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Diagonally switchable 4-cycle systems revisited

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Abstract

A 4-cycle system is said to be diagonally switchable if each 4-cycle can be replaced by another 4-cycle obtained by replacing one pair of non-adjacent edges of the original 4-cycle by its diagonals so that the transformed set of 4-cycles forms another 4-cycle system. The existence of diagonally switchable 4-cycle system of K_v has already been solved [Adams, Bryant, Grannell and Griggs, *Australas. J. Combin.* 34 (2006), 145–152.] In this paper we give an alternative proof of this result and use the method to prove a new result for $K_v - I$, where I is any one factor of K_v.

1 Introduction

A 4-cycle system of G is an ordered pair (V, F) where V is the vertex set of G and F is a partition of the edge set of G, each element of which is a 4-cycle. A 4-cycle system of K_v is said to be a 4-cycle system of order v and is denoted by 4CS(v). It is already known that the set of values of v for which there exists a 4CS(v) is precisely the set of all $v \equiv 1 \pmod 8$ [5]. In this paper we consider a class of 4-cycle systems with the diagonally switchable property.

In order to define the diagonally switchable property, first let (a, b, c, d) denote the 4-cycle induced by the edge set $\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}$. The 4-cycle (a, b, c, d) is said to have diagonals $\{a, c\}$ and $\{b, d\}$. Using the four vertices a, b, c, d two new 4-cycles (a, c, b, d) and (a, b, d, c) can be constructed by replacing a pair of non-adjacent edges of the original 4-cycle (a, b, c, d) by its diagonals. We will call such transformations diagonal switches. A 4-cycle system (V, F) of G is said to be diagonally switchable if each element of F can be replaced by one of its two diagonal switches to get a new set of 4-cycles $\overline{F}$ such that $(V, \overline{F})$ is an another 4-cycle system of G (we use $\overline{F}$ throughout the rest of the paper to denote the set of 4-cycles formed from F after performing diagonal switches). A 4-cycle system (V, F) of K_v which is diagonally switchable is denoted by DS4CS(v).

A pair of 4-cycles (a, b, c, d) and (a', b', c', d') is said to be a double-diamond configuration D if they have a common diagonal. In order for a DS4CS(v) to exist, no two 4-cycles in the original 4CS(v) can share a diagonal because all diagonals of the original 4-cycle system become edges of the transformed system. So diagonally switchable 4-cycle systems must be double-diamond avoiding. Configurations in 4-cycle systems were studied by Bryant, Grannell, Griggs and Mačaj; among other results they proved the following theorem [3].

Theorem 1. *There exists a double-diamond-avoiding* $4CS(v)$ *if and only if* $v \equiv 1 \pmod 8$.

The existence spectrum of DS4CS(v)s was determined by Adams, Bryant, Grannell and Griggs [1]. In this paper we give an alternative proof of their result. This not only solves the case for K_v in a more efficient way, but is also powerful enough to easily prove a new result, considering the case for $K_v - I$, where I is any 1-factor of K_v. The constructions used here are recursive in nature, requiring fewer special cases than the proof in [1]. The basic building blocks in our constructions are holey self-orthogonal latin squares (HSOLS). When the method is then applied to the related problem of finding 4-cycle systems of $K_v - I$ with the diagonally switchable property, self-orthogonal latin squares are used.

A self-orthogonal latin square of order v, or SOLS(v), is a latin square of order v which is orthogonal to its transpose. It is well known [2] that an SOLS(v) exists for all values of v, $v \notin \{2, 3, 6\}$.

Let V be a set and let $H = \{H_1, H_2, \dots, H_k\}$ be a set of nonempty subsets which partitions the set V. A holey SOLS or HSOLS $(h_1^{n_1} h_2^{n_2} \dots h_k^{n_k})$ is a square array L of order $|V| = v = \sum_{1 \le i \le k} n_i h_i$ in which:

(1) every cell of L is either empty or contains one symbol of V;

(2) every symbol of V occurs at most once in each row and each column of L;

(3) the subarrays $H_i \times H_i$ are empty for $1 \le i \le k$ (these subarrays are referred to as holes);

(4) the symbol $x \in V$ occurs in row or column y if and only if $(x, y) \in (V \times V) \setminus \bigcup_{i=1}^{k}(H_i \times H_i)$;

(5) the superposition of L with its transpose yields every ordered pair in $(V \times V) \setminus \bigcup_{i=1}^{k}(H_i \times H_i)$.

When using L, it will be convenient to adopt quasigroup notation, defining $x \circ y$ to be the symbol in cell (x, y) of L (if one exists). Finding necessary and sufficient conditions for the existence of HSOLS$(h_1^{n_1} h_2^{n_2} \dots h_k^{n_k})$ is still an open problem. For the purposes of our proof the following results are sufficient.

Theorem 2.

(1) For $h \geq 2$, there exists an HSOLS(h^n) if and only if $n \geq 4$ [6].

(2) Suppose that n, u are positive integers and $u \neq 12$. Then there exists an HSOLS$(12^n u^1)$ if and only if $n \geq 4$ and $n \geq 1 + u/6$ (Theorem 7.1 in [7]).

2 Preliminary Results

We begin with a result from [1], where each system referred to in the following result is constructed explicitly (they also constructed systems of order 177 and 209 but these special cases are not needed in the constructions presented here).

Lemma 3. *[1] For all $v \equiv 1 \pmod 8$ with $25 \leq v \leq 137$, $v \neq \{97, 121, 129\}$, there exists a DS4CS(v).*

The following result was known to the authors of [1] but was accidentally omitted in [1].

Lemma 4. *There does not exist a diagonally switchable 4-cycle system of order 9.*

Proof. First, note that there are only eight non-isomorphic 4CS(9)s [4], of which seven have double-diamond configurations. In view of the discussion in the introduction, there is, therefore, only one candidate for being diagonally switchable, namely $(V, F) = (Z_9, \{(0, 1, 5, 2) + i \mid i \in Z_9\})$, where each sum is reduced modulo 9.

Observe that F contains the 4-cycles $(0, 1, 5, 2)$, $(5, 6, 1, 7)$ and $(8, 0, 4, 1)$. No matter how $(8, 0, 4, 1)$ is switched, the resulting 4-cycle contains the edge $\{0, 1\}$. So $(0, 1, 5, 2)$ must be switched to $(0, 5, 1, 2)$ (not switched to $(0, 5, 2, 1)$). But this 4-cycle contains the edge $\{5, 1\}$ and hence when the 4-cycle $(5, 6, 1, 7)$ is switched, the edge $\{5, 1\}$ is covered twice. Hence there does not exist a DS4CS(9). □

3 The Main Results

Now we will state and prove the main theorems.

Theorem 5. *There exists a diagonally switchable 4-cycle system of order v (DS4CS(v)) if and only if $v \equiv 1 \pmod 8$, $v \geq 17$, with the possible exception of $v = 17$.*

Proof. In view of Lemmas 3 and 4, we can assume $v \geq 145$ or $v \in \{97, 121, 129\}$. Let $v = 24s + 2h + 1$ with $h \in \{0, 4, 8\}$; so if $h = 0, 4$ or 8 then $s \geq 4, 5$ or 6 respectively. Thus using Theorem 2, let $(Z_{12s+h}, \circ)$ be a HSOLS$(12^{s-1}(12+h)^1)$ having the hole set $H = \{H_i \mid i \in Z_s\}$ where $H_{s-1} = \{12s-12, 12s-11, \dots, 12s+(h-1)\}$ and $H_i = \{12i, 12i+1, \dots, 12i+11\}$ for each $i \in Z_{s-1}$. We will show that $(V, F) = (\{\infty\} \cup (Z_{12s+h} \times \{1, 2\}), F)$ is a 4-cycle system of order $v = 24s + 2h + 1$, where F is defined by:

(1) For each $i \in Z_{s-1}$, let $(\{\infty\} \cup (H_i \times \{1, 2\}), F_i)$ be a DS4CS(25) (see Lemma 3), and let $F_i \subseteq F$.

(2) Let $(\{\infty\} \cup (H_{s-1} \times \{1, 2\}), F_{s-1})$ be a DS4CS$(25 + 2h)$ (see Lemma 3), and let $F_{s-1} \subseteq F$.

(3) For each $\{a, b\} \subseteq Z_{12s+h}$, $\{a, b\} \not\subseteq H_i$ for all $i \in Z_s$, let $((a, 1), (b, 1), (a \circ b, 2), (b \circ a, 2)) \in F$ (note this is a 4-cycle since HSOLS is necessarily antisymmetric).

The number of 4-cycles in F is

$$\begin{aligned}
\sum_{i \in Z_{s-1}} 75|F_i| \;&+\; |F_{s-1}| + \binom{12s+h}{2} - (s-1)\binom{12}{2} - \binom{12+h}{2} \\
&= 75(s-1) + (25+2h)(24+2h)/8 + (144s^2 + 24sh - 144s - 24h)/2 \\
&= (600s + 98h + 4h^2)/8 + (144s^2 + 24sh - 144s - 24h)/2 \\
&= (24s + 2h + 1)(24s + 2h)/8 \\
&= \binom{v}{2}/4
\end{aligned}$$

as required. So to see that (V, F) is a 4-cycle system, it remains to show that each edge e in $E(K_{24s+2h+1})$ occurs in some 4-cycle in F. If $e = \{\infty, (x, j)\}$ or $\{(x, 1), (y, 1)\}$ or $\{(x, 2), (y, 2)\}$ or $\{(x, 1), (y, 2)\}$ where $\{x, y\} \subseteq H_i$ with $i \in Z_s$ and $1 \leq j \leq 2$ then e clearly occurs in a 4-cycle in F_i. Now suppose $\{x, y\} \not\subseteq H_i$ for all $i \in Z_s$. Clearly $\{(x, 1), (y, 1)\}$ occurs in $((x, 1), (y, 1), (x \circ y, 2), (y \circ x, 2))$. If $e = \{(x, 1), (y, 2)\}$ then e occurs in the 4-cycle $((x, 1), (b, 1), (x \circ b, 2), (b \circ x, 2))$ where b is chosen to satisfy $b \circ x = y$. If $e = \{(x, 2), (y, 2)\}$ then e occurs in the 4-cycle $((a, 1), (b, 1), (a \circ b, 2), (b \circ a, 2))$ where a and b are chosen by the self-orthogonal property (5) to satisfy $a \circ b = x$ and $b \circ a = y$.

To see that (V, F) is diagonally switchable, observe that by replacing F_i with $\overline{F_i}$ for each $i \in Z_s$, and replacing each 4-cycle $((a,1),(b,1),(a \circ b,2),(b \circ a,2))$ by $((a,1),(b,1),(b \circ a,2),(a \circ b,2))$ for each $\{a,b\} \subseteq Z_{12s+h}$, $\{a,b\} \not\subseteq H_i$ for all $i \in Z_s$, we get a new set of 4-cycles $\overline{F}$ which can be seen to form another 4-cycle system of $K_{24s+2h+1}$using essentially the same proof that showed (V, F) is a 4-cycle system. □

Now we will use this proof technique to form a diagonally switchable 4-cycle system of $K_v - I$, where I is any 1-factor of K_v. The basic building blocks in our constructions are self-orthogonal latin squares.

Theorem 6. *There exists a 4-cycle system of $K_v - I$ having the diagonally switchable property if and only if v is even and $v \notin \{4, 6\}$.*

Proof. To prove the necessary condition first note that in order to have a 1-factor v has to be even. Secondly, observe that any 4-cycle in $K_4 - I$ is going to cover both the edges of the 1-factor after the diagonal switch. Hence no diagonally switchable 4-cycle system of $K_4 - I$ exists.

Now consider $K_6 - I$ where we can assume $I = \{\{a,b\},\{c,d\},\{e,f\}\}$ is a 1-factor of K_6. Let (V, F) be any diagonally switchable 4-cycle system of $K_6 - I$. Clearly a and b cannot be in the same 4-cycle in F, since $\{a,b\}$ cannot be an edge in any 4-cycle nor in any of the diagonal switches. Similarly c, d and e, f cannot be in the same 4-cycle. Now consider the 4-cycle containing the edge $\{e,d\}$. By the above observation: it cannot contain c, nor f; and it cannot contain both a and b. Hence no diagonally switchable 4-cycle system of $K_6 - I$ exists.

In order to prove the sufficiency, first suppose $v = 12$. Define

$$(V, F) = (\{\infty_1, \infty_2\} \cup (Z_5 \times \{1,2\}), \{C_j(i) \mid 1 \leq j \leq 3, i \in Z_5\})$$

where:

(1) $C_1(i) = ((i,1),(i+1,1),(i+4,1),(i+2,2))$;

(2) $C_2(i) = (\infty_1,(i+1,2),(i+2,2),(i+3,1))$;

(3) $C_3(i) = (\infty_2,(i+1,2),(i+3,2),(i+2,1))$.

We now show $(V, F)$is a DS4CS of $K_{12} - I$ with $I = \{\{(i,1),(i,2)\} \mid i \in Z_5\}$. To see (V, F) is a 4-cycle system, first note it contains 15= $(\binom{12}{2} - 6)/4$ 4-cycles as required. So it remains to show that each edge e in $E(K_{12} - I)$ occurs in some 4-cycle in F. If $e = \{(x,1),(y,1)\}$ with $x, y \in Z_5$ then e occurs in some $C_1(k)$. If $e = \{\infty_i,(x,j)\}$ with $x \in Z_5$ and $1 \leq i,j \leq 2$ then e occurs in some $C_{i+1}(k)$. With $x \in Z_5$: if $e = \{(x,1),(x+2,2)\}$ then e occurs in $C_1(x)$; if $e = \{(x,1),(x+1,2)\}$ then e occurs in $C_3(x-2)$; if $e = \{(x,2),(x+1,1)\}$ then e occurs in $C_2(x-2)$; if $e = \{(x,2),(x+2,1)\}$ then e occurs in $C_1(x-2)$; if $e = \{(x,2),(x+1,2)\}$ then e occurs in $C_2(x-1)$; and if $e = \{(x,2),(x+2,2)\}$ then e occurs in $C_3(x-1)$.

To prove (V, F) is diagonally switchable, for each $i \in Z_5$ replace $C_1(i)$ with $C_1'(i) = ((i,1),(i+4,1),(i+1,1),(i+2,2))$, $C_2(i)$ with $C_2'(i) = (\infty_1,(i+2,2),(i+1,2),(i+3,1))$, and $C_3(i)$ with $C_3'(i) = (\infty_2,(i+3,2),(i+1,2),(i+2,1))$, to form a new set of 4-cycles $\overline{F}$. One can similarly check that $(V, \overline{F})$ forms another 4-cycle system.

Now suppose $v \notin \{4,6,12\}$. Let $(Z_{v/2}, \circ)$ be a SOLS$(v/2)$ (this is known to exist since v is even and $v \neq 4$, 6, 12). Define $(V, F) = (Z_{v/2} \times \{1,2\}, \{((a,1),(b,1),(a \circ b,2),(b \circ a,2)) \mid \{a,b\} \subseteq Z_{v/2}\}$. We now show (V, F) is a DS4CS of $K_v - I$ where $I = \{\{(i,1),(i,2)\} \mid i \in Z_{v/2}\}$. Clearly $|C| = \binom{(v/2)}{2} = (\binom{v}{2} - (v/2))/4$ as required. Using the same arguments as in the proof of Theorem 5 is easy to see that each edge e of $K_v - I$ is in a 4-cycle in F.

To see (V, F) is diagonally switchable, observe that by replacing each 4-cycle $((a,1),(b,1),(a \circ b,2),(b \circ a,2))$ by $((a,1),(b,1),(b \circ a,2),(a \circ b,2))$ for each $\{a,b\} \subseteq Z_{v/2}$ produces a new set of 4-cycles $\overline{F}$ which can similarly be seen to form another 4-cycle system of $K_{v/2} - I$. □

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