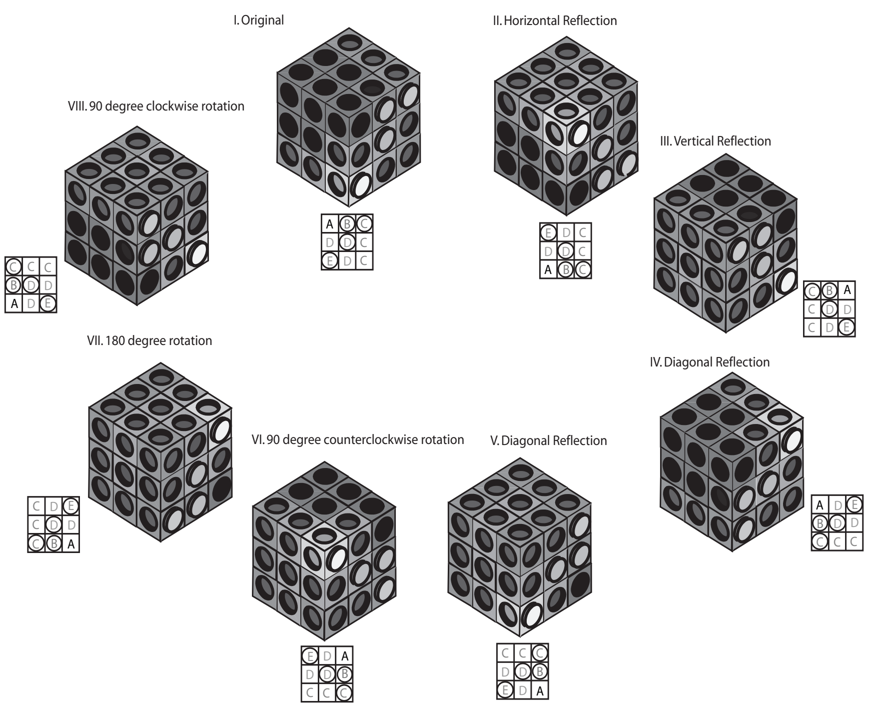

Figure 1.1: The eight equivalent versions of a $3 \times 3 \times 3$ proper array. These eight images are only counted once in the enumeration procedure.

Remark 1.1 *Word Representations are partitions of the preferred face. Hence, two word representations are regarded as the same if they have the same components but their letters are different. In other words, the letters are just labels for the components and the labels are unimportant. For example, take the word representation provided in Figure 1.1. If we replace the A with an X, the resulting word representation is considered to be the same as the original word representation.*

2 Enumerating Word Representations

For the remainder of this paper, we can ignore the three-dimensional context provided by the proper array and work in the two-dimensional setting of the word representation.

Our goal is to fix n, and determine, modulo D_4 symmetry, an upper bound for the number of word representations associated with $n \times n \times p$ proper arrays; i.e. an upper bound for the basis size of $\boldsymbol{M_{n\times n}}$. In order to accomplish this goal, we need to recall the definition of an $n \times n$ circled letter array [4].

Definition: An $\boldsymbol{n \times n}$ **circled letter array** is an $n \times n$ arrangement of letters and circles which obeys the following two conditions.

1. Each distinct letter labels an element in the partition of the $n \times n$ grid of squares.
2. Each element in the partition contains at most one circled letter.

The property of each element in a partition containing at most one circle is so important to the enumeration procedures of this paper, that we will call this property **Condition C**.

Remark 2.1 *An $n \times n$ circled letter array may represent a preferred face of an $n \times n \times p$ proper array. Hence, the $n \times n$ word representations form a proper subset of the $n \times n$ circled letter arrays.*

Definition: Let P_n be the number of $n \times n$ circled letter arrays that obey Condition C. Note that P_n is independent of square symmetry.

Definition: Let $p_j(k, c)$ be the number of partitions of a set of j circles and $k - j$ squares into c blocks subject to Condition C.

Note that

$$P_n = \sum_{c=1}^{n^2} \sum_{j=0}^{c} \binom{n^2}{j} p_j(n^2, c) \tag{2.1}$$

Furthermore, it is easy to show that for $j > 0$,

$$p_j(k, c) = \sum_{p=0}^{k-c} \binom{k-j}{p} \left\{ \begin{matrix} k-j-p \\ c-j \end{matrix} \right\} j^p \tag{2.2}$$

where

1. p counts the number of times the circled letters reappear as uncircled letters, i.e. the number of uncircled spaces contained in the union of partition elements that have a circled letter.
2. $\left\{ \begin{matrix} k-j-p \\ c-j \end{matrix} \right\}$ counts the number of ways to fill the remaining $k-j-p$ squares with arrangements of new uncircled letters.

Remarks 2.2 and 2.3 provide useful algebraic representations for $\left\{ \begin{matrix} n^2 \\ j \end{matrix} \right\}$ and $p_j(n^2, c)$ [1, 5, 6].

Remark 2.2

$$\sum_{n^2=0}^{\infty} \left\{ \begin{matrix} n^2 \\ t \end{matrix} \right\} \frac{y^{n^2}}{n^2!} = \frac{(e^y-1)^t}{t!}.$$

Remark 2.3

$$\sum_{n^2=j}^{\infty} \frac{p_j(n^2,c)z^{n^2-j}}{(n^2-j)!} = \frac{e^{zj}(e^z-1)^{c-j}}{(c-j)!}.$$

Remark 2.4 *By using Remark 2.3 in Equation (2.1) and summing over j and c, we can show that P_n is $(n^2)!$ times the coefficient of z^{n^2} in the expansion of* $\exp(e^z - 1 + ze^z)$.

With all the preliminary information in place, we are ready to develop a formula that provides an upper bound for the number of word representations associated with $n \times n \times p$ proper arrays. Define W_n to be the number of $n \times n$ circled letter arrays modulo D_4 symmetry. Then, W_n is our desired upper bound. We calculate W_n as follows.

1. Let SS_n count $n \times n$ circled letter arrays that are fixed by eight symmetry transformations of D_4.
2. Let S_n count the $n \times n$ circled letter arrays fixed by both horizontal and vertical reflections. The quantity $S_n - SS_n$ counts the $n \times n$ circled letter arrays that have horizontal and vertical reflective symmetry without having 90° rotational symmetry.
3. Let H_n count the $n \times n$ circled letter arrays fixed via horizontal reflection. The quantity $H_n - S_n$ counts the $n \times n$ circled letter arrays that are fixed only by horizontal reflection.
4. Let V_n count the $n \times n$ circled letter arrays fixed via vertical reflection. The quantity $V_n - S_n$ counts the $n \times n$ circled letter arrays that are fixed only by vertical reflection.

5. Let D_n count the $n \times n$ circled letter arrays that are fixed by both diagonal reflections. The quantity $D_n - SS_n$ counts the $n \times n$ circled letter arrays that are fixed by both diagonal reflections but do not have 90° rotational symmetry.

6. Let I_n count the $n \times n$ circled letter arrays that are fixed by a single diagonal reflection. The quantity $I_n - D_n$ counts the $n \times n$ letter representation that are fixed only by one diagonal reflection.

7. Let R_n count the $n \times n$ circled letter arrays fixed via 180° rotation. The quantity $R_n - S_n - D_n - N_n + 2SS_n$ counts the $n \times n$ circled letter arrays that are fixed only by 180° rotation.

8. Let N_n count the $n \times n$ circled letter arrays that are symmetrical with respect to 90° rotational symmetry. The quantity $N_n - SS_n$ counts the $n \times n$ circled letter arrays that are fixed only by 90° rotation.

9. Let $C_n = P_n - H_n - V_n - R_n - 2I_n + 2D_n + 2S_n$ count the $n \times n$ circled letter arrays that have eight distinct symmetry images. In other words, C_n is the number of $n \times n$ circled letter arrays that are not fixed under any symmetry transformation.

Theorem 2.1 *Let $W_n, C_n, P_n, V_n, H_n, R_n, I_n, D_n, S_n, N_n$ and SS_n be as previously defined. Then*

$$W_n = \frac{P_n + H_n + V_n + R_n + 2I_n + 2N_n}{8}.$$

Proof of Theorem 2.1: To calculate the number of $n \times n \times p$ circled letter arrays modulo D_4 symmetry, we first determine whether a given $n \times n$ circled letter array, called A, is fixed via any of the eight symmetry transformations. If A is not fixed by any symmetry, it has eight equivalent images. However, if A is fixed under a symmetry transformation, it has at most four symmetry images. It follows that

$$\begin{aligned} W_n &= \frac{C_n + 8SS_n}{8} + \frac{N_n + D_n + S_n - 3SS_n}{2} \\ &\quad + \frac{H_n + V_n + R_n + 2I_n + 2SSn - 3S_n - 3D_n - N_n}{4} \\ &= \frac{P_n + H_n + V_n + R_n + 2I_n + 2N_n}{8}. \qquad \square \end{aligned}$$

Remark 2.5 *Theorem 2.1 can be considered to be an immediate consequence of Burnside's Lemma.*

2.1 A Numerical Example

In order to understand how Theorem 2.1 provides an upper bound for the basis size of the transition matrix $\boldsymbol{M_{n\times n}}$, look at the following example. Let $n = 2$. It can be shown that $\boldsymbol{M_{2\times 2}}$ is a 28×28 matrix. Theorem 2.1 counts the number of symmetrically distinct $n \times n$ circled letter arrays. Note that $P_2 = 152$. Next, we calculate $H_2 = 16, V_2 = 16, R_2 = 16, I_2 = 32$, and $N_2 = 4$. Theorem 2.1 implies that modulo D_4 symmetry, the number of 2×2 letter representations is $\frac{152+16+16+16+64+8}{8} = 34$. Thus, the transition matrix associated with the $2 \times 2 \times p$ proper arrays can be no larger than a 34×34 matrix. The goal of the author's research is to obtain a formula that calculates the actual basis size of $\boldsymbol{M_{n \times n}}$. As this example demonstrates, Theorem 2.1 provides not actual basis size, but an upper bound on the basis size.

3 Generating Function for Diagonal Symmetry

In order to use Theorem 2.1, we need to find generating functions for H_n, V_n, R_n, I_n, and N_n. In [2] and [4], we have previously calculated the generating functions for H_n, V_n, and R_n. In particular, if n is even, say $n = 2m$, we note that $V_{2m} = H_{2m} = R_{2m}$, where H_{2m} is $(2m^2)!$ times the coefficient of z^m in $\exp(2(e^z - 1) + \frac{1}{2}(e^z - 1)^2 + ze^{2z})$ [4]. Also, if n is odd, say $n = 2m + 1$, we note that $V_{2m+1} = H_{2m+1}$, where H_{2m+1} is $(2m^2 + m)!(2m + 1)!$ times the coefficient of $z^{2m^2+m}y^{2m+1}$ in $\exp((y + 1)e^{y+z} + (z + \frac{1}{2})e^{2z} - \frac{3}{2})$ [4]. Furthermore, let $N = (2m + 2)\lfloor\frac{2m+1}{2}\rfloor$. Then R_{2m+1} is $N!$ times the coefficient of z^n in $2\exp(\frac{1}{2}(e^z - 1)^2 + 2(e^z - 1) + z + ze^{2z})$ [4].

We will now use the next two sections of this paper to determine generating functions for I_n, and N_n. The main technique for determining these generating functions involves the subdivision of the $n \times n$ array into either halves or quarters. In either case, we can arbitrarily fill one of the halves/quarters with any arrangement of circled letters which obey Condition C, and then use symmetry to fill the remaining half/quarters. The trick to this technique is to carefully subdivide around any row and column that will be fixed under the symmetry transformations. Thus, we must take into account whether n is an even or odd integer.

We begin by calculating the generating functions associated with I_n. Assume that the center of the $n \times n$ array is the origin and that the reflection in question is over the line $y = -x$. In this case, we subdivide the $n \times n$ array into two halves, one above and one below the line $y = -x$. In particular, define a **diagonal layer** to be a square whose center lies $y = -x$; i.e. a square fixed by the reflection. Define an **off diagonal layer** to be two squares, each of which is the image of the other via reflection. Define the **bottom half of an off diagonal layer** to be the square whose upper right vertex lies on the line $y = -x$. The other square is said to be the **upper half of the off diagonal layer**. The **bottom half of the $n \times n$ array** is the union of all the squares which occur in the bottom half of an off diagonal layer. The **upper half of the $n \times n$ array** is the union of all the squares which occur in the top half of an off diagonal layer. The geometric strategy for constructing I_n is as follows.

I: Fill the diagonal layers with an arbitrary arrangement of circled letters which obey Condition C.

II: Determine the number of off diagonal layers that are *completely* filled by a letter that occurs in a diagonal layer.

III: Fill the bottom halves of the remaining off diagonal layers with an arbitrary arrangement of circled letters that obey Condition C and use reflective symmetry to determine the top halves of these off diagonal layers.

There are five types of letters that can fill the bottom half of an off diagonal layer. These five possibilities are as follows and are illustrated in Figure 3.1.

1. A circled letter that reflects to a new circled letter.
2. An uncircled letter that reflects to a new uncircled letter.
3. A circled letter that that reflects to an uncircled letter. This uncircled letter also occurs in the bottom half of the array. In this case, we say an **interchange** has occurred.
4. An uncircled letter interchanges with another uncircled letter.
5. An uncircled letter reflects to itself.

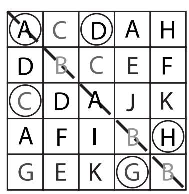

Figure 3.1: An example of a 5×5 array which is symmetrical with respect to reflection over $y = -x$. In this array, A and B occur along the diagonal while C, D, E, F, G, I, and K occur in the bottom halves of various off diagonal layers. We should note that G reflects to H and I reflects to J. Furthermore, C interchanges with D while E interchanges with F. Finally, K reflects to itself.

Thus,

$$I_n = \sum_{\substack{d,D,f,w,W,v,V\\ m,M,p,P,t,R=0}}^{\infty} \frac{n!\left(\frac{n^2-n}{2}\right)!(2v)!p_d(D,d)p_w(W,2w)p_m(M,m)\left\{\begin{matrix} n-D \\ f\end{matrix}\right\}\left\{\begin{matrix} V \\ 2v\end{matrix}\right\}}{d!(D-d)!(n-D)!(W-w)!V!m!(M-m)!P!R!}$$

$$* \frac{\left\{\begin{matrix} P \\ p\end{matrix}\right\}\left\{\begin{matrix} R \\ t\end{matrix}\right\}(d+f)^{\frac{n^2-n}{2}-W-V-M-P-R}}{(\frac{n^2-n}{2}-W-V-M-P-R)!v!2^v}$$

where

1. d counts the circled letters which occur along the diagonal.
2. D counts the spaces in the diagonal filled by these circled letters.

3. f counts the uncircled letters which occur along the diagonal.
4. w counts the interchanges between a circled letter and an uncircled letter.
5. W counts the spaces in the bottom half of the array filled by the $2w$ letters.
6. v counts the interchanges between two uncircled letters.
7. V counts the spaces in the bottom half of the array filled by the $2v$ letters.
8. m counts the circled letters in the bottom half of the array that reflect to a new circled letter.
9. M counts the spaces in the bottom half of the array filled by the m letters.
10. p counts the uncircled letters in the bottom half of the array that reflect to a new uncircled letter.
11. P counts the spaces in the bottom half of the array filled by the p letters.
12. t counts the uncircled letters that reflect to themselves.
13. R counts the spaces in the bottom half of the array filled by these t letters.

Remark 3.1 *In the previous sum, we sum only over a range of values than ensure nonnegative factorials. Instead of explicitly writing the range of summation for each index, we use the shorthand notation of summing each variable from zero to infinity. This convention will be used throughout the paper.*

Theorem 3.1 *Let $L = \frac{n^2-n}{2}$. Let I_n be as previously defined. Then, I_n is $n!L!$ times the coefficient of The coefficient of $T^L y^n$ in the expansion of* $\exp(e^T(e^y-1)+ye^{y+T}+Te^{2T}+\frac{1}{2}(e^T-1)(e^T+3))$.

Proof of Theorem 3.1: Define $I(T,y)$ to be the following sum.

$$I(T,y) = \sum_{n,L=0}^{\infty} I_n \frac{y^n}{n!}\frac{T^L}{L!}.$$

Sum over L. Then, use Remark 2.2 to sum over n, V, P and R. Next, use Remark 2.3 to sum over D, W, and M.

Finally, sum over d, f, m, p, t, and v to obtain the desired result. For a more detailed calculation, see the proof of Theorem 5.1. □

Remark 3.2 *The generating function for I_n is same as the generating function for H_{2m+1}*

In practice, we calculate $I_n^{i,j}$, where $I_n^{i,j}$ calculates the $n \times n$ circled letter arrays which are fixed by diagonal reflection and have exactly i letters, j of which are circled. In other words, $t = i - j - f - 2p - 2v$ and $2m = j - d - 2w$. In particular,

Remark 3.3

$$I_n^{i,j}(T,y) = \sum_{\substack{d,f,w,\\ v,p=0}}^{\infty} \frac{(ye^y)^d(e^y-1)^f(Te^{2T}-Te^T)^w(e^T-1)^{i-j-f-p}(Te^T)^{\frac{j-d-2w}{2}}e^{T(d+f)}}{d!f!w!p!v!2^v\left(\frac{j-d-2w}{2}\right)!(i-j-f-2p-2v)!}.$$

4 Generating Function for 90° Rotational Symmetry

Our next step is to find a generating function associated with N_n. When analyzing rotational symmetry, it is necessary to consider the case of n even as separate from the case of n odd. When n is even, we partition the $n \times n$ array into four quadrants, fill the upper left quadrant with any arbitrary arrangement of circled letters that obeys Condition C and use 90° clockwise rotation symmetry to complete the remaining three quadrants. Let A be a letter in the upper left quadrant. Define **the four letter cycle of A** to be A, a letter in the upper right quadrant that is the 90° clockwise rotational image of A, a letter in the lower right quadrant that is the 180° image of A, and a letter in the lower left quadrant that is the 90° counterclockwise rotational image of A. There are four types of four letter cycles.

Type 1: The four letter cycle contains only one letter that appears in the upper left quadrant. In this case, the letter from the upper left quadrant is called a **singleton**.

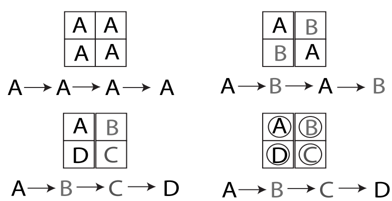

Figure 4.1: The four ways a singleton letter A can be transformed under 90° rotation.

Type 2: The four letter cycle is composed of *three* letters from the upper left quadrant and *one* letter that does not occur in the upper left quadrant.

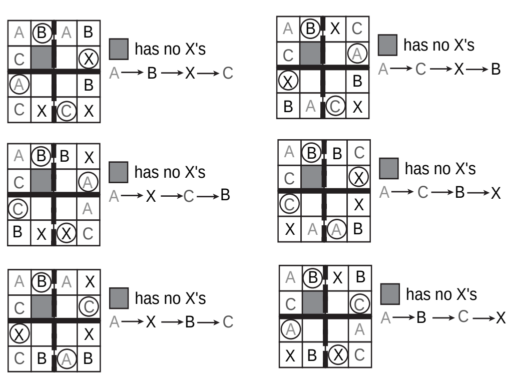

Figure 4.2: The six types of four letter cycles where A, B, and C are the three letters that appear in the upper left quadrant and X is a letter that is new relative to the upper left quadrant. Note that at most one letter in the upper left quadrant may be circled. We have shown B as circled. Four cycles also exist when A is circled, C is circled, or A, B, and C are all uncircled.

Type 3: The four letter cycle is composed of two letters that appear in the upper left quadrant and two letters that are new relative to the upper left quadrant.

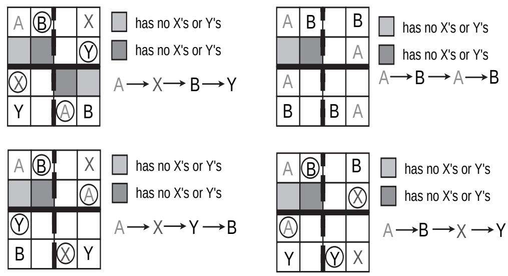

Figure 4.3: The four types of four letter cycles where A and B are from the upper left quadrant and X and Y are new with respect to the upper left quadrant. Except for the diagram in the upper right corner, we note that at most one circle appears in the upper left quadrant.

Type 4: The four letter cycle is composed of four letters from the upper left quadrant.

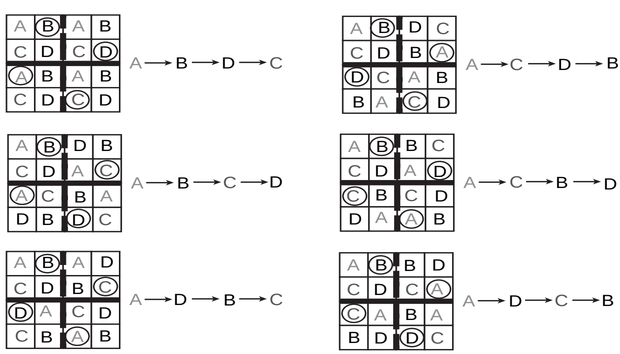

Figure 4.4: The six types of four letter cycles which have the property that the four letters appear in the upper left quadrant. Once again, at most one circle appears in the upper left quadrant.

Let $n = 2m$. The previous discussion implies that

$$N_{2m} = \sum_{\substack{i,j,s,v\\ t,w,u,p=0}}^{\infty} \frac{p_j(m^2,i)\binom{m^2}{j} 2^{t-2w}3^{i-j-3s-4w-2t-v-3u} j!(i-j)!}{w!u!v!s!t!p!(j-v-u-p)!(i-j-3s-2v-4w-2t-p-3u)!}$$

where

1. i counts the letters in the upper left quadrant. Note that $i \equiv 3s + 3v + 4u + 2t + 2p + x$, where x is the number of singletons.
2. j counts how many of these i letters are circled. Note that $j \equiv v+u+w+p+y$, where y is the number of circled singletons
3. s counts the Type 2 cycles which have no circles in the upper left quadrant.
4. v counts the Type 2 cycles which have one circle in the upper left quadrant.
5. t counts the Type 3 cycles which have no circles in the upper left quadrant.
6. p counts the Type 3 cycles which have one circle in the upper left quadrant.
7. w counts the Type 4 cycles which have no circles in the upper left quadrant.
8. u counts the Type 4 cycles which have one circle in the upper left quadrant.

Using the techniques of Theorem 3.1, we easily prove Theorem 4.1.

Theorem 4.1 *Let N_{2m} be as previously defined. Then, N_{2m} is $(m^2)!$ times the coefficient of z^{m^2} in the expansion of* $\exp(\frac{1}{4}e^{4z} + \frac{1}{2}e^{2z} + e^z - \frac{7}{4} + ze^z)$.

Our next step is to compute N_{2q+1}. The only difference between this situation and N_{2m} is the occurrence of a fixed middle square. Otherwise, the $(2q+1) \times (2q+1)$ array is divided into four $(q+1) \times q$ rectangular quadrants. Using a strategy similar to that of N_{2m}, we obtain

$$N_{2q+1} = \sum_{\substack{i,j,s,v \\ t,w,u,p,F=0}}^{\infty} \frac{p_j(q^2+q,i)\binom{q^2+q}{j} 2^{i-j-3s-2v-6w-3u-t-p-F} 3^{v+p} j!(i-j)!(1+F)}{w!u!v!s!t!p!F!(j-v-u-p)!(i-j-3s-2v-4w-2t-p-3u-F)!}$$

where

1. i, j, s, v, w, u, t, and p are as previously defined for N_{2m}.
2. F counts the uncircled singletons that rotate to themselves; i.e. the uncircled letters from the upper left quadrant that may occur in the fixed middle square.

By using a strategy similar to the one that proved Theorem 4.1, we can prove Theorem 4.2.

Theorem 4.2 *Let N_{2q+1} be as previously defined. Then, N_{2q+1} is $(q^2+q)!$ times the coefficient of z^{q^2+q} in the expansion of* $2\exp(z + \frac{1}{4}e^{4z} + \frac{1}{2}e^{2z} + e^z - \frac{7}{4} + ze^z)$.

Remark 4.1 At this point, the author would like to make two small corrections to page 124 in [3]. Both corrections are due to a typing error on the author's part. In particular, we do not need to sum over variable q. This means the sum of N_{2m} on, really should be

$$N_{2m} = \sum_{\substack{j,s \\ t,w=0}}^{\infty} \frac{\left\{ {m^2 \atop j} \right\} j! 3^{j-3s-2t-4w} 2^{t-2w}}{s!t!w!(j-3s-2t-4w)!},$$

while the sum of N_{2m+1} really is

$$N_{2m+1} = \sum_{\substack{j,k \\ s,t,w=0}}^{\infty} \frac{\left\{ \begin{matrix} m^2+m \\ j \end{matrix} \right\} j! 2^{j-3s-2t-4w-k} 2^{t-2w}(1+k)}{k!q!s!t!w!(j-3s-2t-4w-k)!}.$$

Fortunately, in [3], the correct generating functions were provided by Theorems 4.1 and 4.2.

5 Fully Symmetrical Circled Letter Arrays

This section complements Section 5 of [3]. We include this section in the main body of the paper since the techniques for enumerating fully symmetrical circled letter arrays are dependent upon the techniques and terminology of the previous two sections.

We say an $n \times n$ circled letter array is **fully symmetric** if and only if it is fixed via horizontal reflection and 90° rotation; i.e. the circled letter array is counted by SS_n. Once again, it is necessary to consider the case of n even separate from the case of n odd. We begin our discussion with SS_{2n}. In this case, we implement a two step process.

I: We partition the $n \times n$ array into four quadrants and fill the upper left hand quadrant with any arrangement of circled letter obeying Condition C that is symmetrical with respect to reflection over the line $y = -x$.

II: We take letters from the upper left hand quadrant and using the horizontal reflection, along with the 90° clockwise rotational symmetry, complete the other three quadrants.

Step 2 is complex, since we first must classify the upper left quadrant letter via diagonal symmetry. In other words, given any letter in the upper left quadrant, it is either a **diagonal letter**, i.e. a letter that appears in a diagonal layer, or an **off diagonal letter**, i.e. a letter that does not appear in a diagonal letter. Then, based on the type of letter that has been chosen, we use 90° clockwise rotation to determine its four letter cycle. The number of four letter cycles is restricted by the condition that the completed array must obey horizontal reflection. Hence, Step 2 must combine the techniques used to compute I_n and N_{2n}.

In particular,

$$SS_{2n} = \sum_{i=1}^{n^2} I_n^{i,j} T_2 T_3 T_4 S$$

where,

1. $I_n^{i,j}$ counts the $n \times n$ circled letter arrays which are fixed via reflection over $y = -x$ and have exactly i letters, j of which are circled. These are the circled letter array that may fill the upper left quadrant.

2. T_4 counts how many of the i letters form fully symmetrical four letter cycles of Type 4.
3. T_3 counts how many of the i letters form fully symmetrical four letter cycles of Type 3.
4. T_2 counts how many of the i letters form fully symmetrical four letter cycles of Type 2.
5. S counts how many of the i letters are singletons.

We will now describe how to calculate T_4. We should note that T_4 consists of subcases, each of which are determined by the number of diagonal letters and the number of off diagonal letters that occur in the given four cycle. In particular, we have $T_4 = T_{41}T_{42}T_{43}T_{44}$, where,

1. T_{41} counts the fully symmetrical four letter cycles of consisting of four letters from the upper left quadrant distributed as follows: two of the letters are located in one off diagonal layer while the other two letters occur along the diagonal. Examples of these four cycles are illustrated by Figure 5.1, where the four letter cycle is $A \to B \to D \to C$. We should note that if, in the upper left quadrant, the four letter retain their relative positions, equivalent diagrams exist for a four letter cycle given by $A \to C \to D \to B$.

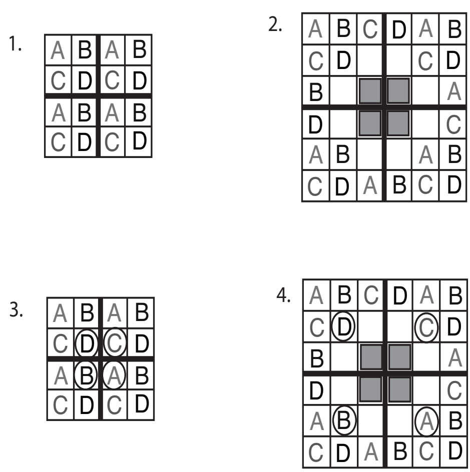

Figure 5.1: Example of four letter cycles enumerated by T_{41}. In Diagrams 1 and 2, A and D are uncircled diagonal letters, while B and C are uncircled off diagonal letters. The shaded squares do not contain either B or C. The difference between Diagrams 1 and 2 is the relative positions, in the upper left quadrant, of B and C. In Diagram 1, only B occurs in the bottom half of the upper left quadrant, while in Diagram 2, both B and C occur in the bottom half of the upper left quadrant. By placing a circle around D, Diagram 1 becomes Diagram 3, and Diagram 2 becomes Diagram 4.

2. T_{42} counts the fully symmetrical four letter cycles consisting of four letters from the upper left quadrant which fill two off diagonal layers and one diagonal layer. One of the off diagonal layers must contain two letters while the other off diagonal layer has only one letter. Figure 5.2 illustrates this situation for the four letter cycle given by $A \to B \to D \to C$. The diagrams in Figure 5.2 have equivalent versions when the four letter cycle is
$A \to C \to D \to B$.

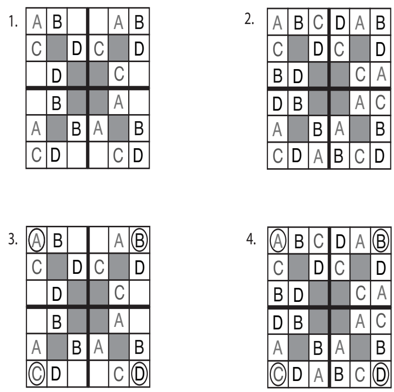

Figure 5.2: Examples of four letter cycles enumerated by T_{42}. In Diagrams 1 and 2, A is an uncircled diagonal letter, while B, C, and D are uncircled off diagonal letters. The shaded squares do not contain D, B, or C. Furthermore, D fills an entire off diagonal layer. The difference between Diagrams 1 and 2 is the relative positions, in the upper left quadrant, of B and C. It is the same difference as described in the caption of Figure 5.1. By placing a circle around A, Diagram 1 becomes Diagram 3, while Diagram 2 becomes Diagram 4.

3. T_{43} counts the fully symmetrical four letter cycles consisting of four letters from the upper left quadrant which fill three off diagonal layers. Two of these off diagonal layers have exactly one letter, while the third off diagonal layer has two letters. We illustrate such four letter cycles in Figure 5.3. In this case, the four letter cycle is $A \to B \to D \to C$. Equivalent versions exist when the four letter cycle is given by $A \to C \to D \to B$.

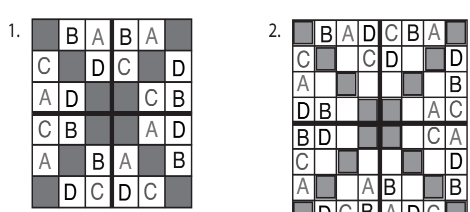

Figure 5.3: Examples of four letter cycles enumerated by T_{43}. All letters are off diagonal letters. Note that both A and D each fill an off diagonal layer. Furthermore, the shaded squares do not contain any of these off diagonal letters. The difference between Diagrams 1 and 2 has been described in the caption of Figure 5.1.

4. T_{44} counts the fully symmetrical four letter cycles consisting of four letters from the upper left quadrant which fill two off diagonal layers. Each of these off diagonal layers have two letters. In Figure 5.4, we illustrate this situation with the four letter cycle given by $A \to B \to C \to D$. Similar diagrams exist for four letter cycles given by
$A \to D \to C \to B, A \to C \to B \to D$, and $A \to D \to B \to C$.

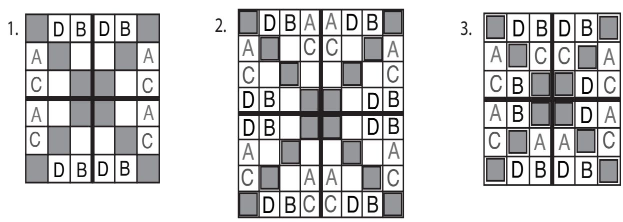

Figure 5.4: Examples of four letter cycles enumerated by T_{44}. All letters are off diagonal letters. Hence, they do not occur in the shaded squares. The difference between these diagrams is the placement of the letters relative to the bottom half of the upper left quadrant. In Diagram 1, only two letters, A and C, occur in the bottom half. In Diagram 2, all four letters occur in the bottom half, while in Diagram 3, only A, B, and C occur in the bottom half.

By using similar arguments, we can show that $T_3 = T_{31}T_{32}T_{33}T_{34}$ and $T_2 = T_{21}T_{32}$, where,

1. T_{31} enumerates the fully symmetrical four letter cycles which have the property that exactly two of the four letters occur along the diagonal of the upper left quadrant.

2. T_{32} enumerates the fully symmetrical four letter cycles which have the property that exactly two of the letters occur in the upper left quadrant. Furthermore, each such letter completely fills an off diagonal layer.

3. T_{33} enumerates the fully symmetrical four letter cycles which have the property that exactly two of the letters occur in the upper left quadrant. One of these letters is along the diagonal while the other letter completely fills an off diagonal layer.

4. T_{34} enumerates the fully symmetrical four letter cycles which have the property that exactly two of the letters occur the upper left quadrant. Both of these letters occur in a single off diagonal layer.

5. T_{21} enumerates the fully symmetrical four letter cycle which have the property that exactly three of the letters come from the upper left quadrant. One of these letters occurs along the diagonal while the other two occur in a single off diagonal layer.

6. T_{22} enumerates the fully four letter cycles which have the property that exactly three of the letters come from the upper left quadrant. These three letters occur in off diagonal layers. One of the off diagonal layers has one letter while the other off diagonal layer has two letters.

By combining the summation form of $I_n^{i,J}$ given by Remark 3.2 along with algebraic expressions for $S, T_{41}, T_{42}, T_{43}, T_{44}, T_{31}, T_{32}, T_{33}, T_{34}, T_{21}$, and T_{22}, we obtain the following sum. By combining the summation form of $I_n^{i,J}$ given by Remark 3.2 along with algebraic expressions for $S, T_{41}, T_{42}, T_{43}, T_{44}, T_{31}, T_{32}, T_{33}, T_{34}, T_{21}$, and T_{22}, we obtain the following sum.

$$SS_{2n}(y,T) \quad = \quad \sum_{\substack{i,J,d,f,w,v,p \\ k,\tilde{k},k',\hat{k} \\ j,\tilde{j},j',\hat{j} \\ l,\tilde{l},m,m',\tilde{m} \\ q,\hat{q},r,u,\hat{u} \\ g,\hat{g},\tilde{p},\hat{p} \\ h,\tilde{h},h',\hat{h}=0}}^{\infty} \frac{2^C 3^D (ye^y)^d (e^y-1)^f (e^T-1)^{i-J-f-p} (Te^{2T}-Te^T)^w (Te^T)^{\frac{(J-d-2w)}{2}} e^{T(d+f)}}{L(f-E)!(d-F)!G!(p-H)!(v-K)!\left(\frac{J-d-2w}{2}\right)!w!2^v} \tag{5.1}$$

where,

1. $C = k' + \hat{k} + j + \tilde{j} + j' + \hat{j} + m + m' + 2\tilde{m} + u + 2g + 2\hat{g} + h + \tilde{h} + h' + \hat{h} + \tilde{p} + \hat{p}$
2. $D = i - J - p - v - 3k - 3\tilde{k} - 2k' - 2\hat{k} - 3j - 3\tilde{j} - 2j' - 2\hat{j} - 3l - 3\tilde{l} - 2m - 2m' - 2\tilde{m} - 2q - \hat{q} - 2r - 2u - \hat{u} - g - \hat{g} - 2h - 2\tilde{h} - h' - \hat{h} - 2\tilde{p} - 2\hat{p}$
3. $E = 2k + 2\tilde{k} + k' + \hat{k} + j + \tilde{j} + 2q + \hat{q} + u + h + \tilde{h}$
4. $F = k' + \hat{k} + j' + \hat{j} + \hat{q} + \hat{u} + h' + \hat{h}$
5. $G = i - J - f - 2p - 2v - j - j' - \tilde{j} - \hat{j} - 2l - 2\tilde{l} - u - \hat{u} - \tilde{p} - \hat{p} - 2r$
6. $H = k + k' + j + j' + l + 2m + \tilde{m} + g + h + h' + \tilde{p}$
7. $K = \tilde{k} + \hat{k} + \tilde{j} + \hat{j} + \tilde{l} + 2m' + \tilde{m} + \hat{g} + \tilde{h} + \hat{h} + \hat{p}$
8. $L = k!\tilde{k}!k'!\hat{k}!j!\tilde{j}!j'!\hat{j}!l!\tilde{l}!m!m'!\tilde{m}!q!\hat{q}!r!u!\hat{u}!g!\hat{g}!h!\tilde{h}!h'!\hat{h}!\tilde{p}!\hat{p}!$

and

1. i counts the letters that appear in the upper left quadrant.
2. J counts how many of these i letters are circled.
3. d, f, w, v, p are associated with $I_{n,n}^{i,J}$ and are defined on Page 11.
4. $k, \tilde{k}, k', \hat{k}$ are indices which enumerate the four cycles of T_{41}.
5. $j, \tilde{j}, j', \hat{j}$ are indices which enumerate the four cycles of T_{42}.
6. $l, \tilde{l}$ are indices which enumerate the four cycles of T_{43}.
7. $m, m', \tilde{m}$ are indices which enumerate the four cycles of T_{44}.
8. $q, \hat{q}$ are indices which enumerate the four cycles of T_{31}.

9. r is an indices which enumerate the four cycles of T_{32}.
10. $u, \hat{u}$ are indices which enumerate the four cycles of T_{33}.
11. $g, \hat{g}$ are indices which enumerate the four cycles of T_{34}.
12. $h, \tilde{h}, h', \hat{h}$ are indices which enumerate the four cycles of T_{21}.
13. $\tilde{p}, \hat{p}$ are indices which enumerate the four cycles of T_{22}.

Theorem 5.1 *Let SS_{2n} be as previously defined. Then, SS_{2n} is $\left(\frac{n^2-n}{2}\right)!n!$ times the coefficient of $T^{\frac{n^2-n}{2}}y^n$ in the expansion of* $\exp(ye^{2y+4T} + e^{y+T} + \frac{1}{2}e^{2y+2T} + \frac{1}{2}e^{2y+4T} + Te^{2T} + 2e^{2T} + \frac{1}{2}e^{4T} - \frac{9}{2})$.

Proof of Theorem 5.1: We use standard combinatorial techniques to simplify $SS_{2n}(y,T)$ [5], [6]. In particular, sum (5.1) over i to obtain

$$SS_{2n} = e^{3(e^T-1)} \sum_{\substack{J,d,f,w,v,p\\ k,\tilde{k},k',\hat{k}\\ j,\tilde{j},j',\hat{j}\\ l,\tilde{l},m,m',\tilde{m}\\ q,\hat{q},r,u,\hat{u}\\ g,\hat{g},\tilde{p},\hat{p}\\ h,\tilde{h},h',\hat{h}=0}}^{\infty} \frac{2^C 3^{D_1} e^{T(d+f)}(e^y-1)^f(ye^y)^d(e^T-1)^{A_1}(Te^{2T}-Te^T)^w(Te^T)^{\frac{J-d-2w}{2}}}{L(f-E)!(d-F)!(p-H)!(v-K)!\left(\frac{J-d-2w}{2}\right)!w!2^v} \tag{5.2}$$

where

$$\begin{aligned} D_1 &= f+p+v-3k-3\tilde{k}-2k'-2\hat{k}-2j-2\tilde{j}-j'-\hat{j}-l-\tilde{l}-2m-2m' \\ &\quad -2\tilde{m}-u-g-\hat{g}-2h-2\tilde{h}-h'-\hat{h}-\tilde{p}-\hat{p}-2q-\hat{q}. \\ A_1 &= 2v+p+j+j'+\tilde{j}+\hat{j}+2l+2\tilde{l}+2r+u+\hat{u}+\tilde{p}+\hat{p}. \end{aligned}$$

Next, sum (5.2) over J and w to obtain,

$$SS_{2n} = e^{3(e^T-1)+Te^{2T}} \sum_{\substack{d,f,v,p\\ k,\tilde{k},k',\hat{k}\\ j,\tilde{j},j',\hat{j}\\ l,\tilde{l},m,m',\tilde{m}\\ q,\hat{q},r,u,\hat{u}\\ g,\hat{g},\tilde{p},\hat{p}\\ h,\tilde{h},h',\hat{h}=0}}^{\infty} \frac{2^C 3^{D_1} e^{T(d+f)}(e^y-1)^f(ye^y)^d(e^T-1)^{A_1}}{L(f-E)!(d-F)!(p-H)!(v-K)!2^v} \tag{5.3}$$

Take (5.3) and sum over p to obtain

$$SS_{2n} = e^{6(e^T-1)+Te^{2T}} \sum_{\substack{d,f,v\\ k,\tilde{k},k',\hat{k}\\ j,\tilde{j},j',\hat{j}\\ l,\tilde{l},m,m',\tilde{m}\\ q,\hat{q},r,u,\hat{u}\\ g,\hat{g},\tilde{p},\hat{p}\\ h,\tilde{h},h',\hat{h}=0}}^{\infty} \frac{2^C 3^{D_2} e^{T(d+f)}(e^y-1)^f(ye^y)^d(e^T-1)^{A_2}}{L(f-E)!(d-F)!(v-K)!2^v}, \tag{5.4}$$

where

$$D_2 = f+v-2k-3\tilde{k}-k'-2\hat{k}-j-2\tilde{j}-\hat{j}-\tilde{l}-2m'-\tilde{m}-u-\hat{g}-h-2\tilde{h}-\hat{h}-\hat{p}-2q-\hat{q}$$
$$A_2 = 2v+2j+2j'+\tilde{j}+\hat{j}+3l+2\tilde{l}+2r+u+\hat{u}+2\tilde{p}+\hat{p}+k+k'+2m+\tilde{m}+g+h+h'.$$

Now, sum (5.4) over v to obtain

$$SS_{2n} = e^{6(e^T-1)+Te^{2T}+\frac{3}{2}(e^T-1)^2} \sum_{\substack{d,\,f \\ k,\,\tilde{k},\,k',\,\hat{k} \\ j,\,\tilde{j},\,j',\,\hat{j} \\ l,\,\tilde{l},\,m,\,m',\,\tilde{m} \\ q,\,\hat{q},\,r,\,u,\,\hat{u} \\ g,\,\hat{g},\,\tilde{p},\,\hat{p} \\ h,\,\tilde{h},\,h',\,\hat{h}=0}}^{\infty} \frac{2^{C_1}3^{f-E}e^{T(d+f)}(e^y-1)^f(ye^y)^d(e^T-1)^{A_3}}{L(f-E)!(d-F)!}, \tag{5.5}$$

where

$$C_1 = k'-\tilde{k}+j+j'+m-m'+\tilde{m}+u+2g+\hat{g}+h+h'+\tilde{p}-\tilde{l}$$
$$A_3 = 2j+2j'+3\tilde{j}+3\hat{j}+3l+4\tilde{l}+2r+u+\hat{u}+2\tilde{p}+3\hat{p}+k+k'+2m+3\tilde{m}+g+h+h' + 2\tilde{k}+2\hat{k}+4m'+2\hat{g}+2\tilde{h}+2\hat{h}.$$

Next, sum (5.5) over f and d to obtain

$$SS_{2n} = e^{6(e^T-1)+Te^{2T}+\frac{3}{2}(e^T-1)^2+3e^T(e^y-1)+ye^{y+T}} \sum_{\substack{k,\,\tilde{k},\,k',\,\hat{k} \\ j,\,\tilde{j},\,j',\,\hat{j} \\ l,\,\tilde{l},\,m,\,m',\,\tilde{m} \\ q,\,\hat{q},\,r,\,u,\,\hat{u} \\ g,\,\hat{g},\,\tilde{p},\,\hat{p} \\ h,\,\tilde{h},\,h',\,\hat{h}=0}}^{\infty} \frac{2^{C_1}(ye^{y+T})^F(e^T(e^y-1))^E(e^T-1)^{A_3}}{L} \tag{5.6}$$

Finally, we sum over the remaining 26 indices contained in L. All of these indices lead to simple exponential sums. After simplifying the results of the summation, we obtain the generating function of Theorem 5.1. □

We end this paper by calculating the generating function for SS_{2q+1}. We begin by subdividing the $(2p+1)\times(2p+1)$ array into four quadrants, each a $q\times q$ array that avoids the central row and central column. The union of central row and central column is called the **central cross**.

We can then implement the following five step process to arrive at the geometric sum associated with SS_{2q+1}.

I: We fill in the first q spots of the central column with an arbitrary arrangement of circled letters that obeys Condition C.

II: We use 90° clockwise rotation and vertical reflection to fill in all the spaces of the central cross except for the fixed middle position, where the **fixed middle position** is the square that occurs both in the central column and the central row. There are seven ways to fill the central cross minus the fixed middle position. These possibilities are illustrated in Figure 5.5.

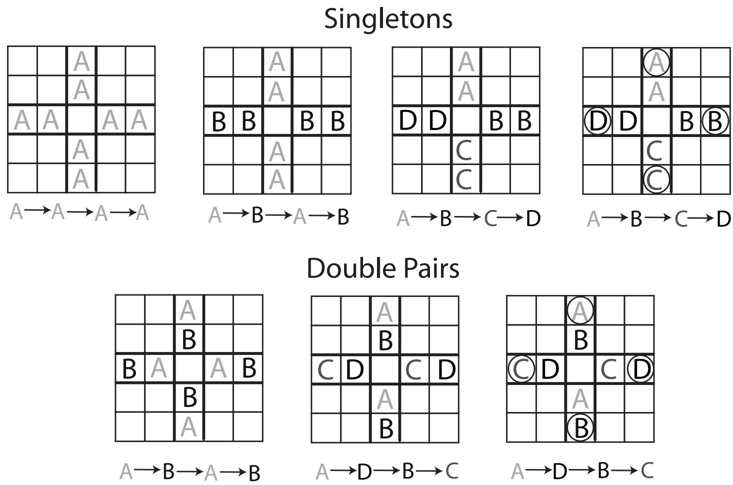

Figure 5.5: The seven ways to complete the central cross minus the fixed middle position in a fully symmetrical manner. Below each diagram, we record the four letter cycle. In the first row, only A occurs in the top part of the central cross and is considered to be a singleton. In the second row, the image of A under the rotation and reflections contains B, where B also occurs in the top part of the central cross. Hence, A and B form a double pair.

III: We take letters that occur in the central cross and place them either in the diagonal spaces of the upper left quadrant or in the bottom half of the upper left quadrant. When we transfer the letters to the quadrant, we do not transfer any possible circles. The only letters that occur in diagonal spaces are those singletons that go to themselves under rotation. The 90° clockwise rotation and vertical reflection uniquely determine how the letters appear in the remaining three quadrants.

IV: We now complete the free spaces of the quadrants with letters that do not occur in the central cross. This is done by determining T_4, T_3, T_2, and S, wh ere T_4, T_3, T_3, and S are as defined previously for SS_{2n}.

V: We fill the fixed middle position of the central cross. There are four ways to fill this position: by a singleton from the cross that goes to itself under rotation and reflection, by a singleton in the diagonal of the upper left quadrant that goes to itself under rotation and reflection, by a letter in the upper left quadrant that completely fills an off diagonal layer and goes to itself under the symmetry mappings, or a letter that does not previously occur. We should note that a circle may occur in the central square.

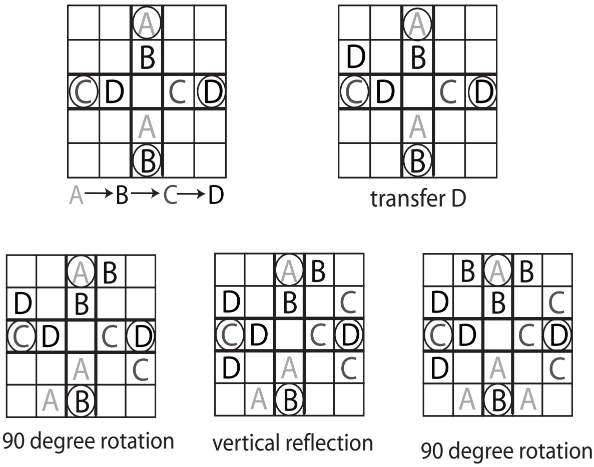

Figure 5.6: An illustration of how a letter that appears in the cross may uni quely occur in the quadrants. We begin by transferring D into the bottom half of t he upper left quadrant. Then, using 90° rotation and vertical reflection, we determi ne the image of D in the other three quadrants.

Following the steps outlined above, we arrive at a summation of 56 variables. The derivation of the exponential generating function utilizes the techniques of Theorem 5.1. Details are available, upon request, from the author.

Theorem 5.2 *Let SS_{2q+1} be as previously defined. Let* $C = \frac{3}{2}e^{2y}+xe^{2x+4y}+\frac{1}{2}e^{2y+2z}+\frac{1}{2}e^{4y+2z}+ye^{2y}+ze^{4y+2z}-\frac{9}{2}+\frac{1}{2}e^{2x+4y}+e^{x+y+z}+\frac{1}{2}e^{2y+2x}$. *Then, SS_{2q+1} is $(q!)^2(\frac{q^2-q}{2})!$ times the coefficient of $y^{\frac{q^2-q}{2}}z^q x^q$ in the expansion of $2\exp(C)\exp(y+z+x)$.*

6 Open Questions

By using a particular decomposition of the Bell Numbers [1, 5, 6] and applying various symmetry transformations to $n \times n$ arrays, we discovered a formula which calculates the number of circled letter arrays modulo D_4 symmetry. The number only provides an upper bound for the basis size of the transition matrix $\boldsymbol{M_{n\times n}}$. One research question is to find an asymptotic formula for the basis size of $\boldsymbol{M_{n\times n}}$. From computer calculations, the author conjectures that the basis size approaches W_n for large n. Another open question is to find a formula, similar in nature to Theorem 2.1, that exactly calculates the basis size of $\boldsymbol{M_{n\times n}}$. The author is attempting to find such a formula by applying an additional condition which will restrict the placement of circled even further than Remark 2.1.

Another promising area of research involves exploring the connections between $n \times n \times p$ proper arrays and percolation theory. At the present time, the author has not explored the connection in any depth but realizes that the stochastic and probabilistic techniques of percolation theory could, when applied to the representation of an $n \times n \times p$ proper array as a bond percolation on Z^3 with an open cluster at the origin, give rise to a whole new category of results.

Appendix A: Numerical Data

The following table provides, for small integer values of n, numerical values of $I_n, N_n, H_n, V_n, R_n, S_n$ and SS_n. All the values came from the generating functions given by Sections 3 through 5 of this paper or Sections 3 and 4 of [4].

$n \times n$	P_n	I_n	N_n	H_n
2×2	152	32	4	16
3×3	2527984	7978	68	7978
4×4	31919113081724	56240084	1824	11035100
5×5	7489625716055922816112 74	17237787579670	762586	17237787579670

$n \times n$	V_n	R_n	SS_n	S_n
2×2	16	16	4	6
3×3	7978	2776	52	320
4×4	11035100	11035100	392	7890
5×5	17237787579670	2119744205010	50818	45997602

Table 1: Numerical Data for certain $n \times n$ circled letter arrays

$n \times n$	L_n	W_n	$M_{n\times n}$
2×2	7	34	28
3×3	2966	320351	
4×4	1310397193	398990733385	
5×5	579823814813639193	936203214595828972075 17	

Table 2: The lower bound L_n, the upper bound W_n, and the basis size for the transition matrix $\boldsymbol{M_{n\times n}}$. Note that L_n is given by Theorem 2.1 of [3] while W_n is computed via Theorem 2.1. The author has written a Maple program that will compute the basis size for $\boldsymbol{M_{3\times 3}}$. It is available upon request. However, the user will need at least 1G of Maple memory in a single computer, or perferrably, can arrange to have parallel processing situation. The difficulty in running this Maple program underscores the importance of having a calculation that provides an upper bound.

Appendix B: Generating Function for $\boldsymbol{D_n}$

In this appendix, we will enumerate those $n \times n$ circled letter arrays which are symmetrical with respect to both diagonal reflections, i.e. those circled letter representations enumerated by D_n. The technique used to calculate D_n is similar to the technique used in the computation of $S_{m,n}$ [4], namely, subdividing the $n \times n$ array into an X and four quadrants. The X is the union of the squares whose centers lie on the line $y = -x$ and the line $y = x$. We must analyze the case of n even as separate from n odd since, if n is odd, the X contains a central square that is fixed by the two diagonal reflections.

First, we will work with a $2n \times 2n$ array. Assume the center of this array is at the origin. In order to calculate those $2n \times 2n$ circled letter arrays fixed by the two diagonal mapping, we employ the following four part strategy.

I. Fill in the top half of the $y = -x$ diagonal with an arbitrary arrangement of circled letters which obey Condition C, where by **top half** of the $y = -x$ diagonal, we mean those squares, occurring in the first n rows, whose centers lie on $y = -x$. Then, by reflecting over $y = x$, we can fill in the remainder of the $y = -x$ diagonal, which we will call the **bottom half** of the diagonal. There are five possible ways a letter in the top half of the diagonal is transformed, via reflection, into the bottom half. It can be an uncircled letter that reflects to itself; it can be an uncircled letter that reflects to uncircled letter which does not appear in the top half; it can be an circled letter that reflects to circled letter which does not appear in the top half; it can be an uncircled letter that reflect to another uncircled letter which appears in the top half; it can be an uncircled letter that reflect to circled letter which appears in the top half. These are the same three possibilities that occurred in the computation of I_n.

II. Fill in the top half of the $y = x$ diagonal. There are two possibilities. The first possibility involves letters occurring in the $y = -x$ diagonal; these letters *must* be fixed by reflection over the $y = x$ diagonal. The second possibility involves letters which do not occur in $y = -x$ diagonal. For these letters, we use the argument of Step I to complete the bottom half of the $y = x$ diagonal. The only difference is that reflect occurs over $y = -x$.

III. Transfer the letters that occur along the diagonal into the left quadrant, where the **left quadrant** consists of those $n^2 - n$ squares whose upper right corners lie on $y = -x$ and whose lower left corners lie on $y = x$. Then, by using the two diagonal reflections, we are able to *uniquely* determine the images which occur in the other three quadrants. Notice, that when we transfer the letters to the quadrants, we never gain any circles.

IV. Fill in the remaining squares of the left quadrant with an arbitrary arrangement of letters, none of which occur along the diagonals. Such a letter is either a singleton or part of a double pair. A **singleton letter** is a letter whose image, in the remaining three quadrants, is never another letter that appears in the left quadrant. There are six possible ways a letter can be a singleton. These six ways are illustrated in Figure 1.

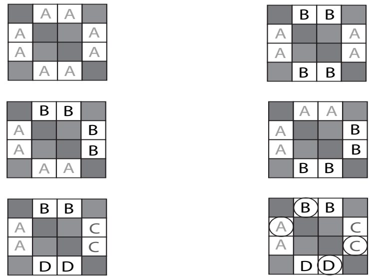

Figure 1: The six ways a singleton letter can be transformed under symmetry. None of the letters occur in the diagonal $y = -x$ or the the diagonal $y = x$. Notice that only one of the six ways involves circled letters.

The second type of letter present in the left quadrant can be considered to be part of a double pair. A letter is part of **double pair** when its image in one of the other three quadrants is another letter originally present in the left quadrant. Figure 2 illustrates the six ways double pairs, consisting of two uncircled letters, transform in a manner fixed by the two diagonal reflections. Figure 3 illustrates the three ways double pairs, consisting of a circled *and* an uncircled letter, transform via the two diagonal reflections

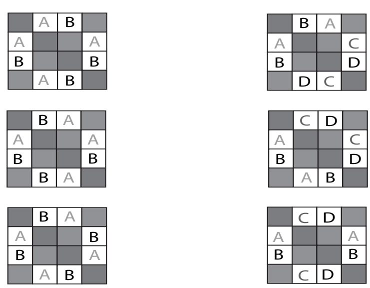

Figure 2: The six ways two uncircled letters A and B, each appearing in the left quadrant, form double pairs. Once again, these letters do not appear in the shaded squares.

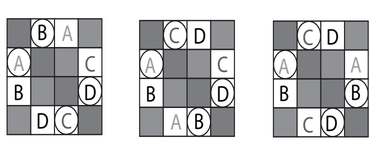

Figure 3: The three ways a circled A and an uncircled B, each appearing in the left quadrant, form double pairs. Once again, these letters do not appear in the shaded squares.

By utilizing this four step process, we obtain

$$D_{2n} = \sum_{\substack{q,j,t,s,w,v,Q,J\\ T,S,W,V,r,P,u,U=0}}^{\infty} \frac{(n!)^2(n^2-n)!p_j(n,q)p_J(n-v,Q)w^v(q-j)!(Q-J)!}{W!(n-j)!(j-t)!(q-j-t-2s-w)!t!w!s!v!(n-v-J)!(J-T)!}$$
$$* \frac{(r-P)!(2q+2Q-2t-2T-2s-2S-w-W)^V p_P(n^2-n-V,r)3^{u+U}5^{r-P-u-2U}}{(Q-J-T-2S-W)!T!S!V!(n^2-n-V-P)!u!(P-u)!U!(r-P-u-2U)!2^{S+s}}$$

where,

1. q counts the letters in the top half of the $y=-x$ diagonal.
2. j counts how many of the q letters are circled.
3. t counts the interchanges that occur between the j circled letters and the $q-j$ uncircled letters.
4. s counts the interchanges which occur among the $q-j-t$ uncircled letters.
5. w counts those $q-j-t-2s$ uncircled letters that are fixed by reflection over the $y=x$ diagonal.
6. v counts the squares in the top half of the $y=x$ diagonal filled by the w letters.
7. Q counts the the letters which occur in the top half of the $y=x$ diagonal, but *not* in the $y=-x$ diagonal.
8. J counts how many of the Q letters are circled.
9. T counts the interchanges that occur between the J circled letters and the $Q-J$ uncircled letters.
10. S counts the interchanges which occur among the $Q-J-T$ uncircled letters.
11. W counts those $Q-J-T-2S$ uncircled letters that are fixed by reflection over the $y=-x$ diagonal.
12. V counts the squares in the left quadrant filled by letters which occur along the diagonal.
13. r counts the letters which occur in the left quadrant, but *not* in the two diagonals.
14. P counts how many of these r letters are circled.
15. u counts the double pairs consisting of a circled letter and an uncircled letter.
16. U counts the double pairs consisting of two uncircled letters.

If we multiply the previous sum by $\frac{x^n}{n!}$, $\frac{z^n}{n!}$, $\frac{y^{n^2-n}}{(n^2-n)!}$, apply Remark 2.3 to sum over $n-J, n-v-J, n^2-n-V-P$, and then note that the remaining indices are exponential sums, we can prove Theorem 6.1. For more detail, see the proof of Theorem 5.1 or [4, p. 17].

Theorem 6.1 *Let D_{2n} be as previously defined. Then, D_{2n} is $(n)!^2(n^2-n)!$ times the coefficient of $x^n y^n z^{n^2-n}$ in the expansion of* $\exp(ye^{2y+2z}+xe^{2x+2z}+3ze^{2z}-2ze^z+e^{x+y+z}-2e^z+2e^{2z}+\frac{1}{2}e^{2x+2z}+\frac{1}{2}e^{2y+2z}-2)$

The strategy for calculating D_{2n+1} is akin to that of calculating D_{2n}. The only difference is that we must contend with a fixed middle square. This middle square can be filled with a letter that maps to itself under both diagonal reflections or a letter that appears only in that square. Note that the letter in the middle square *can* have a circle. In either case, a slight adjustment to the earlier sum allows us to prove the following theorem. Upon request, details are available from the author.

Theorem 6.2 *Let D_{2n+1} be as previously defined. Then, D_{2n+1} is $(n!)^2(n^2)!$ times the coefficient of $x^n y^n z^{n^2}$ in the expansion of* $2*\exp(y+z+x)\exp(ye^{2y+2z}+xe^{2x+2z}+3ze^{2z}-2ze^z+e^{x+y+z}-2e^z+2e^{2z}+\frac{1}{2}e^{2x+2z}+\frac{1}{2}e^{2y+2z}-2)$.

Remark 6.1 *The generating function for D_{2n+1} is identical to the generating function for $S_{2m+1,2n+1}$. (See Theorem 5.3 in [4].)*

n	2	3	4	5
D_n	10	320	44328	45997602

Table 3: Numerical Data for D_n

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