AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **43** (2009), Pages 133–162

On projective planes of order 12 with a collineation group of order 9*

Kenzi Akiyama

Department of Applied Mathematics
Fukuoka University
Fukuoka 814-0180
Japan
akiyama@sm.fukuoka-u.ac.jp

Chihiro Suetake

Department of Mathematics
Faculty of Engineering, Oita University
Oita 870-1192
Japan
suetake@csis.oita-u.ac.jp

Abstract

In this paper, we prove that if π is a projective plane of order 12 admitting a collineation group G of order 9, then G is an elementary abelian group and is not planar.

1 Introduction

There is a famous conjecture that any finite projective plane has a prime power order. The smallest non-prime power order for which the conjecture has not been proved is 12. In the 1980's, Janko and Trung have studied projective planes of order 12 in a series of papers [5, 6, 7, 8, 9, 10, 11, 12]. Horvatic-Baldasar, Kramer and Matulic-Bedenic [2, 3] showed that the order of any collineation group of a projective plane of order 12 divides 16 or 9. Recently the authors [1] proved that there does not exist a projective plane of order 12 admitting a collineation group of order 8. This improves the result of [14]. In this paper, we investigate projective planes of order 12 with a collineation group of order 9. We obtain the following theorem.

Theorem A *Let G be a collineation group of order* 9 *of a projective plane π of order* 12*. Then G is an elementary abelian group and is not planar.*

* This research was partially supported by Grant-in-Aid for Scientific Research (No. 18540132), Ministry of Education, Culture, Sports, Science and Technology, Japan.

2 Preliminaries

DEFINITION 2.1 A *symmetric transversal design* $\mathrm{STD}_\lambda[k;u]$ (STD) is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ satisfying the following three conditions:

(i) Each block contains exactly k points.

(ii) The point set $\mathcal{P}$ is partitioned into k point sets $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{k-1}$ of equal size u such that any two distinct points are incident with exactly λ blocks or no block according as they are contained in different $\mathcal{P}_i$'s or not. $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{k-1}$ are said to be the *point classes* of $\mathcal{D}$.

(iii) The dual structure of $\mathcal{D}$ also satisfies the above conditions (i) and (ii). The point classes of the dual structure of $\mathcal{D}$ are said to be the *block classes* of $\mathcal{D}$.

We remark that if $\mathcal{D}$ is an $\mathrm{STD}_\lambda[k;u]$, then $k = \lambda u$.

NOTATION 2.2 Let G be a permutation group on a finite set Λ and H a non empty set of G. Then set $F_\Lambda(H) = \{x \in \Lambda | x^\mu = x \text{ for all } \mu \in H\}$ and $\theta_\Lambda(H) = |F_\Lambda(H)|$. Especially when $H = \{\varphi\}$, set $F_\Lambda(\{\varphi\}) = F_\Lambda(\varphi)$ and $\theta_\Lambda(\{\varphi\}) = \theta_\Lambda(\varphi)$. Let $t_\Lambda(G) = t_\Lambda$ be the number of orbits of (G, Λ).

The following lemma is well known as an orbit theorem on projective planes (see Theorem 4.2 and Corollary 4.2.1 of [13]).

LEMMA 2.3 *Let $\pi = (\mathcal{Q}, \mathcal{L})$ be a finite projective plane, φ an automorphism of π and G an automorphism group of π. Then $|F_\mathcal{Q}(\varphi)| = |F_\mathcal{L}(\varphi)|$ and $t_\mathcal{Q}(G) = t_\mathcal{L}(G)$.*

DEFINITION 2.4 Let $\pi = (\mathcal{Q}, \mathcal{L})$ be a finite projective plane and G an automorphism group of π. Then G is said to be *planar*, if the substructure $(F_\mathcal{Q}(G), F_\mathcal{L}(G))$ of π is a subplane of π.

3 Projective planes of order 12 admitting a collineation group of order 9

In the rest of this paper we assume the following.

HYPOTHESIS 3.1 Let $\pi = (\mathcal{Q}, \mathcal{L})$ be a projective plane of order 12 and G an automorphism group of order 9 of π.

LEMMA 3.2 *Let $H (\neq \{1\})$ be a subgroup of G. If H is planar, then $(F_\mathcal{Q}(H), F_\mathcal{L}(H))$ is a subplane of π of order 3.*

PROOF. Let n be the order of the subplane $(F_\mathcal{Q}(H), F_\mathcal{L}(H))$. Since $n^2 + n \leq 12$ by Bruck's theorem (see Theorem 1.5 of [13]), $n = 2$ or 3. Let $L \in F_\mathcal{L}(H)$. Since $|H| = 3$ or 9, $|F_{(L)}(H)| \neq 3$. Therefore $n = 3$. □

Since $|\mathcal{L}| = 12^2 + 12 + 1 \equiv 1 \pmod 3$, G fixes a line L_∞. Since $|(L_\infty)| = 13 \equiv 1 \pmod 3$, G also fixes a point r_∞ on L_∞. Here we choose the line L_∞ such that $|F_{(L_\infty)}(G)|$ is maximal.

Lemma 3.3 *If G is cyclic, then one of the following statements holds.*
(i) $F_{\mathcal{Q}}(G) = \{r_\infty\}$, $F_{\mathcal{L}}(G) = \{L_\infty\}$, $r_\infty \in (L_\infty)$ *and G has a point orbit of size* 3 *and a ponit orbit of size* 9 *on* $(L_\infty) - \{r_\infty\}$.
(ii) $|F_{\mathcal{Q}}(G)| = |F_{\mathcal{L}}(G)| = 4$, $F_{\mathcal{Q}}(G) \ni r_\infty$, $F_{\mathcal{L}}(G) \ni L_\infty$, $F_{\mathcal{Q}}(G) \subseteq (L_\infty)$, $F_{\mathcal{L}}(G) \subseteq (r_\infty)$ *and G has* 3 *point orbits of size* 1 *and a point orbit of size* 9 *on* $(L_\infty) - \{r_\infty\}$.
(iii) $(F_{\mathcal{Q}}(G), F_{\mathcal{L}}(G))$ *is a subplane of order* 3 *of* π.

Proof. Let $G = \langle \varphi \rangle$. Suppose that G is not planar. By [7], φ^3 is not an elation. Therefore G has a point orbit of size 9 on L_∞. Thus $|F_{(L_\infty)}(\varphi)| = 1$ or 4. Since φ is a generalized elation, (i) and (ii) hold by Lemma 2.3.

If G is planar, (iii) holds by Lemma 3.2. □

In the rest of this paper, we use the following notation.

Notation 3.4 Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be an $\mathrm{STD}_1[12; 12]$ obtained by deleting lines (r_∞) and points (L_∞) from π. Then G induces an automorphism group on $\mathcal{D}$. Let $L_0, L_1, \cdots, L_{11}$ be the lines of π through the point r_∞ except L_∞ and $r_0, r_1, \cdots, r_{11}$ the points of π on the line L_∞ except r_∞. Set $\mathcal{P}_i = (L_i) - \{r_\infty\}$ and $\mathcal{B}_j = (r_j) - \{L_\infty\}$ for $0 \le i, j \le 11$. Then $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{11}$ are point classes of $\mathcal{D}$ and $\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_{11}$ are block classes of $\mathcal{D}$. Set $\Omega = \{\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{11}\}$ and $\Delta = \{\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_{11}\}$. Let $\mathcal{P}_0 = \{p_0, p_1, \cdots, p_{11}\}$, $\mathcal{P}_1 = \{p_{12}, p_{13}, \cdots, p_{23}\}, \cdots, \mathcal{P}_{11} = \{p_{132}, p_{133}, \cdots, p_{143}\}$,
$\mathcal{B}_0 = \{B_0, B_1, \cdots, B_{11}\}$, $\mathcal{B}_1 = \{B_{12}, B_{13}, \cdots, B_{23}\}, \cdots$,
$\mathcal{B}_{11} = \{B_{132}, B_{133}, \cdots, B_{143}\}$. Let

$$N = \begin{pmatrix} N_{0\ 0} & N_{0\ 1} & \cdots & N_{0\ 11} \\ N_{1\ 0} & N_{1\ 1} & \cdots & N_{1\ 11} \\ \vdots & \vdots & & \vdots \\ N_{11\ 0} & N_{11\ 1} & \cdots & N_{11\ 11} \end{pmatrix}$$

be the incidence matrix of $\mathcal{D}$ corresponding to these numberings of the point and the block set, where each $N_{i\ j}$ $(0 \le i, j \le 11)$ is a permutation matrix of degree 12.

Let I be the identity matrix of degree 12 and J the 12×12 all one matrix. Then

$$(*)\quad N^tN = {}^tNN = \begin{pmatrix} 12I & J & \cdots & J \\ J & 12I & \ddots & \vdots \\ \vdots & \ddots & \ddots & J \\ J & \cdots & J & 12I \end{pmatrix}.$$

Notation 3.5 For any $\tau \in G$, let $\widetilde{\tau}$ be a permutation on Ω and $\widetilde{\widetilde{\tau}}$ a permutation on Δ induced by τ.

4 The case of Lemma 3.3 (i)

In this section we consider the case of Lemma 3.3 (i) and assume the following.

Hypothesis 4.1 Let $G = \langle\varphi\rangle$ be a cyclic automorphism group of π. $F_{\mathcal{Q}}(G) = \{r_\infty\}$, $F_{\mathcal{L}}(G) = \{L_\infty\}$ and $r_\infty \in (L_\infty)$. G has one orbit of size 3 and one orbit of size 9 on $(L_\infty) - \{r_\infty\}$.

Lemma 4.2 *After changing appropriately the indexes of the point classes and the block classes, we have* $\widetilde{\varphi} = (\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_8)(\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11})$ *and* $\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_8)(\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11})$.

Proof. Suppose that $G = \langle\varphi\rangle$ does not have an orbit of size 9 on (r_∞). Then, φ^3 fixes any line through the point r_∞ and therefore φ^3 is an elation of order 3. This is a contradiction, because π does not have an elation of order 3. Thus, $G = \langle\varphi\rangle$ has an orbit of size 9 on Δ. From this, $F_{\mathcal{Q}}(G) = \{q_\infty\}$ and $F_{\mathcal{L}}(G) = \{L_\infty\}$, the lemma holds. □

Lemma 4.3 (i) $F_\Omega(\varphi^3) = \{\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11}\}$ *and* $F_\Delta(\varphi^3) = \{\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11}\}$.
(ii) $|F_{\mathcal{P}}(\varphi^3) \cap \mathcal{P}_i| = |F_{\mathcal{B}}(\varphi^3) \cap \mathcal{B}_j| = 0 \quad (i, j \in \{9, 10, 11\})$ *or* $|F_{\mathcal{P}}(\varphi^3) \cap \mathcal{P}_i| = |F_{\mathcal{B}}(\varphi^3) \cap \mathcal{B}_j| = 3 \quad (i, j \in \{9, 10, 11\})$.

Proof. By Lemma 4.2, (i) holds. If φ^3 is a generalized elation, the former case of (ii) holds. If φ is a planar, the latter case of (ii) holds. □

By Lemma 4.3, the following lemma holds.

Lemma 4.4 *One of the following* (i) *and* (ii) *must occur after changing appropriately the indexes of the points and the blocks.*
(i) $\varphi = (x_0, x_{12}, x_{24}, x_{36}, x_{48}, x_{60}, x_{72}, x_{84}, x_{96})$
$(x_1, x_{13}, x_{25}, x_{37}, x_{49}, x_{61}, x_{73}, x_{85}, x_{97})$
$(x_2, x_{14}, x_{26}, x_{38}, x_{50}, x_{62}, x_{74}, x_{86}, x_{98})$
$(x_3, x_{15}, x_{27}, x_{39}, x_{51}, x_{63}, x_{75}, x_{87}, x_{99})$
$(x_4, x_{16}, x_{28}, x_{40}, x_{52}, x_{64}, x_{76}, x_{88}, x_{100})$
$(x_5, x_{17}, x_{29}, x_{41}, x_{53}, x_{65}, x_{77}, x_{89}, x_{101})$
$(x_6, x_{18}, x_{30}, x_{42}, x_{54}, x_{66}, x_{78}, x_{90}, x_{102})$
$(x_7, x_{19}, x_{31}, x_{43}, x_{55}, x_{67}, x_{79}, x_{91}, x_{103})$
$(x_8, x_{20}, x_{32}, x_{44}, x_{56}, x_{68}, x_{80}, x_{92}, x_{104})$
$(x_9, x_{21}, x_{33}, x_{45}, x_{57}, x_{69}, x_{81}, x_{93}, x_{105})$
$(x_{10}, x_{22}, x_{34}, x_{46}, x_{58}, x_{70}, x_{82}, x_{94}, x_{106})$
$(x_{11}, x_{23}, x_{35}, x_{47}, x_{59}, x_{71}, x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132}, x_{109}, x_{121}, x_{133}, x_{110}, x_{122}, x_{134})$
$(x_{111}, x_{123}, x_{135}, x_{112}, x_{124}, x_{136}, x_{113}, x_{125}, x_{137})$
$(x_{114}, x_{126}, x_{138}, x_{115}, x_{127}, x_{139}, x_{116}, x_{128}, x_{140})$
$(x_{117}, x_{129}, x_{141}, x_{118}, x_{130}, x_{142}, x_{119}, x_{131}, x_{143})$ *and*

(ii) $\varphi = (x_0, x_{12}, x_{24}, x_{36}, x_{48}, x_{60}, x_{72}, x_{84}, x_{96})$
$(x_1, x_{13}, x_{25}, x_{37}, x_{49}, x_{61}, x_{73}, x_{85}, x_{97})$
$(x_2, x_{14}, x_{26}, x_{38}, x_{50}, x_{62}, x_{74}, x_{86}, x_{98})$
$(x_3, x_{15}, x_{27}, x_{39}, x_{51}, x_{63}, x_{75}, x_{87}, x_{99})$
$(x_4, x_{16}, x_{28}, x_{40}, x_{52}, x_{64}, x_{76}, x_{88}, x_{100})$

$(x_5, x_{17}, x_{29}, x_{41}, x_{53}, x_{65}, x_{77}, x_{89}, x_{101})$
$(x_6, x_{18}, x_{30}, x_{42}, x_{54}, x_{66}, x_{78}, x_{90}, x_{102})$
$(x_7, x_{19}, x_{31}, x_{43}, x_{55}, x_{67}, x_{79}, x_{91}, x_{103})$
$(x_8, x_{20}, x_{32}, x_{44}, x_{56}, x_{68}, x_{80}, x_{92}, x_{104})$
$(x_9, x_{21}, x_{33}, x_{45}, x_{57}, x_{69}, x_{81}, x_{93}, x_{105})$
$(x_{10}, x_{22}, x_{34}, x_{46}, x_{58}, x_{70}, x_{82}, x_{94}, x_{106})$
$(x_{11}, x_{23}, x_{35}, x_{47}, x_{59}, x_{71}, x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{120}, x_{132}, x_{109}, x_{121}, x_{133}, x_{110}, x_{122}, x_{134})$
$(x_{111}, x_{123}, x_{135}, x_{112}, x_{124}, x_{136}, x_{113}, x_{125}, x_{137})$
$(x_{114}, x_{126}, x_{138}, x_{115}, x_{127}, x_{139}, x_{116}, x_{128}, x_{140})$
$(x_{117}, x_{129}, x_{141})(x_{118}, x_{130}, x_{142})(x_{119}, x_{131}, x_{143}),$

where $x \in \{p, B\}$.

Assume that Lemma 4.4(i) occurs.

NOTATION 4.5 For a permutation matrix $A = (a_{ij})_{0 \le i,j \le 11}$ of degree 12 set

$$A^{(1)} = \left(\begin{array}{ccc|ccc|c|ccc} a_{0\ 2} & a_{0\ 0} & a_{0\ 1} & a_{0\ 5} & a_{0\ 3} & a_{0\ 4} & \cdots & a_{0\ 11} & a_{0\ 9} & a_{0\ 10} \\ a_{1\ 2} & a_{1\ 0} & a_{1\ 1} & a_{1\ 5} & a_{1\ 3} & a_{1\ 4} & \cdots & a_{1\ 11} & a_{1\ 9} & a_{1\ 10} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{11\ 2} & a_{11\ 0} & a_{11\ 1} & a_{11\ 5} & a_{11\ 3} & a_{11\ 4} & \cdots & a_{11\ 11} & a_{11\ 9} & a_{11\ 10} \end{array}\right),$$

$A^{(2)} = (A^{(1)})^{(1)}$,

$$A^{(3)} = \left(\begin{array}{cccc} a_{2\ 0} & a_{2\ 1} & \cdots & a_{2\ 11} \\ a_{0\ 0} & a_{0\ 1} & \cdots & a_{0\ 11} \\ a_{1\ 0} & a_{1\ 1} & \cdots & a_{1\ 11} \\ \hline a_{5\ 0} & a_{5\ 1} & \cdots & a_{5\ 11} \\ a_{3\ 0} & a_{3\ 1} & \cdots & a_{3\ 11} \\ a_{4\ 0} & a_{4\ 1} & \cdots & a_{4\ 11} \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline a_{11\ 0} & a_{11\ 1} & \cdots & a_{11\ 11} \\ a_{9\ 0} & a_{9\ 1} & \cdots & a_{9\ 11} \\ a_{10\ 0} & a_{10\ 1} & \cdots & a_{10\ 11} \end{array}\right)$$

and $A^{(4)} = (A^{(3)})^{(3)}$.

Let Φ_1 be the set of permutation matrices of degree 12 with the form

$$B = \begin{pmatrix} B_{00} & B_{01} & B_{02} & B_{03} \\ B_{10} & B_{11} & B_{12} & B_{13} \\ B_{20} & B_{21} & B_{22} & B_{23} \\ B_{30} & B_{31} & B_{32} & B_{33} \end{pmatrix},$$

where

$$B_{ij} = \begin{pmatrix} a_{ij} & b_{ij} & c_{ij} \\ c_{ij} & a_{ij} & b_{ij} \\ b_{ij} & c_{ij} & a_{ij} \end{pmatrix} \quad (0 \le i, j \le 3).$$

Lemma 4.6 *$N=$*

$$\left(\begin{array}{ccccccccc|ccc}
N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & A_0 & A_1 & A_2 \\
N_8 & N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & {A_2}^{(1)} & A_0 & A_1 \\
N_7 & N_8 & N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & {A_1}^{(1)} & {A_2}^{(1)} & A_0 \\
N_6 & N_7 & N_8 & N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & {A_0}^{(1)} & {A_1}^{(1)} & {A_2}^{(1)} \\
N_5 & N_6 & N_7 & N_8 & N_0 & N_1 & N_2 & N_3 & N_4 & {A_2}^{(2)} & {A_0}^{(1)} & {A_1}^{(1)} \\
N_4 & N_5 & N_6 & N_7 & N_8 & N_0 & N_1 & N_2 & N_3 & {A_1}^{(2)} & {A_2}^{(2)} & {A_0}^{(1)} \\
N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_0 & N_1 & N_2 & {A_0}^{(2)} & {A_1}^{(2)} & {A_2}^{(2)} \\
N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_0 & N_1 & A_2 & {A_0}^{(2)} & {A_1}^{(2)} \\
N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_0 & A_1 & A_2 & {A_0}^{(2)} \\
\hline
A_3 & {A_5}^{(3)} & {A_4}^{(3)} & {A_3}^{(3)} & {A_5}^{(4)} & {A_4}^{(4)} & {A_3}^{(4)} & A_5 & A_4 & A_6 & A_7 & A_8 \\
A_4 & A_3 & {A_5}^{(3)} & {A_4}^{(3)} & {A_3}^{(3)} & {A_5}^{(4)} & {A_4}^{(4)} & {A_3}^{(4)} & A_5 & {A_8}^{(1)} & A_6 & A_7 \\
A_5 & A_4 & A_3 & {A_5}^{(3)} & {A_4}^{(3)} & {A_3}^{(3)} & {A_5}^{(4)} & {A_4}^{(4)} & {A_3}^{(4)} & {A_7}^{(1)} & {A_8}^{(1)} & A_6
\end{array}\right),$$

where $N_0, \cdots, N_8, A_0, \cdots, A_5$ are permutation matrices of degree 12 and $A_6, A_7, A_8 \in \Phi_1$.

Proof. The lemma holds from Lemma 4.4 (i) and Notation 4.5. □

Set G-orbits on $\mathcal{P}$ and on $\mathcal{B}$ as follows:

$\mathcal{Y}_0 = \{x_0, x_{12}, x_{24}, x_{36}, x_{48}, x_{60}, x_{72}, x_{84}, x_{96}\}$,
$\mathcal{Y}_1 = \{x_1, x_{13}, x_{25}, x_{37}, x_{49}, x_{61}, x_{73}, x_{85}, x_{97}\}$,
$\mathcal{Y}_2 = \{x_2, x_{14}, x_{26}, x_{38}, x_{50}, x_{62}, x_{74}, x_{86}, x_{98}\}$,
$\mathcal{Y}_3 = \{x_3, x_{15}, x_{27}, x_{39}, x_{51}, x_{63}, x_{75}, x_{87}, x_{99})\}$,
$\mathcal{Y}_4 = \{x_4, x_{16}, x_{28}, x_{40}, x_{52}, x_{64}, x_{76}, x_{88}, x_{100}\}$,
$\mathcal{Y}_5 = \{x_5, x_{17}, x_{29}, x_{41}, x_{53}, x_{65}, x_{77}, x_{89}, x_{101}\}$,
$\mathcal{Y}_6 = \{x_6, x_{18}, x_{30}, x_{42}, x_{54}, x_{66}, x_{78}, x_{90}, x_{102}\}$,
$\mathcal{Y}_7 = \{x_7, x_{19}, x_{31}, x_{43}, x_{55}, x_{67}, x_{79}, x_{91}, x_{103}\}$,
$\mathcal{Y}_8 = \{x_8, x_{20}, x_{32}, x_{44}, x_{56}, x_{68}, x_{80}, x_{92}, x_{104}\}$,
$\mathcal{Y}_9 = \{x_9, x_{21}, x_{33}, x_{45}, x_{57}, x_{69}, x_{81}, x_{93}, x_{105}\}$,
$\mathcal{Y}_{10} = \{x_{10}, x_{22}, x_{34}, x_{46}, x_{58}, x_{70}, x_{82}, x_{94}, x_{106}\}$,
$\mathcal{Y}_{11} = \{x_{11}, x_{23}, x_{35}, x_{47}, x_{59}, x_{71}, x_{83}, x_{95}, x_{107}\}$,
$\mathcal{Y}_{12} = \{x_{108}, x_{120}, x_{132}, x_{109}, x_{121}, x_{133}, x_{110}, x_{122}, x_{134}\}$,
$\mathcal{Y}_{13} = \{x_{111}, x_{123}, x_{135}, x_{112}, x_{124}, x_{136}, x_{113}, x_{125}, x_{137}\}$,
$\mathcal{Y}_{14} = \{x_{114}, x_{126}, x_{138}, x_{115}, x_{127}, x_{139}, x_{116}, x_{128}, x_{140}\}$ and
$\mathcal{Y}_{15} = \{x_{117}, x_{129}, x_{141}, x_{118}, x_{130}, x_{142}, x_{119}, x_{131}, x_{143}\}$,
where $(\mathcal{Y}, x) \in \{(\mathcal{Q}, p),\ (\mathcal{C}, B)\}$.

Set $q_0 = p_0,\ q_1 = p_1, \cdots, q_{11} = p_{11}, q_{12} = p_{108}, q_{13} = p_{111}, q_{14} = p_{114}, q_{15} = p_{117}$ and $C_0 = B_0,\ C_1 = B_1, \cdots, C_{11} = B_{11}, C_{12} = B_{108}, C_{13} = B_{111}, C_{14} = B_{114}, C_{15} = B_{117}$.

For i, j with $0 \le i, j \le 15$ set $m_{ij} = |\mathcal{Q}_i \cap (C_j)|$ and $D_{ij} = \{\alpha \in G | {q_i}^{\alpha} \in (C_j)\}$. Then, $m_{ij} = |D_{ij}| \quad (0 \le i, j \le 15)$. We remark that each number m_{ij} depends only on $\mathcal{Q}_i$ and $\mathcal{C}_j$ not on C_j. Set $M = (m_{ij})_{0 \le i,j \le 15}$.

When H is a non empty set of G, we denote the element $\displaystyle\sum_{h \in H} h$ of the group ring $\mathbb{Z}[G]$ by H for simplicity and set $H^{(-1)} = \displaystyle\sum_{h \in H} h^{-1}$.

Lemma 4.7 (i) *If* $i, i'(\neq) \in \{0, 1, \cdots, 11\}$,

$$\sum_{0 \leq j \leq 15} D_{ij} {D_{i'j}}^{(-1)} = G - \{1\}.$$

(ii) *If* $i \in \{0, 1, \cdots, 11\}$,

$$\sum_{0 \leq j \leq 15} D_{ij} {D_{ij}}^{(-1)} = 12.$$

Proof. Let $i, i'(\neq) \in \{0, 1, \cdots, 11\}$. Set $A = \sum_{0 \leq j \leq 15} D_{ij} {D_{i'j}}^{(-1)}$. Let $\alpha \in G$. Then we want to know the number $|\{(\beta, \gamma) \in D_{ij} \times D_{i'j} | \alpha = \beta\gamma^{-1}\}|$. Let $(\beta, \gamma) \in D_{ij} \times D_{i'j}$ such that $\alpha = \beta\gamma^{-1}$. Since $\alpha\gamma = \beta \in D_{ij}$ and $\gamma \in D_{i'j}$, ${q_i}^\alpha \in ({C_j}^{\gamma^{-1}})$ and $q_{i'} \in ({C_j}^{\gamma^{-1}})$. If $\alpha = 1$, there is no such γ, because ${q_i}^\alpha$ and $q_{i'}$ are contained in a same point class. If $\alpha \neq 1$, there exists only one such γ, because ${q_i}^\alpha$ and $q_{i'}$ are contained in distinct point classes. Therefore, since $|\{(\beta, \gamma) \in D_{ij} \times D_{i'j} | \alpha = \beta\gamma^{-1}\}| = |\{\gamma \in G | {q_i}^\alpha \in ({C_j}^{\gamma^{-1}}),\ q_{i'} \in ({C_j}^{\gamma^{-1}})\}|$, $A = G - \{1\}$. Thus we have (i). By the similar argument, we also have (ii). □

Lemma 4.8 (i) *If* $i, i'(\neq) \in \{0, 1, \cdots, 11\}$,

$$\sum_{0 \leq j \leq 15} m_{ij} m_{i'j} = 8.$$

(ii) *If* $i \in \{0, 1, \cdots, 11\}$,

$$\sum_{0 \leq j \leq 15} {m_{ij}}^2 = 12.$$

(iii) *If* $i \in \{0, 1, \cdots, 11\}$,

$$\sum_{0 \leq j \leq 15} m_{ij} = 12.$$

Proof. We have the equations of (i) and (ii) by considering the action of the trivial character of G on equations of Lemma 4.7. (iii) holds from Lemma 4.6. □

Lemma 4.9 Lemma 4.4 (i) *does not occur.*

Proof. By Lemma 4.8 (ii) and (iii), it follows that $m_{ij} \in \{0, 1\}$ for i, j with $0 \leq i, j \leq 15$. Interchanging columns of M appropriately, we may assume that

$$(m_{ij})_{0 \leq i \leq 3,\ 0 \leq j \leq 15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

But there does not exist $(m_{4\ 0}, m_{4\ 1}, \cdots, m_{4\ 15})$ satisfying Lemma 4.8. Therefore Lemma 4.4 (i) does not occur. □

Assume that Lemma 4.4(ii) occurs.

NOTATION 4.10 For a permutation matrix $A = (a_{ij})_{0 \leq i,j \leq 11}$ of degree 12 set

$$A^{(5)} = \left(\begin{array}{ccc|ccc|ccc|ccc} a_{0\,2} & a_{0\,0} & a_{0\,1} & a_{0\,5} & a_{0\,3} & a_{0\,4} & a_{0\,8} & a_{0\,6} & a_{0\,7} & a_{0\,9} & a_{0\,10} & a_{0\,11} \\ a_{1\,2} & a_{1\,0} & a_{1\,1} & a_{1\,5} & a_{1\,3} & a_{1\,4} & a_{1\,8} & a_{1\,6} & a_{1\,7} & a_{1\,9} & a_{1\,10} & a_{1\,11} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{11\,2} & a_{11\,0} & a_{11\,1} & a_{11\,5} & a_{11\,3} & a_{11\,4} & a_{11\,8} & a_{11\,6} & a_{11\,7} & a_{11\,9} & a_{11\,10} & a_{11\,11} \end{array}\right),$$

$A^{(6)} = (A^{(5)})^{(5)}$,

$$A^{(7)} = \left(\begin{array}{cccc} a_{2\,0} & a_{2\,1} & \cdots & a_{2\,11} \\ a_{0\,0} & a_{0\,1} & \cdots & a_{0\,11} \\ a_{1\,0} & a_{1\,1} & \cdots & a_{1\,11} \\ \hline a_{5\,0} & a_{5\,1} & \cdots & a_{5\,11} \\ a_{3\,0} & a_{3\,1} & \cdots & a_{3\,11} \\ a_{4\,0} & a_{4\,1} & \cdots & a_{4\,11} \\ \hline a_{8\,0} & a_{8\,1} & \cdots & a_{8\,11} \\ a_{6\,0} & a_{6\,1} & \cdots & a_{6\,11} \\ a_{7\,0} & a_{7\,1} & \cdots & a_{7\,11} \\ \hline a_{9\,0} & a_{9\,1} & \cdots & a_{9\,11} \\ a_{10\,0} & a_{10\,1} & \cdots & a_{10\,11} \\ a_{11\,0} & a_{11\,1} & \cdots & a_{11\,11} \end{array}\right)$$

and $A^{(8)} = (A^{(7)})^{(7)}$.

Let Φ_2 be the set of permutation matrices of degree 12 with the form

$$B = \begin{pmatrix} B_{00} & B_{01} & B_{02} & O_3 \\ B_{10} & B_{11} & B_{12} & O_3 \\ B_{20} & B_{21} & B_{22} & O_3 \\ O_3 & O_3 & O_3 & B_{33} \end{pmatrix},$$

where

$$B_{ij} = \begin{pmatrix} a_{ij} & b_{ij} & c_{ij} \\ c_{ij} & a_{ij} & b_{ij} \\ b_{ij} & c_{ij} & a_{ij} \end{pmatrix}$$

for each i, j and O_3 is the 3×3 zero matrix.

LEMMA 4.11 $N =$

$$\left(\begin{array}{ccccccccc|ccc} N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & A_0 & A_1 & A_2 \\ N_8 & N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & {A_2}^{(5)} & A_0 & A_1 \\ N_7 & N_8 & N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & {A_1}^{(5)} & {A_2}^{(5)} & A_0 \\ N_6 & N_7 & N_8 & N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & {A_0}^{(5)} & {A_1}^{(5)} & {A_2}^{(5)} \\ N_5 & N_6 & N_7 & N_8 & N_0 & N_1 & N_2 & N_3 & N_4 & {A_2}^{(6)} & {A_0}^{(5)} & {A_1}^{(5)} \\ N_4 & N_5 & N_6 & N_7 & N_8 & N_0 & N_1 & N_2 & N_3 & {A_1}^{(6)} & {A_2}^{(6)} & {A_0}^{(5)} \\ N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_0 & N_1 & N_2 & {A_0}^{(6)} & {A_1}^{(6)} & {A_2}^{(6)} \\ N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_0 & N_1 & A_2 & {A_0}^{(6)} & {A_1}^{(6)} \\ N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_0 & A_1 & A_2 & {A_0}^{(6)} \\ \hline A_3 & {A_5}^{(7)} & {A_4}^{(7)} & {A_3}^{(7)} & {A_5}^{(8)} & {A_4}^{(8)} & {A_3}^{(8)} & A_5 & A_4 & A_6 & A_7 & A_8 \\ A_4 & A_3 & {A_5}^{(7)} & {A_4}^{(7)} & {A_3}^{(7)} & {A_5}^{(8)} & {A_4}^{(8)} & {A_3}^{(8)} & A_5 & {A_8}^{(5)} & A_6 & A_7 \\ A_5 & A_4 & A_3 & {A_5}^{(7)} & {A_4}^{(7)} & {A_3}^{(7)} & {A_5}^{(8)} & {A_4}^{(8)} & {A_3}^{(8)} & {A_7}^{(5)} & {A_8}^{(5)} & A_6 \end{array}\right),$$

where $N_0, \cdots, N_8, A_0, \cdots, A_5$ *are permutation matrices of degree* 12 *and* $A_6, A_7, A_8 \in \Phi_2$.

PROOF. The lemma holds from Lemma 4.4 (ii) and Notation 4.10. $\square$

LEMMA 4.12 *After changing appropriately the indexes of the points and the blocks, we have* $A_0 = \begin{pmatrix} O_3 & O_3 & O_3 & E_3 \\ & & & O_3 \\ & *_1 & & O_3 \\ & & & O_3 \end{pmatrix}$, $A_1 = \begin{pmatrix} & *_2 & & O_3 \\ O_3 & O_3 & O_3 & E_3 \\ & *_3 & & O_3 \\ & & & O_3 \end{pmatrix}$ *and* $A_2 = \begin{pmatrix} & *_4 & & O_3 \\ & & & O_3 \\ O_3 & O_3 & O_3 & E_3 \\ & *_5 & & O_3 \end{pmatrix}$, *where* O_3 *is the* 3×3 *zero matrix and* E_3 *is the identity matrix of degree* 3.

PROOF. Since $(F_{\mathcal{Q}}(\varphi^3), F_{\mathcal{L}}(\varphi^3))$ is a subplane of order 3 of π, $F_{\mathcal{P}}(\varphi^3) = \{p_{117}, p_{118}, p_{119}, p_{129}, p_{130}, p_{131}, p_{141}, p_{142}, p_{143}\}$ and $F_{\mathcal{B}}(\varphi^3) = \{B_{117}, B_{118}, B_{119}, B_{129}, B_{130}, B_{131}, B_{141}, B_{142}, B_{143}\}$, the lemma holds. □

Set G-orbits of size 9 on $\mathcal{P}$ and on $\mathcal{B}$ as follows:
$\mathcal{Y}_0 = \{x_0, x_{12}, x_{24}, x_{36}, x_{48}, x_{60}, x_{72}, x_{84}, x_{96}\}$,
$\mathcal{Y}_1 = \{x_1, x_{13}, x_{25}, x_{37}, x_{49}, x_{61}, x_{73}, x_{85}, x_{97}\}$,
$\mathcal{Y}_2 = \{x_2, x_{14}, x_{26}, x_{38}, x_{50}, x_{62}, x_{74}, x_{86}, x_{98}\}$,
$\mathcal{Y}_3 = \{x_3, x_{15}, x_{27}, x_{39}, x_{51}, x_{63}, x_{75}, x_{87}, x_{99})\}$,
$\mathcal{Y}_4 = \{x_4, x_{16}, x_{28}, x_{40}, x_{52}, x_{64}, x_{76}, x_{88}, x_{100}\}$,
$\mathcal{Y}_5 = \{x_5, x_{17}, x_{29}, x_{41}, x_{53}, x_{65}, x_{77}, x_{89}, x_{101}\}$,
$\mathcal{Y}_6 = \{x_6, x_{18}, x_{30}, x_{42}, x_{54}, x_{66}, x_{78}, x_{90}, x_{102}\}$,
$\mathcal{Y}_7 = \{x_7, x_{19}, x_{31}, x_{43}, x_{55}, x_{67}, x_{79}, x_{91}, x_{103}\}$,
$\mathcal{Y}_8 = \{x_8, x_{20}, x_{32}, x_{44}, x_{56}, x_{68}, x_{80}, x_{92}, x_{104}\}$,
$\mathcal{Y}_9 = \{x_9, x_{21}, x_{33}, x_{45}, x_{57}, x_{69}, x_{81}, x_{93}, x_{105}\}$,
$\mathcal{Y}_{10} = \{x_{10}, x_{22}, x_{34}, x_{46}, x_{58}, x_{70}, x_{82}, x_{94}, x_{106}\}$,
$\mathcal{Y}_{11} = \{x_{11}, x_{23}, x_{35}, x_{47}, x_{59}, x_{71}, x_{83}, x_{95}, x_{107}\}$,
$\mathcal{Y}_{12} = \{x_{108}, x_{120}, x_{132}, x_{109}, x_{121}, x_{133}, x_{110}, x_{122}, x_{134}\}$,
$\mathcal{Y}_{13} = \{x_{111}, x_{123}, x_{135}, x_{112}, x_{124}, x_{136}, x_{113}, x_{125}, x_{137}\}$ and
$\mathcal{Y}_{14} = \{x_{114}, x_{126}, x_{138}, x_{115}, x_{127}, x_{139}, x_{116}, x_{128}, x_{140}\}$,
where $(\mathcal{Y}, x) \in \{(\mathcal{Q}, p),\ (\mathcal{C}, B)\}$.

Set $q_0 = p_0,\ q_1 = p_1, \cdots, q_{11} = p_{11}, q_{12} = p_{108}, q_{13} = p_{111}, q_{14} = p_{114}$ and $C_0 = B_0,\ C_1 = B_1, \cdots, C_{11} = B_{11}, C_{12} = B_{108}, C_{13} = B_{111}, C_{14} = B_{114}$.

For i, j with $0 \leq i, j \leq 14$ set $m_{ij} = |\mathcal{Q}_i \cap (C_j)|$ and $D_{ij} = \{\alpha \in G | q_i{}^{\alpha} \in C_j\}$. Then, $m_{ij} = |D_{ij}| \quad (0 \leq i, j \leq 14)$. Set $M = (m_{ij})_{0 \leq i,j \leq 14}$.

LEMMA 4.13 (i) *Let* $i, i'(\neq) \in \{0, 1, \cdots, 11\}$.
(a) *If* $i \not\equiv i' \pmod 3$ *or* ($i \geq 9$ or $i' \geq 9$),

$$\sum_{0 \leq j \leq 14} D_{ij} D_{i'j}{}^{(-1)} = G - \{1\}.$$

(b) *If* $i' - i = 3$ *or* $i' - i = -6$,

$$\sum_{0 \leq j \leq 14} D_{ij} D_{i'j}{}^{(-1)} = G - \{1, \varphi, \varphi^4, \varphi^7\}.$$

(c) *If* $i'-i=6$ *or* $i'-i=-3$,

$$\sum_{0\le j\le 14} D_{ij}{D_{i'j}}^{(-1)} = G-\{1,\varphi^2,\varphi^5,\varphi^8\}.$$

(ii) *Let* $i\in\{0,1,\cdots,11\}$.
(a) *If* $i\in\{0,1,\cdots,8\}$,

$$\sum_{0\le j\le 14} D_{ij}{D_{ij}}^{(-1)} = 11+\{G-\{1,\varphi^3,\varphi^6\}\}.$$

(b) *If* $i\in\{9,10,11\}$,

$$\sum_{0\le j\le 14} D_{ij}{D_{ij}}^{(-1)} = 12+\{G-\{1\}\}.$$

PROOF. Since $\mathcal{D}$ is an $\mathrm{STD}_1[12;12]$, using the same argument as in the proof of Lemma 4.7, the lemma holds from Lemmas 4.11 and 4.12. □

LEMMA 4.14 (i) *Let* $i,i'(\neq)\in\{0,1,\cdots,11\}$.
(a) *If* $i\not\equiv i' \pmod 3$ *or* ($i\ge 9$ or $i'\ge 9$),

$$\sum_{0\le j\le 14} m_{ij}m_{i'j}=8.$$

(b) *If* $i\equiv i' \pmod 3$ *and* $0\le i,i'\le 8$,

$$\sum_{0\le j\le 14} m_{ij}m_{i'j}=5.$$

(ii) *Let* $i\in\{0,1,\cdots,11\}$.
(a) *If* $i\in\{0,1,\cdots,8\}$,

$$\sum_{0\le j\le 14} {m_{ij}}^2=17.$$

(b) *If* $i\in\{9,10,11\}$,

$$\sum_{0\le j\le 14} {m_{ij}}^2=20.$$

(iii) *Let* $i\in\{0,1,\cdots,11\}$.
(a) *If* $i\in\{0,1,\cdots,8\}$,

$$\sum_{0\le j\le 11} m_{ij}=9 \ \textit{and} \sum_{12\le j\le 14} m_{ij}=2.$$

(b) *If* $i\in\{9,10,11\}$,

$$\sum_{0\le j\le 11} m_{ij}=9 \ \textit{and} \sum_{12\le j\le 14} m_{ij}=3.$$

PROOF. We have the equations of (i) and (ii) by considering the action of the trivial character of G on equations of Lemma 4.13. (iii) holds from Lemmas 4.11 and 4.12 . □

LEMMA 4.15 (i) *Let* $i \in \{0, 1, \cdots, 8\}$. *Then* $(m_{i\ 0}, m_{i\ 1}, \cdots, m_{i\ 14})$ *is equal to*

$$(\underbrace{0, \cdots, 0}_{5}, \underbrace{1, \cdots, 1}_{6}, 3, 0, 1, 1),$$

$$(\underbrace{0, \cdots, 0}_{6}, 1, 1, 1, 2, 2, 2, 0, 1, 1) \text{ or}$$

$$(\underbrace{0, \cdots, 0}_{5}, \underbrace{1, \cdots, 1}_{5}, 2, 2, 0, 0, 2)$$

up to ordering for from the 0*th column to the* 11*th column and for from the* 12*th column to the* 14*th column.*
(ii) *Let* $i \in \{9, 10, 11\}$. *Then* $(m_{i\ 0}, m_{i\ 1}, \cdots, m_{i\ 14})$ *is equal to*

$$(\underbrace{0, \cdots, 0}_{6}, \underbrace{1, \cdots, 1}_{4}, 2, 3, 1, 1, 1),$$

$$(\underbrace{0, \cdots, 0}_{5}, \underbrace{1, \cdots, 1}_{6}, 3, 0, 1, 2),$$

$$(\underbrace{0, \cdots, 0}_{4}, \underbrace{1, \cdots, 1}_{7}, 2, 0, 0, 3),$$

$$(\underbrace{0, \cdots, 0}_{7}, 1, \underbrace{2, \cdots, 2}_{4}, 1, 1, 1) \text{ or}$$

$$(\underbrace{0, \cdots, 0}_{6}, 1, 1, 1, 2, 2, 2, 0, 1, 2)$$

up to ordering for from the 0*th column to* 11*th column and for from the* 12*th column to the* 14*th column.*

PROOF. The lemma holds from Lemma 4.14 (ii), (iii). □

LEMMA 4.16 *There does not exist a projective plane of order* 12 *satisfying Hypothesis* 4.1.

PROOF. We use a computer. There are exactly 59 $(m_{ij})_{0 \le i \le 1,\ 0 \le j \le 14}$s up to ordering for from the 0th column to 11th column and for from the 12th column to the 14th column. Any one of these matrices is extended to $(m_{ij})_{0 \le i \le 2,\ 0 \le j \le 14}$s, but there is no $(m_{9\ 0}, m_{9\ 1}, \cdots, m_{9\ 14})$ satisfying Lemma 4.14 for each $(m_{ij})_{0 \le i \le 2,\ 0 \le j \le 14}$. Therefore Lemma 4.4 (ii) does not occur. From this and Lemma 4.9, the lemma holds. □

5 The case of Lemma 3.3(ii)

In this section we consider Lemma 3.3(ii) and assume the following.

HYPOTHESIS 5.1 Let $G = \langle\varphi\rangle$ be a cyclic automorphism group of π of order 9. $|F_{\mathcal{Q}}(G)| = |F_{\mathcal{L}}(G)| = 4$, $r_\infty \in F_{\mathcal{Q}}(G)$, $L_\infty \in F_{\mathcal{L}}(G)$, $r \in (L_\infty)$ for all $r \in F_{\mathcal{Q}}(G)$ and $L \in (r_\infty)$ for all $L \in F_{\mathcal{L}}(G)$. G has 3 orbits of size 1 and one orbit of size 9 on $(L_\infty) - \{r_\infty\}$.

LEMMA 5.2 *After changing appropriately the indexes of the point classes and the block classes, we have* $\widetilde{\varphi} = (\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_8)(\mathcal{P}_9)(\mathcal{P}_{10})(\mathcal{P}_{11})$, $\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_8)(\mathcal{B}_9)(\mathcal{B}_{10})(\mathcal{B}_{11})$, $F_{\mathcal{Q}}(G) = \{r_\infty, r_9, r_{10}, r_{11}\}$ *and* $F_{\mathcal{L}}(G) = \{L_\infty, L_9, L_{10}, L_{11}\}$.

PROOF. By the same argument as in the proof of Lemma 4.2, it follows that $G = \langle\varphi\rangle$ has an orbits of size 9 on Δ. The rest of the lemma holds from $|F_{\mathcal{Q}}(G)| = |F_{\mathcal{L}}(G)| = 4$. □

LEMMA 5.3 $F_\Omega(\varphi^3) = \{\mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11}\}$, $F_\Delta(\varphi^3) = \{\mathcal{B}_9, \mathcal{B}_{10}, \mathcal{B}_{11}\}$ *and* $|F_\Omega(\varphi^3) \cap \mathcal{P}_i| = |F_\Delta(\varphi^3) \cap \mathcal{B}_j| = 3$ *for* $i, j \in \{9, 10, 11\}$.

PROOF. Since $(F_{\mathcal{Q}}(\varphi^3), F_{\mathcal{L}}(\varphi^3))$ is a subplane of order 3 of π, the lemma holds from Lemma 5.2. □

By Lemmas 5.2 and 5.3, the following lemma holds.

LEMMA 5.4 *After changing appropriately the indexes of the points and the blocks, we have*
$\varphi = (x_0, x_{12}, x_{24}, x_{36}, x_{48}, x_{60}, x_{72}, x_{84}, x_{96})$
$(x_1, x_{13}, x_{25}, x_{37}, x_{49}, x_{61}, x_{73}, x_{85}, x_{97})$
$(x_2, x_{14}, x_{26}, x_{38}, x_{50}, x_{62}, x_{74}, x_{86}, x_{98})$
$(x_3, x_{15}, x_{27}, x_{39}, x_{51}, x_{63}, x_{75}, x_{87}, x_{99})$
$(x_4, x_{16}, x_{28}, x_{40}, x_{52}, x_{64}, x_{76}, x_{88}, x_{100})$
$(x_5, x_{17}, x_{29}, x_{41}, x_{53}, x_{65}, x_{77}, x_{89}, x_{101})$
$(x_6, x_{18}, x_{30}, x_{42}, x_{54}, x_{66}, x_{78}, x_{90}, x_{102})$
$(x_7, x_{19}, x_{31}, x_{43}, x_{55}, x_{67}, x_{79}, x_{91}, x_{103})$
$(x_8, x_{20}, x_{32}, x_{44}, x_{56}, x_{68}, x_{80}, x_{92}, x_{104})$
$(x_9, x_{21}, x_{33}, x_{45}, x_{57}, x_{69}, x_{81}, x_{93}, x_{105})$
$(x_{10}, x_{22}, x_{34}, x_{46}, x_{58}, x_{70}, x_{82}, x_{94}, x_{106})$
$(x_{11}, x_{23}, x_{35}, x_{47}, x_{59}, x_{71}, x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{109}, x_{110}, x_{111}, x_{112}, x_{113}, x_{114}, x_{115}, x_{116})$
$(x_{120}, x_{121}, x_{122}, x_{123}, x_{124}, x_{125}, x_{126}, x_{127}, x_{128})$
$(x_{132}, x_{133}, x_{134}, x_{135}, x_{136}, x_{137}, x_{138}, x_{139}, x_{140})$
$(x_{117}, x_{118}, x_{119})(x_{129}, x_{130}, x_{131})(x_{141}, x_{142}, x_{143})$,
where $x \in \{p, B\}$.

NOTATION 5.5 For a permutation matrix of degree 12 $A = (a_{ij})_{0 \le i,j \le 11}$, set

$$A^{(1)} = \left(\begin{array}{ccccc|ccc} a_{0\ 8} & a_{0\ 0} & a_{0\ 1} & \cdots & a_{0\ 7} & a_{0\ 11} & a_{0\ 9} & a_{0\ 10} \\ a_{1\ 8} & a_{1\ 0} & a_{1\ 1} & \cdots & a_{1\ 7} & a_{1\ 11} & a_{1\ 9} & a_{1\ 10} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ a_{11\ 8} & a_{11\ 0} & a_{11\ 1} & \cdots & a_{11\ 7} & a_{11\ 11} & a_{11\ 9} & a_{11\ 10} \end{array} \right),$$

$A^{(i)} = (A^{(i-1)})^{(1)}$ for $i \in \{2, 3, \cdots, 8\}$.

$$A^{(9)} = \left(\begin{array}{cccc} a_{8\ 0} & a_{8\ 1} & \cdots & a_{8\ 11} \\ a_{0\ 0} & a_{0\ 1} & \cdots & a_{0\ 11} \\ a_{1\ 0} & a_{1\ 1} & \cdots & a_{1\ 11} \\ \vdots & \vdots & & \vdots \\ a_{7\ 0} & a_{7\ 1} & \cdots & a_{7\ 11} \\ \hline a_{11\ 0} & a_{11\ 1} & \cdots & a_{11\ 11} \\ a_{9\ 0} & a_{9\ 1} & \cdots & a_{9\ 11} \\ a_{10\ 0} & a_{10\ 1} & \cdots & a_{10\ 11} \end{array} \right) \text{ and}$$

$A^{(i)} = (A^{(i-1)})^{(9)}$ for $i \in \{10, 11, \cdots, 16\}$.

Let Φ_1 be the set of permutation matrices of degree 12 with the form

$$B = \left(\begin{array}{ccc|ccc|ccc|ccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & 0 & 0 & 0 \\ a_8 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & 0 & 0 & 0 \\ a_7 & a_8 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 & 0 & 0 \\ \hline a_6 & a_7 & a_8 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 \\ a_4 & a_5 & a_6 & a_7 & a_8 & a_0 & a_1 & a_2 & a_3 & 0 & 0 & 0 \\ \hline a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_0 & a_1 & a_2 & 0 & 0 & 0 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_0 & a_1 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & b_0 & b_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & b_2 & b_0 \end{array} \right).$$

LEMMA 5.6 $N =$

$$\left(\begin{array}{ccccccccc|ccc} N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & A_0 & A_1 & A_2 \\ N_8 & N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & {A_0}^{(1)} & {A_1}^{(1)} & {A_2}^{(1)} \\ N_7 & N_8 & N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & {A_0}^{(2)} & {A_1}^{(2)} & {A_2}^{(2)} \\ N_6 & N_7 & N_8 & N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & {A_0}^{(3)} & {A_1}^{(3)} & {A_2}^{(3)} \\ N_5 & N_6 & N_7 & N_8 & N_0 & N_1 & N_2 & N_3 & N_4 & {A_0}^{(4)} & {A_1}^{(4)} & {A_2}^{(4)} \\ N_4 & N_5 & N_6 & N_7 & N_8 & N_0 & N_1 & N_2 & N_3 & {A_0}^{(5)} & {A_1}^{(5)} & {A_2}^{(5)} \\ N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_0 & N_1 & N_2 & {A_0}^{(6)} & {A_1}^{(6)} & {A_2}^{(6)} \\ N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_0 & N_1 & {A_0}^{(7)} & {A_1}^{(7)} & {A_2}^{(7)} \\ N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_0 & {A_0}^{(8)} & {A_1}^{(8)} & {A_2}^{(8)} \\ \hline A_3 & {A_3}^{(9)} & {A_3}^{(10)} & {A_3}^{(11)} & {A_3}^{(12)} & {A_3}^{(13)} & {A_3}^{(14)} & {A_3}^{(15)} & {A_3}^{(16)} & A_6 & A_7 & A_8 \\ A_4 & {A_4}^{(9)} & {A_4}^{(10)} & {A_4}^{(11)} & {A_4}^{(12)} & {A_4}^{(13)} & {A_4}^{(14)} & {A_4}^{(15)} & {A_4}^{(16)} & A_8 & A_6 & A_7 \\ A_5 & {A_5}^{(9)} & {A_5}^{(10)} & {A_5}^{(11)} & {A_5}^{(12)} & {A_5}^{(13)} & {A_5}^{(14)} & {A_5}^{(15)} & {A_5}^{(16)} & A_7 & A_8 & A_6 \end{array} \right),$$

where $N_0, \cdots, N_8, A_0, \cdots, A_5$ *are permutation matrices of degree* 12 *and* $A_6, A_7, A_8 \in \Phi_1$.

PROOF. The lemma holds from Lemma 5.4 and Notation 5.5. □

LEMMA 5.7 *The same statement as Lemma* 4.12 *holds.*

Set G-orbits of size 9 on $\mathcal{P}$ and on $\mathcal{B}$ as follows:

$\mathcal{Y}_0 = \{x_0, x_{12}, x_{24}, x_{36}, x_{48}, x_{60}, x_{72}, x_{84}, x_{96}\}$,
$\mathcal{Y}_1 = \{x_1, x_{13}, x_{25}, x_{37}, x_{49}, x_{61}, x_{73}, x_{85}, x_{97}\}$,
$\mathcal{Y}_2 = \{x_2, x_{14}, x_{26}, x_{38}, x_{50}, x_{62}, x_{74}, x_{86}, x_{98}\}$,
$\mathcal{Y}_3 = \{x_3, x_{15}, x_{27}, x_{39}, x_{51}, x_{63}, x_{75}, x_{87}, x_{99})\}$,
$\mathcal{Y}_4 = \{x_4, x_{16}, x_{28}, x_{40}, x_{52}, x_{64}, x_{76}, x_{88}, x_{100}\}$,
$\mathcal{Y}_5 = \{x_5, x_{17}, x_{29}, x_{41}, x_{53}, x_{65}, x_{77}, x_{89}, x_{101}\}$,
$\mathcal{Y}_6 = \{x_6, x_{18}, x_{30}, x_{42}, x_{54}, x_{66}, x_{78}, x_{90}, x_{102}\}$,
$\mathcal{Y}_7 = \{x_7, x_{19}, x_{31}, x_{43}, x_{55}, x_{67}, x_{79}, x_{91}, x_{103}\}$,
$\mathcal{Y}_8 = \{x_8, x_{20}, x_{32}, x_{44}, x_{56}, x_{68}, x_{80}, x_{92}, x_{104}\}$,
$\mathcal{Y}_9 = \{x_9, x_{21}, x_{33}, x_{45}, x_{57}, x_{69}, x_{81}, x_{93}, x_{105}\}$,
$\mathcal{Y}_{10} = \{x_{10}, x_{22}, x_{34}, x_{46}, x_{58}, x_{70}, x_{82}, x_{94}, x_{106}\}$,
$\mathcal{Y}_{11} = \{x_{11}, x_{23}, x_{35}, x_{47}, x_{59}, x_{71}, x_{83}, x_{95}, x_{107}\}$,
$\mathcal{Y}_{12} = \{x_{108}, x_{109}, x_{110}, x_{111}, x_{112}, x_{113}, x_{114}, x_{115}, x_{116}\}$,
$\mathcal{Y}_{13} = \{x_{120}, x_{121}, x_{122}, x_{123}, x_{124}, x_{125}, x_{126}, x_{127}, x_{128}\}$ and
$\mathcal{Y}_{14} = \{x_{132}, x_{133}, x_{134}, x_{135}, x_{136}, x_{137}, x_{138}, x_{139}, x_{140}\}$,
where $(\mathcal{Y}, x) \in \{(\mathcal{Q}, p),\ (\mathcal{C}, B)\}$.

Set $q_0 = p_0,\ q_1 = p_1, \cdots, q_{11} = p_{11}, q_{12} = p_{108}, q_{13} = p_{120}, q_{14} = p_{132}$ and $C_0 = B_0,\ C_1 = B_1, \cdots, C_{11} = B_{11}, C_{12} = B_{108}, C_{13} = B_{120}, C_{14} = B_{132}$.

For i, j with $0 \le i \le 11$ and $0 \le j \le 14$ set $m_{ij} = |\mathcal{Q}_i \cap (C_j)|$ and $D_{ij} = \{\alpha \in G | {q_i}^{\alpha} \in C_j\}$. Then, $m_{ij} = |D_{ij}|$ for i, j with $0 \le i \le 11$ and $0 \le j \le 14$. Set $M = (m_{ij})_{0 \le i \le 11,\ 0 \le j \le 14}$.

Lemma 5.8 (i) *Let $i, i' \in \{0, 1, \cdots, 11\}$ and $i < i'$.*
(a) *If $(i, i') \in \{(0,1), (1,2), (3,4), (4,5), (6,7), (7,8)\}$,*

$$\sum_{0 \le j \le 14} D_{ij} {D_{i'j}}^{(-1)} = G - \{1, \varphi, \varphi^4, \varphi^7\}.$$

(b) *If $(i, i') \in \{(0,2), (3,5), (6,8)\}$,*

$$\sum_{0 \le j \le 14} D_{ij} {D_{i'j}}^{(-1)} = G - \{1, \varphi^2, \varphi^5, \varphi^8\}.$$

(c) *If i, i' do not satisfy any one of two assumptions of* (a) *and* (b),

$$\sum_{0 \le j \le 14} D_{ij} {D_{i'j}}^{(-1)} = G - \{1\}.$$

(ii) *Let $i \in \{0, 1, \cdots, 11\}$.*
(a) *If $i \in \{0, 1, \cdots, 8\}$,*

$$\sum_{0 \le j \le 14} D_{ij} {D_{ij}}^{(-1)} = 11 + \{G - \{1, \varphi^3, \varphi^6\}\}.$$

(b) *If $i \in \{9, 10, 11\}$,*

$$\sum_{0 \le j \le 14} D_{ij} {D_{ij}}^{(-1)} = 12 + \{G - \{1\}\}.$$

PROOF. Since $\mathcal{D}$ is an $\mathrm{STD}_1[12;12]$, Using the same argument as in the proof of Lemma 4.7 the lemma holds from Lemmas 5.6 and 5.7. □

LEMMA 5.9 (i) *Let $i, i' \in \{0, 1, \cdots, 11\}$ and $i < i'$.*
(a) *If $(i, i') \in \{(0,1), (1,2), (3,4), (4,5), (6,7), (7,8), (0,2), (3,5), (6,8)\}$,*

$$\sum_{0 \le j \le 14} m_{ij} m_{i'j} = 5.$$

(b) *If (i, i') does not satisfy the assumption of* (a)*,*

$$\sum_{0 \le j \le 14} m_{ij} m_{i'j} = 8.$$

(ii) *Let $i \in \{0, 1, \cdots, 11\}$.*
(a) *If $i \in \{0, 1, \cdots, 8\}$,*

$$\sum_{0 \le j \le 14} {m_{ij}}^2 = 17.$$

(b) *If $i \in \{9, 10, 11\}$,*

$$\sum_{0 \le j \le 14} {m_{ij}}^2 = 20.$$

(iii) *Let $i \in \{0, 1, \cdots, 11\}$.*
(a) *If $i \in \{0, 1, \cdots, 8\}$,*

$$\sum_{0 \le j \le 11} m_{ij} = 9 \quad \textit{and} \quad \sum_{12 \le j \le 14} m_{ij} = 2.$$

(b) *If $i \in \{9, 10, 11\}$,*

$$\sum_{0 \le j \le 11} m_{ij} = 9 \quad \textit{and} \quad \sum_{12 \le j \le 14} m_{ij} = 3.$$

(iv) *For $j \in \{0, 1, \cdots, 14\}$*

$$\sum_{0 \le i \le 11} m_{ij} = 9.$$

PROOF. We have the equations of (i) and (ii) by considering the action of the trivial character of G on equations of Lemma 5.8 (i), (ii). The two equations of (iii) hold from Lemma 5.6. Since $D_{ij} \cap D_{i'j} = \emptyset$ for $i, i'(\neq) \in \{0, 1, \cdots, 11\}$, $j \in \{0, 1, \cdots, 14\}$, $\sum_{0 \le i \le 11} m_{ij} \le 9$ and $11 \times 9 + 12 \times 3 = 9 \times 15$, the equation of (iv) holds. □

LEMMA 5.10 (i) *Let $i \in \{0, 1, \cdots, 8\}$. Then $(m_{i\ 0}, m_{i\ 1}, \cdots, m_{i\ 14})$ is equal to*

$$(\underbrace{0, \cdots, 0}_{6}, 1, 1, 1, 1, 2, 2, 2, 0, 1, 1) \ \textit{or}$$

$$(\underbrace{0,\cdots,0}_{5},\underbrace{1,\cdots,1}_{5},2,2,0,0,2)$$

up to ordering for from the 0*th column to the* 11*th column and for from the* 12*th column to the* 14*th column.*
(ii) *Let* $i \in \{9,10,11\}$. *Then* $(m_{i\ 0}, m_{i\ 1}, \cdots, m_{i\ 14})$ *is equal to*

$$(\underbrace{0,\cdots,0}_{6},\underbrace{1,\cdots,1}_{4},2,3,1,1,1),$$

$$(\underbrace{0,\cdots,0}_{5},\underbrace{1,\cdots,1}_{6},3,0,1,2),$$

$$(\underbrace{0,\cdots,0}_{4},\underbrace{1,\cdots,1}_{7},2,0,0,3),$$

$$(\underbrace{0,\cdots,0}_{7},1,\underbrace{2,\cdots,2}_{4},1,1,1) \ \textit{or}$$

$$(\underbrace{0,\cdots,0}_{6},1,1,1,2,2,2,0,1,2)$$

up to ordering for from the 0*th column to* 11*th column and for from the* 12*th column to the* 14*th column.*

PROOF. (i): Let $i \in \{0,1,\cdots,8\}$. Then, by Lemma 5.9 (ii), (iii) we have the two cases stated in the lemma and the another case

$$(\underbrace{0,\cdots,0}_{5},\underbrace{1,\cdots,1}_{6},3,0,1,1).$$

But the last case does not occur by Lemma 5.8(ii). Actually, let $\varphi^k \longleftrightarrow k$ for $k \in \{0,1,\cdots,8\}$ (mod 9), then there exists a subset of $\{0,1,\cdots,8\}$ such that

$$\{\pm(a-b),\ \pm(a-c),\ \pm(b-c)\} = \{1,2,4,5,7,8\} \pmod 9.$$

But it follows that there does not exist such $\{a,b,c\}$ by easy calculations.
(ii): If $i \in \{9,10,11\}$, it follows that the five cases stated in the lemma occur by Lemma 5.9 (ii), (iii). □

LEMMA 5.11 *For* $i \in \{0,3,6\}$ *and* $j \in \{0,1,\cdots,11\}$

$$\begin{pmatrix} m_{i\ j} \\ m_{i+1\ j} \\ m_{i+2\ j}\end{pmatrix} \notin \{\begin{pmatrix} 2 \\ 1 \\ 1\end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1\end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2\end{pmatrix}\}.$$

PROOF. Suppose that $\begin{pmatrix} m_{i\ j} \\ m_{i+1\ j} \\ m_{i+2\ j}\end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1\end{pmatrix}$ for some $i \in \{0,3,6\}$ and $j \in \{0,1,\cdots,11\}$. Let $\varphi^k \longleftrightarrow k$ for $k \in \{0,1,\cdots,8\}$ (mod 9) and $\begin{pmatrix} D_{i\ j} \\ D_{i+1\ j} \\ D_{i+2\ j}\end{pmatrix} =$

$\begin{pmatrix} \{a,b\} \\ \{c\} \\ \{d\} \end{pmatrix}$. In the rest of the proof, we consider $a, b, c, d \pmod 3$. By Lemma 5.8 (ii), $a - b \not\equiv 0 \pmod 3$. Adding an appropriate integer (mod 3) to $\mathcal{D}$, we may assume that $a = 0$ and $b = 1$. By Lemma 5.8 (i), $a - c \not\equiv 1,\ b - c \not\equiv 1 \pmod 3$. Therefore $c \equiv 1 \pmod 3$. Again by Lemma 5.8 (i), $d - a \not\equiv 1,\ d - b \not\equiv 1 \pmod 3$. Therefore $d \equiv 0 \pmod 3$. But by Lemma 5.8 (i), $c - d \not\equiv 1 \pmod 3$. This is a contradiction. Therefore $\begin{pmatrix} m_{i\ j} \\ m_{i+1\ j} \\ m_{i+2\ j} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ does not occur. It follows that the other two cases also do not occur by a similar argument to that stated above. □

Notation 5.12 Let $A = (a_{ij})_{0 \le i \le 11,\ 0 \le j \le 14}$ be a matrix on $\{0, 1, 2\}$ satisfying the following conditions (i), (ii), (iii) and (iv).
(i) Let $i \in \{0, 1, \cdots, 8\}$. Then, $(a_{i\ 0}, a_{i\ 1}, \cdots, a_{i\ 14})$ is equal to

$$(\underbrace{0, \cdots, 0}_{6}, 1, 1, 1, 2, 2, 2, 0, 1, 1) \text{ or}$$

$$(\underbrace{0, \cdots, 0}_{5}, \underbrace{1, \cdots, 1}_{5}, 2, 2, 0, 0, 2)$$

after permuting from the 0th column to the 11th column and from the 12th column to the 14th column appropriately. (We call that $(a_{i\ 0}, a_{i\ 1}, \cdots, a_{i\ 14})$ is of *type* R for the former case and of *type* S for the latter case.)
(ii) Let $i \in \{9, 10, 11\}$. Then, $(a_{i\ 0}, a_{i\ 1}, \cdots, a_{i\ 14})$ is equal to

$$(\underbrace{0, \cdots, 0}_{6}, \underbrace{1, \cdots, 1}_{4}, 2, 3, 1, 1, 1),$$

$$(\underbrace{0, \cdots, 0}_{5}, \underbrace{1, \cdots, 1}_{6}, 3, 0, 1, 2),$$

$$(\underbrace{0, \cdots, 0}_{4}, \underbrace{1, \cdots, 1}_{7}, 2, 0, 0, 3),$$

$$(\underbrace{0, \cdots, 0}_{7}, 1, \underbrace{2, \cdots, 2}_{4}, 1, 1, 1) \text{ or}$$

$$(\underbrace{0, \cdots, 0}_{6}, 1, 1, 1, 2, 2, 2, 0, 1, 2)$$

after permuting from the 0th column to the 11th column and from the 12th column to the 14th column appropriately.
(iii) Let $i, i' \in \{0, 1, \cdots, 11\}$ and $i < i'$.
If $(i, i') \in \{(0,1), (1,2), (3,4), (4,5), (6,7), (7,8), (0,2), (3,5), (6,8)\}$, then

$$\sum_{0 \le j \le 14} a_{ij} a_{i'j} = 5,$$

otherwise

$$\sum_{0\le j\le 14} a_{ij}a_{i'j} = 8.$$

(iv) For $i \in \{0, 3, 6\}$ and $j \in \{0, 1, \cdots, 11\}$

$$\begin{pmatrix} a_{i\ j} \\ a_{i+1\ j} \\ a_{i+2\ j} \end{pmatrix} \notin \{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\}.$$

Let Φ be the set of matrices A's satisfying the conditions (i), (ii), (iii) and (iv). We remark that $M = (m_{ij})_{0\le i\le 11, 0\le j\le 14} \in \Phi$.

Definition 5.13 Let $A = (a_{ij})_{0\le i\le 11, 0\le j\le 14}$ and $B = (b_{ij})_{0\le i\le 11, 0\le j\le 14} \in \Phi$. Then, if there exists a permutation σ on $\{0, 1, \cdots, 11\}$ such that $\{\{0^\sigma, 1^\sigma, 2^\sigma\},\ \{3^\sigma, 4^\sigma, 5^\sigma\},\ \{6^\sigma, 7^\sigma, 8^\sigma\}\} = \{\{0, 1, 2\},\ \{3, 4, 5\},\ \{6, 7, 8\}\}$, $\{9^\sigma, 10^\sigma, 11^\sigma\} = \{9, 10, 11\}$ and there exists a permutation τ on $\{0, 1, \cdots, 14\}$ such that $\{0^\tau, 1^\tau, \cdots, 11^\tau\} = \{0, 1, \cdots, 11\}$, $\{12^\tau, 13^\tau, 14^\tau\} = \{12, 13, 14\}$, we say that A is *equivalent* to B and we denote this by $A \sim B$. (Then, $\sim$is an equivalence relation on Φ.)

We now want to determine $\Phi/\sim$.

Definition 5.14 Let $A = (a_{ij})_{0\le i\le 11, 0\le j\le 14} \in \Phi$. Then, if for i with $0 \le i \le 8$, $(a_{i\ 0}, a_{i\ 1}, \cdots, a_{i\ 14})$ is of type X_i ($X_i = R$ or S), we say that A is of type ${}^t(X_0, X_1, X_2|X_3, X_4, X_5|X_6, X_7, X_8)$.

We may assume that a representative element of Φ by $\sim$ is of one of the following 20 types from Definition 5.13.

(1) ${}^t(R, R, R|R, R, R|R, R, R)$,
(2) ${}^t(R, R, R|R, R, R|R, R, S)$,
(3) ${}^t(R, R, R|R, R, R|R, S, S)$,
(4) ${}^t(R, R, R|R, R, R|S, S, S)$,
(5) ${}^t(R, R, R|R, R, S|R, R, S)$,
(6) ${}^t(R, R, R|R, R, S|R, S, S)$,
(7) ${}^t(R, R, R|R, R, S|S, S, S)$,
(8) ${}^t(R, R, R|R, S, S|R, S, S)$,
(9) ${}^t(R, R, R|R, S, S|S, S, S)$,
(10) ${}^t(R, R, R|S, S, S|S, S, S)$,
(11) ${}^t(R, R, S|R, R, S|R, R, S)$,
(12) ${}^t(R, R, S|R, R, S|R, S, S)$,
(13) ${}^t(R, R, S|R, R, S|S, S, S)$,
(14) ${}^t(R, R, S|R, S, S|R, S, S)$,
(15) ${}^t(R, R, S|R, S, S|S, S, S)$,
(16) ${}^t(R, R, S|S, S, S|S, S, S)$,

(17) ${}^t(R,S,S|R,S,S|R,S,S)$,
(18) ${}^t(R,S,S|R,S,S|S,S,S)$,
(19) ${}^t(R,S,S|S,S,S|S,S,S)$ and
(20) ${}^t(S,S,S|S,S,S|S,S,S)$.

Using a computer, we have the following lemma.

LEMMA 5.15 (i) *There are exactly the following* 3 *M's up to equivalence for type* (5).

$$M_1 = \left(\begin{array}{cccccccccccc|ccc}
0&0&0&0&0&0&1&1&1&2&2&2&0&1&1\\
0&0&0&2&2&2&1&1&1&0&0&0&0&1&1\\
2&2&2&0&0&1&0&0&1&0&0&1&0&1&1\\
0&0&2&0&0&1&1&2&2&0&1&0&1&0&1\\
0&2&0&0&2&1&0&1&0&0&1&2&1&1&0\\
1&1&1&2&1&0&0&0&0&2&1&0&0&0&2\\
0&2&0&1&0&1&0&1&2&2&0&0&1&1&0\\
2&0&0&1&0&2&0&1&0&0&2&1&1&0&1\\
1&0&2&0&2&0&1&1&0&1&1&0&0&2&0\\
\hline
0&0&2&2&0&1&1&0&0&1&0&2&2&1&0\\
1&2&0&1&0&0&3&1&0&0&1&0&1&1&1\\
2&0&0&0&2&0&1&0&2&1&0&1&2&0&1
\end{array}\right),$$

$$M_2 = \left(\begin{array}{cccccccccccc|ccc}
0&0&0&0&0&0&1&1&1&2&2&2&0&1&1\\
0&0&0&2&2&2&1&1&1&0&0&0&0&1&1\\
2&2&2&0&0&1&0&0&1&0&0&1&0&1&1\\
0&0&2&0&0&2&1&2&0&0&1&1&1&0&1\\
0&2&0&0&2&0&0&1&2&0&1&1&1&1&0\\
1&1&1&2&1&0&0&0&0&2&1&0&0&0&2\\
0&2&0&1&0&2&0&1&0&2&0&1&1&1&0\\
2&0&0&1&0&1&0&1&2&0&2&0&1&0&1\\
1&0&2&0&2&0&1&1&0&1&1&0&0&2&0\\
\hline
0&0&2&2&0&0&1&0&2&1&0&1&2&1&0\\
1&2&0&1&0&0&3&1&0&0&1&0&1&1&1\\
2&0&0&0&2&1&1&0&0&1&0&2&2&0&1
\end{array}\right) \quad \textit{and}$$

$$M_3 = \left(\begin{array}{cccccccccccc|ccc}
0&0&0&0&0&0&1&1&1&2&2&2&0&1&1\\
0&0&1&1&1&2&0&2&2&0&0&0&1&0&1\\
1&2&0&1&2&0&2&0&0&0&0&1&1&0&1\\
0&0&2&1&2&0&1&0&1&0&2&0&0&1&1\\
0&2&0&1&0&1&1&0&2&2&0&0&1&1&0\\
2&1&1&1&0&0&0&2&0&1&0&1&0&0&2\\
1&2&0&0&1&0&0&2&1&0&2&0&1&1&0\\
2&0&1&0&0&1&2&0&2&0&0&1&0&1&1\\
0&0&2&1&1&0&1&1&0&2&0&1&2&0&0\\
\hline
0&2&2&0&0&2&0&0&0&0&1&2&1&1&1\\
1&0&0&3&0&1&1&1&0&0&1&1&1&2&0\\
2&0&0&0&2&2&0&0&0&2&1&0&1&1&1
\end{array}\right).$$

(ii) *There is exactly the following one M up to equivalence for type* (6).

$$M_4 = \left(\begin{array}{cccccccccccc|ccc}
0&0&0&0&0&0&1&1&1&2&2&2&0&1&1\\
0&0&1&1&1&2&0&2&2&0&0&0&1&0&1\\
1&2&0&1&2&0&2&0&0&0&0&1&1&0&1\\
0&1&2&0&0&1&2&0&1&0&2&0&1&0&1\\
2&1&0&1&0&1&0&0&2&0&0&2&0&1&1\\
1&1&1&0&1&0&0&2&0&2&0&1&2&0&0\\
2&0&0&0&2&1&0&0&1&1&2&0&1&0&1\\
0&2&1&0&1&0&1&1&2&1&0&0&0&2&0\\
1&0&2&2&0&0&1&1&0&1&0&1&0&0&2\\
\hline
0&0&2&1&2&1&0&0&0&0&1&2&1&2&0\\
0&2&0&2&0&2&0&0&0&2&1&0&1&1&1\\
2&0&0&1&0&1&2&2&0&0&1&0&1&2&0
\end{array}\right).$$

(iii) $|\Phi/\sim| = 4$.

For each M_i of Lemma 5.15, we have to consider 4 $(D_{i,j})_{0\le 11, 0\le j\le 14}$'s. For example, for M_1 we have to consider 4 $(D_{i,j})_{0\le 11, 0\le j\le 14}$'s of Appendix A. Here we consider $D_1, \cdots, D_4$ on $\mathbb{Z}/9\mathbb{Z}$. But it follows that from the 0th row to the 5th row of any one of $D_1, \cdots, D_4$ do not satisfy Lemma 5.8 using a computer. By the similar argument, it follows that D's corresponding to any other M_i $(i = 2, 3, 4)$ do not exist. Therefore we have the following lemma.

Lemma 5.16 *There is no projective plane of order* 12 *satisfying Hypothesis* 5.1.

6 The case that G is planar

Let $\pi = (\mathcal{Q}, \mathcal{L})$ be a projective plane of order 12. In this section we assume the following.

Hypothesis 6.1 Let G be an automorphism group of π of order 9 and $(F_{\mathcal{Q}}(G), F_{\mathcal{L}}(G))$ a subplane of π.

Lemma 6.2 $(F_{\mathcal{Q}}(G), F_{\mathcal{L}}(G))$ *is a subplane of* π *of order* 3.

Proof. It follows from Lemma 3.2. □

Lemma 6.3 *G acts semiregularly on $\mathcal{Q} - F_{\mathcal{Q}}(G)$ and on $\mathcal{L} - F_{\mathcal{L}}(G)$.*

Proof. Suppose that there exists $\tau \in G - \{1\}$ such that $r^\tau = r$ for some $r \in \mathcal{Q} - F_{\mathcal{Q}}(G)$. Then $(F_{\mathcal{Q}}(\langle\tau\rangle), F_{\mathcal{L}}(\langle\tau\rangle))$ is a proper subplane of π. If let m be the order of the subplane, $m \ge 4$. This is contrary to Lemma 3.2. Therefore, G acts semiregularly on $\mathcal{Q} - F_{\mathcal{Q}}(G)$. By the similar argument, it follows that G acts semiregularly on $\mathcal{L} - F_{\mathcal{L}}(G)$. □

Notation being as in Notation 3.4, we may assume that
$F_{\mathcal{P}}(G) = \{p_{117}, p_{118}, p_{119}, p_{129}, p_{130}, p_{131}, p_{141}, p_{142}, p_{143}\}$ and
$F_{\mathcal{B}}(G) = \{B_{117}, B_{118}, B_{119}, B_{129}, B_{130}, B_{131}, B_{141}, B_{142}, B_{143}\}$.

Lemma 6.4 *If $\mu (\ne 1) \in G$, then ${\mathcal{P}_i}^\mu \ne \mathcal{P}_i$, and ${\mathcal{B}_i}^\mu \ne \mathcal{B}_i$ for i with $0 \le i \le 8$.*

Proof. If for example ${\mathcal{P}_0}^\mu = \mathcal{P}_0$, then μ fixes the line L_0 of π. Since $(F_{\mathcal{Q}}(\mu), F_{\mathcal{L}}(\mu))$ is a subplane of π containing the subplane $(F_{\mathcal{Q}}(G), F_{\mathcal{L}}(G))$ properly, $(F_{\mathcal{Q}}(\mu), F_{\mathcal{L}}(\mu)) = \pi$ by Lemma 3.2. Therefore, $\mu = 1$. This is a contradiction. Thus ${\mathcal{P}_0}^\mu \ne \mathcal{P}_0$. By the similar argument the remaining assertion can be proved. □

Let $\mathcal{Q}_0, \mathcal{Q}_1, \cdots, \mathcal{Q}_{14}$ be G-orbits on $\mathcal{P}$ of size 9 and $\mathcal{C}_0, \mathcal{C}_1, \cdots, \mathcal{C}_{14}$ G-orbits on $\mathcal{B}$ of size 9. For i, j with $0 \le i, j \le 14$, choose a point $q_i \in \mathcal{Q}_i$ and a block $C_j \in \mathcal{C}_j$. For i, j with $0 \le i, j \le 14$ set $m_{ij} = |\mathcal{Q}_i \cap (C_j)|$ and $D_{ij} = \{\alpha \in G | {q_i}^\alpha \in (C_j)\}$. Then $m_{ij} = |D_{ij}|$ $(0 \le i, j \le 14)$. Set $M = (m_{ij})_{0\le i,j\le 14}$.

Case A. G is a cyclic group.
Let $G = \langle\varphi\rangle$. By Lemmas 6.3 and 6.4 we may assume that
$\widetilde{\varphi} = (\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_8)(\mathcal{P}_9)(\mathcal{P}_{10})(\mathcal{P}_{11})$ and $\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_8)(\mathcal{B}_9)(\mathcal{B}_{10})(\mathcal{B}_{11})$.

Case B. G is an elementary abelian group.
Let $G = \langle \varphi, \tau | \varphi^3 = \tau^3 = 1,\ \varphi\tau = \tau\varphi \rangle$. By Lemmas 6.3 and 6.4 we may assume that
$\widetilde{\varphi} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)(\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5)(\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8)(\mathcal{P}_9)(\mathcal{P}_{10})(\mathcal{P}_{11})$,
$\widetilde{\widetilde{\varphi}} = (\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)(\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)(\mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8)(\mathcal{B}_9)(\mathcal{B}_{10})(\mathcal{B}_{11})$,
$\widetilde{\tau} = (\mathcal{P}_0, \mathcal{P}_3, \mathcal{P}_6)(\mathcal{P}_1, \mathcal{P}_4, \mathcal{P}_7)(\mathcal{P}_2, \mathcal{P}_5, \mathcal{P}_8)(\mathcal{P}_9)(\mathcal{P}_{10})(\mathcal{P}_{11})$ and
$\widetilde{\widetilde{\tau}} = (\mathcal{B}_0, \mathcal{B}_3, \mathcal{B}_6)(\mathcal{B}_1, \mathcal{B}_4, \mathcal{B}_7)(\mathcal{B}_2, \mathcal{B}_5, \mathcal{B}_8)(\mathcal{B}_9)(\mathcal{B}_{10})(\mathcal{B}_{11})$.

Assume that **Case A** occurs.
We may assume that
$\varphi = (x_0, x_{12}, x_{24}, x_{36}, x_{48}, x_{60}, x_{72}, x_{84}, x_{96})$
$(x_1, x_{13}, x_{25}, x_{37}, x_{49}, x_{61}, x_{73}, x_{85}, x_{97})$
$(x_2, x_{14}, x_{26}, x_{38}, x_{50}, x_{62}, x_{74}, x_{86}, x_{98})$
$(x_3, x_{15}, x_{27}, x_{39}, x_{51}, x_{63}, x_{75}, x_{87}, x_{99})$
$(x_4, x_{16}, x_{28}, x_{40}, x_{52}, x_{64}, x_{76}, x_{88}, x_{100})$
$(x_5, x_{17}, x_{29}, x_{41}, x_{53}, x_{65}, x_{77}, x_{89}, x_{101})$
$(x_6, x_{18}, x_{30}, x_{42}, x_{54}, x_{66}, x_{78}, x_{90}, x_{102})$
$(x_7, x_{19}, x_{31}, x_{43}, x_{55}, x_{67}, x_{79}, x_{91}, x_{103})$
$(x_8, x_{20}, x_{32}, x_{44}, x_{56}, x_{68}, x_{80}, x_{92}, x_{104})$
$(x_9, x_{21}, x_{33}, x_{45}, x_{57}, x_{69}, x_{81}, x_{93}, x_{105})$
$(x_{10}, x_{22}, x_{34}, x_{46}, x_{58}, x_{70}, x_{82}, x_{94}, x_{106})$
$(x_{11}, x_{23}, x_{35}, x_{47}, x_{59}, x_{71}, x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{109}, x_{110}, x_{111}, x_{112}, x_{113}, x_{114}, x_{115}, x_{116})$
$(x_{117})(x_{118})(x_{119})$
$(x_{120}, x_{121}, x_{122}, x_{123}, x_{124}, x_{125}, x_{126}, x_{127}, x_{128})$
$(x_{129})(x_{130})(x_{131})$
$(x_{132}, x_{133}, x_{134}, x_{135}, x_{136}, x_{137}, x_{138}, x_{139}, x_{140})$
$(x_{141})(x_{142})(x_{143})$, where $x \in \{p, B\}$.

Set G-orbits of size 9 on $\mathcal{P}$ and on $\mathcal{B}$ as follows:
$\mathcal{Y}_0 = \{x_0, x_{12}, x_{24}, x_{36}, x_{48}, x_{60}, x_{72}, x_{84}, x_{96}\}$,
$\mathcal{Y}_1 = \{x_1, x_{13}, x_{25}, x_{37}, x_{49}, x_{61}, x_{73}, x_{85}, x_{97}\}$,
$\mathcal{Y}_2 = \{x_2, x_{14}, x_{26}, x_{38}, x_{50}, x_{62}, x_{74}, x_{86}, x_{98}\}$,
$\mathcal{Y}_3 = \{x_3, x_{15}, x_{27}, x_{39}, x_{51}, x_{63}, x_{75}, x_{87}, x_{99}\}$,
$\mathcal{Y}_4 = \{x_4, x_{16}, x_{28}, x_{40}, x_{52}, x_{64}, x_{76}, x_{88}, x_{100}\}$,
$\mathcal{Y}_5 = \{x_5, x_{17}, x_{29}, x_{41}, x_{53}, x_{65}, x_{77}, x_{89}, x_{101}\}$,
$\mathcal{Y}_6 = \{x_6, x_{18}, x_{30}, x_{42}, x_{54}, x_{66}, x_{78}, x_{90}, x_{102}\}$,
$\mathcal{Y}_7 = \{x_7, x_{19}, x_{31}, x_{43}, x_{55}, x_{67}, x_{79}, x_{91}, x_{103}\}$,
$\mathcal{Y}_8 = \{x_8, x_{20}, x_{32}, x_{44}, x_{56}, x_{68}, x_{80}, x_{92}, x_{104}\}$,
$\mathcal{Y}_9 = \{x_9, x_{21}, x_{33}, x_{45}, x_{57}, x_{69}, x_{81}, x_{93}, x_{105}\}$,
$\mathcal{Y}_{10} = \{x_{10}, x_{22}, x_{34}, x_{46}, x_{58}, x_{70}, x_{82}, x_{94}, x_{106}\}$,
$\mathcal{Y}_{11} = \{x_{11}, x_{23}, x_{35}, x_{47}, x_{59}, x_{71}, x_{83}, x_{95}, x_{107}\}$,
$\mathcal{Y}_{12} = \{x_{108}, x_{109}, x_{110}, x_{111}, x_{112}, x_{113}, x_{114}, x_{115}, x_{116}\}$,
$\mathcal{Y}_{13} = \{x_{120}, x_{121}, x_{122}, x_{123}, x_{124}, x_{125}, x_{126}, x_{127}, x_{128}\}$ and
$\mathcal{Y}_{14} = \{x_{132}, x_{133}, x_{134}, x_{135}, x_{136}, x_{137}, x_{138}, x_{139}, x_{140}\}$,
where $(\mathcal{Y}, x) \in \{(\mathcal{Q}, p),\ (\mathcal{C}, B)\}$.

Set $q_0 = p_0,\ q_1 = p_1, \cdots, q_{11} = p_{11}, q_{12} = p_{108}, q_{13} = p_{120}, q_{14} = p_{132}$ and $C_0 = B_0,\ C_1 = B_1, \cdots, C_{11} = B_{11}, C_{12} = B_{108}, C_{13} = B_{120}, C_{14} = B_{132}$.

Since $(F_{\mathcal{Q}}(G), F_{\mathcal{L}}(G))$ is a subplane of π,
$F_{\mathcal{P}}(G) = \{p_{117}, p_{118}, p_{119}, p_{129}, p_{130}, p_{131}, p_{141}, p_{142}, p_{143}\}$ and
$F_{\mathcal{B}}(G) = \{B_{117}, B_{118}, B_{119}, B_{129}, B_{130}, B_{131}, B_{141}, B_{142}, B_{143}\}$,
we may assume that $N_{0\ 9},\ N_{0\ 10},\ N_{0\ 11}$ have the same forms as $A_9,\ A_{10},\ A_{11}$ stated in Lemma 4.12, respectively.

Lemma 6.5 (i) *Let $i, i'(\neq) \in \{0, 1, \cdots, 14\}$.*
(a) *If $i, i'(\neq) \in \{0, 1, \cdots, 11\}$,*

$$\sum_{0 \le j \le 14} D_{ij} {D_{i'j}}^{(-1)} = G - \{1\}.$$

(b) *If $i \in \{0, 1, \cdots, 11\}$ and $i' \in \{12, 13, 14\}$,*

$$\sum_{0 \le j \le 14} D_{ij} {D_{i'j}}^{(-1)} = G.$$

(c) *If $i, i'(\neq) \in \{12, 13, 14\}$,*

$$\sum_{0 \le j \le 14} D_{ij} {D_{i'j}}^{(-1)} = G.$$

(ii) *Let $i \in \{0, 1, \cdots, 14\}$.*
(a) *If $i \in \{0, 1, \cdots, 8\}$,*

$$\sum_{0 \le j \le 14} D_{ij} {D_{ij}}^{(-1)} = 11.$$

(b) *If $i \in \{9, 10, 11\}$,*

$$\sum_{0 \le j \le 14} D_{ij} {D_{ij}}^{(-1)} = 12 + (G - \{1\}).$$

(c) *If $i \in \{12, 13, 14\}$,*

$$\sum_{0 \le j \le 14} D_{ij} {D_{ij}}^{(-1)} = 12.$$

Proof. Since $\mathcal{D}$ is an $\mathrm{STD}_1[12; 12]$, using the same argument as in the proof of Lemma 4.7 the lemma holds by considering the form of N. □

Lemma 6.6 (i) *Let $i, i'(\neq) \in \{0, 1, \cdots, 14\}$.*
(a) *If $i, i'(\neq) \in \{0, 1, \cdots, 11\}$,*

$$\sum_{0 \le j \le 14} m_{ij} m_{i'j} = 8.$$

(b) *If* $i \in \{0, 1, \cdots, 11\}$ *and* $i' \in \{12, 13, 14\}$,

$$\sum_{0 \leq j \leq 14} m_{ij} m_{i'j} = 9.$$

(c) *If* $i, i'(\neq) \in \{12, 13, 14\}$,

$$\sum_{0 \leq j \leq 14} m_{ij} m_{i'j} = 9.$$

(ii) *Let* $i \in \{0, 1, \cdots, 14\}$.
(a) *If* $i \in \{0, 1, \cdots, 8\}$,

$$\sum_{0 \leq j \leq 14} {m_{ij}}^2 = 11.$$

(b) *If* $i \in \{9, 10, 11\}$,

$$\sum_{0 \leq j \leq 14} {m_{ij}}^2 = 20.$$

(c) *If* $i \in \{12, 13, 14\}$,

$$\sum_{0 \leq j \leq 14} {m_{ij}}^2 = 12.$$

(iii) *Let* $i \in \{0, 1, \cdots, 14\}$.
(a) *If* $i \in \{0, 1, \cdots, 8\}$,

$$\sum_{0 \leq j \leq 14} m_{ij} = 11.$$

(b) *If* $i \in \{9, 10, \cdots, 14\}$,

$$\sum_{0 \leq j \leq 14} m_{ij} = 12.$$

PROOF. We have the equations of (i) and (ii) by considering the action of the trivial character of G on equations of Lemma 6.5 (i) and (vi), respectively. The equations of (ii) hold from the form of N. □

Using a computer the following lemma holds from Lemma 6.6.

LEMMA 6.7 *Set* $P = (m_{ij})_{0 \leq i \leq 8, 0 \leq j \leq 14}$. *Then there are exactly* 13 *P's of Appendix B up to ordering of rows and columns.*

LEMMA 6.8 *There is no projective plane of order* 12 *admitting a planar cyclic automorphism group of order* 9.

PROOF. Any one of matrices $P_1, \cdots, P_{13}$ stated in Lemma 6.7 can not be extended to $(m_{ij})_{0 \leq i \leq 11, 0 \leq j \leq 14}$ satisfying Lemma 6.6. Therefore we have the lemma. □

Assume that **Case B** occurs.
We may assume that
$\varphi = (x_0, x_{12}, x_{24})(x_1, x_{13}, x_{25})(x_2, x_{14}, x_{26})$

$(x_3, x_{15}, x_{27})(x_4, x_{16}, x_{28})(x_5, x_{17}, x_{29})$
$(x_6, x_{18}, x_{30})(x_7, x_{19}, x_{31})(x_8, x_{20}, x_{32})$
$(x_9, x_{21}, x_{33})(x_{10}, x_{22}, x_{34})(x_{11}, x_{23}, x_{35})$
$(x_{36}, x_{48}, x_{60})(x_{37}, x_{49}, x_{61})(x_{38}, x_{50}, x_{62})$
$(x_{39}, x_{51}, x_{63})(x_{40}, x_{52}, x_{64})(x_{41}, x_{53}, x_{65})$
$(x_{42}, x_{54}, x_{66})(x_{43}, x_{55}, x_{67})(x_{44}, x_{56}, x_{68})$
$(x_{45}, x_{57}, x_{69})(x_{46}, x_{58}, x_{70})(x_{47}, x_{59}, x_{71})$
$(x_{72}, x_{84}, x_{96})(x_{73}, x_{85}, x_{97})(x_{74}, x_{86}, x_{98})$
$(x_{75}, x_{87}, x_{99})(x_{76}, x_{88}, x_{100})(x_{77}, x_{89}, x_{101})$
$(x_{78}, x_{90}, x_{102})(x_{79}, x_{91}, x_{103})(x_{80}, x_{92}, x_{104})$
$(x_{81}, x_{93}, x_{105})(x_{82}, x_{94}, x_{106})(x_{83}, x_{95}, x_{107})$
$(x_{108}, x_{109}, x_{110})(x_{111}, x_{112}, x_{113})(x_{114}, x_{115}, x_{116})$
$(x_{117})(x_{118})(x_{119})(x_{120}, x_{121}, x_{122})(x_{123}, x_{124}, x_{125})(x_{126}, x_{127}, x_{128})$
$(x_{129})(x_{130})(x_{131})$
$(x_{132}, x_{133}, x_{134})(x_{135}, x_{136}, x_{137})(x_{138}, x_{139}, x_{140})$
$(x_{141})(x_{142})(x_{143})$ and
$\tau = (x_0, x_{36}, x_{72})(x_1, x_{37}, x_{73})(x_2, x_{38}, x_{74})$
$(x_3, x_{39}, x_{75})(x_4, x_{40}, x_{76})(x_5, x_{41}, x_{77})$
$(x_6, x_{42}, x_{78})(x_7, x_{43}, x_{79})(x_8, x_{44}, x_{80})$
$(x_9, x_{45}, x_{81})(x_{10}, x_{46}, x_{82})(x_{11}, x_{47}, x_{83})$
$(x_{12}, x_{48}, x_{84})(x_{13}, x_{49}, x_{85})(x_{14}, x_{50}, x_{86})$
$(x_{15}, x_{51}, x_{87})(x_{16}, x_{52}, x_{88})(x_{17}, x_{53}, x_{89})$
$(x_{18}, x_{54}, x_{90})(x_{19}, x_{55}, x_{91})(x_{20}, x_{56}, x_{92})$
$(x_{21}, x_{57}, x_{93})(x_{22}, x_{58}, x_{94})(x_{23}, x_{59}, x_{95})$
$(x_{24}, x_{60}, x_{96})(x_{25}, x_{61}, x_{97})(x_{26}, x_{62}, x_{98})$
$(x_{27}, x_{63}, x_{99})(x_{28}, x_{64}, x_{100})(x_{29}, x_{65}, x_{101})$
$(x_{30}, x_{66}, x_{102})(x_{31}, x_{67}, x_{103})(x_{32}, x_{68}, x_{104})$
$(x_{33}, x_{69}, x_{105})(x_{34}, x_{70}, x_{106})(x_{35}, x_{71}, x_{107})$
$(x_{108}, x_{111}, x_{114})(x_{109}, x_{112}, x_{115})(x_{110}, x_{113}, x_{116})$
$(x_{117})(x_{118})(x_{119})$
$(x_{120}, x_{123}, x_{126})(x_{121}, x_{124}, x_{127})(x_{122}, x_{125}, x_{128})$
$(x_{129})(x_{130})(x_{131})$
$(x_{132}, x_{135}, x_{138})(x_{133}, x_{136}, x_{139})(x_{134}, x_{137}, x_{140})$
$(x_{141})(x_{142})(x_{143})$, where $x \in \{p, B\}$.

Set G-orbits of size 9 on $\mathcal{P}$ and on $\mathcal{B}$ as follows:
$\mathcal{Y}_0 = \{X_0, X_{12}, X_{24}, X_{36}, X_{48}, X_{60}, X_{72}, X_{84}, X_{96}\}$,
$\mathcal{Y}_1 = \{X_1, X_{13}, X_{25}, X_{37}, X_{49}, X_{61}, X_{73}, X_{85}, X_{97}\}$,
$\mathcal{Y}_2 = \{X_2, X_{14}, X_{26}, X_{38}, X_{50}, X_{62}, X_{74}, X_{86}, X_{98}\}$,
$\mathcal{Y}_3 = \{X_3, X_{15}, X_{27}, X_{39}, X_{51}, X_{63}, X_{75}, X_{87}, X_{99}\}$,
$\mathcal{Y}_4 = \{X_4, X_{16}, X_{28}, X_{40}, X_{52}, X_{64}, X_{76}, X_{88}, X_{100}\}$,
$\mathcal{Y}_5 = \{X_5, X_{17}, X_{29}, X_{41}, X_{53}, X_{65}, X_{77}, X_{89}, X_{101}\}$,
$\mathcal{Y}_6 = \{X_6, X_{18}, X_{30}, X_{42}, X_{54}, X_{66}, X_{78}, X_{90}, X_{102}\}$,
$\mathcal{Y}_7 = \{X_7, X_{19}, X_{31}, X_{43}, X_{55}, X_{67}, X_{79}, X_{91}, X_{103}\}$,
$\mathcal{Y}_8 = \{X_8, X_{20}, X_{32}, X_{44}, X_{56}, X_{68}, X_{80}, X_{92}, X_{104}\}$,

$\mathcal{Y}_9 = \{X_9, X_{21}, X_{33}, X_{45}, X_{57}, X_{69}, X_{81}, X_{93}, X_{105}\}$,
$\mathcal{Y}_{10} = \{X_{10}, X_{22}, X_{34}, X_{46}, X_{58}, X_{70}, X_{82}, X_{94}, X_{106}\}$,
$\mathcal{Y}_{11} = \{X_{11}, X_{23}, X_{35}, X_{47}, X_{59}, X_{71}, X_{83}, X_{95}, X_{107}\}$,
$\mathcal{Y}_{12} = \{X_{108}, X_{109}, X_{110}, X_{111}, X_{112}, X_{113}, X_{114}, X_{115}, X_{116}\}$,
$\mathcal{Y}_{13} = \{X_{120}, X_{121}, X_{122}, X_{123}, X_{124}, X_{125}, X_{126}, X_{127}, X_{128}\}$ and
$\mathcal{Y}_{14} = \{X_{132}, X_{133}, X_{134}, X_{135}, X_{136}, X_{137}, X_{138}, X_{139}, X_{140}\}$,
where $(\mathcal{Y}, X) \in \{(\mathcal{Q}, p), (\mathcal{C}, B)\}$.
Set $q_0 = p_0, q_1 = p_1, q_2 = p_2, q_3 = p_3, q_4 = p_4, q_5 = p_5, q_6 = p_6, q_7 = p_7, q_8 = p_8, q_9 = p_9, q_{10} = p_{10}, q_{11} = p_{11}, q_{12} = p_{108}, q_{13} = p_{120}, q_{14} = p_{132}$ and $C_0 = B_0, C_1 = B_1, C_2 = B_2, C_3 = B_3, C_4 = B_4, C_5 = B_5, C_6 = B_6, C_7 = B_7, C_8 = B_8, C_9 = B_9, C_{10} = B_{10}, C_{11} = B_{11}, C_{12} = B_{108}, C_{13} = B_{120}, C_{14} = B_{132}$.

In this case, we have Lemma 6.5 by the similar argument as in the Case A. Therefore, by Lammas 6.6 and 6.7 we have the following lemma.

Lemma 6.9 *There is no projective plane of order* 12 *admitting a planar elementary abelian automorphism group of order* 9.

Proof of Theorem A; By Lemmas 4.16, 5.16, 6.8 and 6.9 the theorem holds.

Appendix A

$D_1 =$

$$
\left(\begin{array}{cccccccc|}
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{a_0\} & \{a_1\} \\
\emptyset & \emptyset & \emptyset & \{b_0,b_1\} & \{b_2,b_3\} & \{b_4,b_5\} & \{b_6\} & \{b_7\} \\
\{c_0,c_1\} & \{c_2,c_3\} & \{c_4,c_5\} & \emptyset & \emptyset & \{c_6\} & \emptyset & \emptyset \\
\hline
\emptyset & \emptyset & \{d_0,d_1\} & \emptyset & \emptyset & \{d_2\} & \{d_3\} & \{d_4,d_5\} \\
\emptyset & \{e_0,e_1\} & \emptyset & \emptyset & \{e_2,e_3\} & \{e_4\} & \emptyset & \{e_5\} \\
\{f_0\} & \{f_1\} & \{f_2\} & \{f_3,f_4\} & \{f_5\} & \emptyset & \emptyset & \emptyset \\
\hline
\emptyset & \{g_0,g_1\} & \emptyset & \{g_2\} & \emptyset & \{g_3\} & \emptyset & \{g_4\} \\
\{h_0,h_1\} & \emptyset & \emptyset & \{h_2\} & \emptyset & \{h_3,h_4\} & \emptyset & \{h_5\} \\
\{k_0\} & \emptyset & \{k_1,k_2\} & \emptyset & \{k_3,k_4\} & \emptyset & \{k_5\} & \{k_6\} \\
\hline
\emptyset & \emptyset & \{l_0,l_1\} & \{l_2,l_3\} & \emptyset & \{l_4\} & \{l_5\} & \emptyset \\
\{m_0\} & \{m_1,m_2\} & \emptyset & \{m_3\} & \emptyset & \emptyset & \{m_4,m_5,m_6\} & \{m_7\} \\
\{n_0,n_1\} & \emptyset & \emptyset & \emptyset & \{n_2,n_3\} & \emptyset & \{n_4\} & \emptyset
\end{array}\right.
$$

$$
\left.\begin{array}{|ccccccc}
\{a_2\} & \{a_3,a_4\} & \{a_5,a_6\} & \{a_7,a_8\} & \emptyset & \{a_9\} & \{a_{10}\} \\
\{b_8\} & \emptyset & \emptyset & \emptyset & \emptyset & \{b_9\} & \{b_{10}\} \\
\{c_7\} & \emptyset & \emptyset & \{c_8\} & \emptyset & \{c_9\} & \{c_{10}\} \\
\hline
\{d_6,d_7\} & \emptyset & \{d_8\} & \emptyset & \{d_9\} & \emptyset & \{d_{10}\} \\
\emptyset & \emptyset & \{e_6\} & \{e_7,e_8\} & \{e_9\} & \{e_{10}\} & \emptyset \\
\emptyset & \{f_6,f_7\} & \{f_8\} & \emptyset & \emptyset & \emptyset & \{f_9,f_{10}\} \\
\hline
\{g_5,g_6\} & \{g_7,g_8\} & \emptyset & \emptyset & \{g_9\} & \{g_{10}\} & \emptyset \\
\emptyset & \emptyset & \{h_6,h_7\} & \{h_8\} & \{h_9\} & \emptyset & \{h_{10}\} \\
\emptyset & \{k_7\} & \{k_8\} & \emptyset & \emptyset & \{k_9,k_{10}\} & \emptyset \\
\hline
\emptyset & \{l_6\} & \emptyset & \{l_7,l_8\} & \{l_9,l_{10}\} & \{l_{11}\} & \emptyset \\
\emptyset & \emptyset & \{m_8\} & \emptyset & \{m_9\} & \{m_{10}\} & \{m_{11}\} \\
\{n_5,n_6\} & \{n_7\} & \emptyset & \{n_8\} & \{n_9,n_{10}\} & \emptyset & \{n_{11}\}
\end{array}\right)
$$

$$
D_2 = \left(\begin{array}{cccccccc|ccccccc}
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{a_0\} & \{a_1\} & \{a_2\} & \{a_3,a_4\} & \{a_5,a_6\} & \{a_7,a_8\} & \emptyset & \{a_9\} & \{a_{10}\} \\
\emptyset & \emptyset & \emptyset & \{b_0,b_1\} & \{b_2,b_3\} & \{b_4,b_5\} & \{b_6\} & \{b_7\} & \{b_8\} & \emptyset & \emptyset & \emptyset & \emptyset & \{b_9\} & \{b_{10}\} \\
\{c_0,c_1\} & \{c_2,c_3\} & \{c_4,c_5\} & \emptyset & \emptyset & \{c_6\} & \emptyset & \emptyset & \{c_7\} & \emptyset & \emptyset & \{c_8\} & \emptyset & \{c_9\} & \{c_{10}\} \\
\hline
\emptyset & \emptyset & \{d_0,d_1\} & \emptyset & \emptyset & \{d_2\} & \{d_3\} & \{d_4,d_5\} & \{d_6,d_7\} & \emptyset & \{d_8\} & \emptyset & \{d_9\} & \emptyset & \{d_{10}\} \\
\emptyset & \{e_0,e_1\} & \emptyset & \emptyset & \{e_2,e_3\} & \{e_4\} & \emptyset & \{e_5\} & \emptyset & \emptyset & \{e_6\} & \{e_7,e_8\} & \{e_9\} & \{e_{10}\} & \emptyset \\
\{f_0\} & \{f_1\} & \{f_2\} & \{f_3,f_4\} & \{f_5\} & \emptyset & \emptyset & \emptyset & \emptyset & \{f_6,f_7\} & \{f_8\} & \emptyset & \emptyset & \emptyset & \{f_9,f_{10}\} \\
\hline
\emptyset & \{g_0,g_1\} & \emptyset & \{g_2\} & \emptyset & \{g_3\} & \emptyset & \{g_4\} & \{g_5,g_6\} & \{g_7,g_8\} & \emptyset & \emptyset & \{g_9\} & \{g_{10}\} & \emptyset \\
\{h_0\} & \emptyset & \{h_1,h_2\} & \emptyset & \{h_3,h_4\} & \emptyset & \{h_5\} & \{h_6\} & \emptyset & \{h_7\} & \{h_8\} & \emptyset & \emptyset & \{h_9,h_{10}\} & \emptyset \\
\{k_0,k_1\} & \emptyset & \emptyset & \{k_2\} & \emptyset & \{k_3,k_4\} & \emptyset & \{k_5\} & \emptyset & \emptyset & \{k_6,k_7\} & \{k_8\} & \{k_9\} & \emptyset & \{k_{10}\} \\
\hline
\emptyset & \emptyset & \{l_0,l_1\} & \{l_2,l_3\} & \emptyset & \{l_4\} & \{l_5\} & \emptyset & \emptyset & \{l_6\} & \emptyset & \{l_7,l_8\} & \{l_9,l_{10}\} & \{l_{11}\} & \emptyset \\
\{m_0\} & \{m_1,m_2\} & \emptyset & \{m_3\} & \emptyset & \emptyset & \{m_4,m_5,m_6\} & \{m_7\} & \emptyset & \emptyset & \{m_8\} & \emptyset & \{m_9\} & \{m_{10}\} & \{m_{11}\} \\
\{n_0,n_1\} & \emptyset & \emptyset & \emptyset & \{n_2,n_3\} & \emptyset & \{n_4\} & \emptyset & \{n_5,n_6\} & \{n_7\} & \emptyset & \{n_8\} & \{n_9,n_{10}\} & \emptyset & \{n_{11}\}
\end{array}\right)
$$

$$
D_3 = \left(\begin{array}{cccccccc|ccccccc}
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{a_0\} & \{a_1\} & \{a_2\} & \{a_3,a_4\} & \{a_5,a_6\} & \{a_7,a_8\} & \emptyset & \{a_9\} & \{a_{10}\} \\
\emptyset & \emptyset & \emptyset & \{b_0,b_1\} & \{b_2,b_3\} & \{b_4,b_5\} & \{b_6\} & \{b_7\} & \{b_8\} & \emptyset & \emptyset & \emptyset & \emptyset & \{b_9\} & \{b_{10}\} \\
\{c_0,c_1\} & \{c_2,c_3\} & \{c_4,c_5\} & \emptyset & \emptyset & \{c_6\} & \emptyset & \emptyset & \{c_7\} & \emptyset & \emptyset & \{c_8\} & \emptyset & \{c_9\} & \{c_{10}\} \\
\hline
\emptyset & \emptyset & \{d_0,d_1\} & \emptyset & \emptyset & \{d_2\} & \{d_3\} & \{d_4,d_5\} & \{d_6,d_7\} & \emptyset & \{d_8\} & \emptyset & \{d_9\} & \emptyset & \{d_{10}\} \\
\{e_0\} & \{e_1\} & \{e_2\} & \{e_3,e_4\} & \{e_5\} & \emptyset & \emptyset & \emptyset & \emptyset & \{e_6,e_7\} & \{e_8\} & \emptyset & \emptyset & \emptyset & \{e_9,e_{10}\} \\
\emptyset & \{f_0,f_1\} & \emptyset & \emptyset & \{f_2,f_3\} & \{f_4\} & \emptyset & \{f_5\} & \emptyset & \emptyset & \{f_6\} & \{f_7,f_8\} & \{f_9\} & \{f_{10}\} & \emptyset \\
\hline
\emptyset & \{g_0,g_1\} & \emptyset & \{g_2\} & \emptyset & \{g_3\} & \emptyset & \{g_4\} & \{g_5,g_6\} & \{g_7,g_8\} & \emptyset & \emptyset & \{g_9\} & \{g_{10}\} & \emptyset \\
\{h_0,h_1\} & \emptyset & \emptyset & \{h_2\} & \emptyset & \{h_3,h_4\} & \emptyset & \{h_5\} & \emptyset & \emptyset & \{h_6,h_7\} & \{h_8\} & \{h_9\} & \emptyset & \{h_{10}\} \\
\{k_0\} & \emptyset & \{k_1,k_2\} & \emptyset & \{k_3,k_4\} & \emptyset & \{k_5\} & \{k_6\} & \emptyset & \{k_7\} & \{k_8\} & \emptyset & \emptyset & \{k_9,k_{10}\} & \emptyset \\
\hline
\emptyset & \emptyset & \{l_0,l_1\} & \{l_2,l_3\} & \emptyset & \{l_4\} & \{l_5\} & \emptyset & \emptyset & \{l_6\} & \emptyset & \{l_7,l_8\} & \{l_9,l_{10}\} & \{l_{11}\} & \emptyset \\
\{m_0\} & \{m_1,m_2\} & \emptyset & \{m_3\} & \emptyset & \emptyset & \{m_4,m_5,m_6\} & \{m_7\} & \emptyset & \emptyset & \{m_8\} & \emptyset & \{m_9\} & \{m_{10}\} & \{m_{11}\} \\
\{n_0,n_1\} & \emptyset & \emptyset & \emptyset & \{n_2,n_3\} & \emptyset & \{n_4\} & \emptyset & \{n_5,n_6\} & \{n_7\} & \emptyset & \{n_8\} & \{n_9,n_{10}\} & \emptyset & \{n_{11}\}
\end{array}\right)
$$

$$
D_4 = \left(\begin{array}{cccccccc}
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{a_0\} & \{a_1\} \\
\emptyset & \emptyset & \emptyset & \{b_0,b_1\} & \{b_2,b_3\} & \{b_4,b_5\} & \{b_6\} & \{b_7\} \\
\{c_0,c_1\} & \{c_2,c_3\} & \{c_4,c_5\} & \emptyset & \emptyset & \{c_6\} & \emptyset & \emptyset \\
\hline
\emptyset & \emptyset & \{d_0,d_1\} & \emptyset & \emptyset & \{d_2\} & \{d_3\} & \{d_4,d_5\} \\
\{e_0\} & \{e_1\} & \{e_2\} & \{e_3,e_4\} & \{e_5\} & \emptyset & \emptyset & \emptyset \\
\emptyset & \{f_0,f_1\} & \emptyset & \emptyset & \{f_2,f_3\} & \{f_4\} & \emptyset & \{f_5\} \\
\hline
\emptyset & \{g_0,g_1\} & \emptyset & \{g_2\} & \emptyset & \{g_3\} & \emptyset & \{g_4\} \\
\{h_0\} & \emptyset & \{h_1,h_2\} & \emptyset & \{h_3,h_4\} & \emptyset & \{h_5\} & \{h_6\} \\
\{k_0,k_1\} & \emptyset & \emptyset & \{k_2\} & \emptyset & \{k_3,k_4\} & \emptyset & \{k_5\} \\
\hline
\emptyset & \emptyset & \{l_0,l_1\} & \{l_2,l_3\} & \emptyset & \{l_4\} & \{l_5\} & \emptyset \\
\{m_0\} & \{m_1,m_2\} & \emptyset & \{m_3\} & \emptyset & \emptyset & \{m_4,m_5,m_6\} & \{m_7\} \\
\{n_0,n_1\} & \emptyset & \emptyset & \emptyset & \{n_2,n_3\} & \emptyset & \{n_4\} & \emptyset
\end{array}\right|
$$

$$
\left|\begin{array}{ccccccc}
\{a_2\} & \{a_3,a_4\} & \{a_5,a_6\} & \{a_7,a_8\} & \emptyset & \{a_9\} & \{a_{10}\} \\
\{b_8\} & \emptyset & \emptyset & \emptyset & \emptyset & \{b_9\} & \{b_{10}\} \\
\{c_7\} & \emptyset & \emptyset & \{c_8\} & \emptyset & \{c_9\} & \{c_{10}\} \\
\hline
\{d_6,d_7\} & \emptyset & \{d_8\} & \emptyset & \{d_9\} & \emptyset & \{d_{10}\} \\
\emptyset & \{e_6,e_7\} & \{e_8\} & \emptyset & \emptyset & \emptyset & \{e_9,e_{10}\} \\
\emptyset & \emptyset & \{f_6\} & \{f_7,f_8\} & \{f_9\} & \{f_{10}\} & \emptyset \\
\hline
\{g_5,g_6\} & \{g_7,g_8\} & \emptyset & \emptyset & \{g_9\} & \{g_{10}\} & \emptyset \\
\emptyset & \{h_7\} & \{h_8\} & \emptyset & \emptyset & \{h_9,h_{10}\} & \emptyset \\
\emptyset & \emptyset & \{k_6,k_7\} & \{k_8\} & \{k_9\} & \emptyset & \{k_{10}\} \\
\hline
\emptyset & \{l_6\} & \emptyset & \{l_7,l_8\} & \{l_9,l_{10}\} & \{l_{11}\} & \emptyset \\
\emptyset & \emptyset & \{m_8\} & \emptyset & \{m_9\} & \{m_{10}\} & \{m_{11}\} \\
\{n_5,n_6\} & \{n_7\} & \emptyset & \{n_8\} & \{n_9,n_{10}\} & \emptyset & \{n_{11}\}
\end{array}\right)
$$

Appendix B

$$
P_1 = \left(\begin{array}{ccccc|ccccc|ccccc}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&0&0&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&0&0&0&1&1\\
1&0&1&1&0&1&1&0&1&1&0&1&1&1&1\\
1&0&1&1&1&0&1&1&0&1&1&0&1&1&1\\
1&0&1&1&1&1&0&1&1&0&1&1&0&1&1\\
1&1&0&1&0&1&1&1&0&1&1&1&0&1&1\\
1&1&0&1&1&1&0&0&1&1&1&0&1&1&1
\end{array}\right)
$$

$$
P_2 = \left(\begin{array}{ccccc|ccccc|ccccc}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&0&0&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&0&0&0&1&1\\
1&0&1&1&0&1&1&0&1&1&0&1&1&1&1\\
1&0&1&1&1&0&1&1&0&1&1&0&1&1&1\\
1&0&1&1&1&1&0&1&1&0&1&1&0&1&1\\
1&1&0&1&0&1&1&1&0&1&1&1&0&1&1\\
1&1&1&0&0&1&1&1&1&0&1&0&1&1&1
\end{array}\right)
$$

$$
P_3 = \left(\begin{array}{ccccc|ccccc|ccccc}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&0&0&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&0&0&0&1&1\\
1&0&1&1&0&1&1&0&1&1&0&1&1&1&1\\
1&0&1&1&1&0&1&1&0&1&1&0&1&1&1\\
1&0&1&1&1&1&0&1&1&0&1&1&0&1&1\\
1&1&0&1&0&1&1&1&0&1&1&1&0&1&1\\
1&1&1&0&1&0&1&0&1&1&1&1&0&1&1
\end{array}\right)
$$

$$
P_4 = \left(\begin{array}{ccccc|ccccc|ccccc}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&0&0&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&0&0&0&1&1\\
1&0&1&1&0&1&1&0&1&1&0&1&1&1&1\\
1&0&1&1&1&0&1&1&0&1&1&0&1&1&1\\
1&0&1&1&1&1&0&1&1&0&1&1&0&1&1\\
1&1&0&1&0&1&1&1&0&1&1&1&0&1&1\\
1&1&1&0&1&0&1&1&1&0&0&1&1&1&1
\end{array}\right)
$$

$$
P_5 = \left(\begin{array}{ccccc|ccccc|ccccc}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&0&0&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&0&0&0&1&1\\
1&0&1&1&0&1&1&0&1&1&0&1&1&1&1\\
1&0&1&1&1&0&1&1&0&1&1&0&1&1&1\\
1&1&0&1&0&1&1&1&0&1&1&1&0&1&1\\
1&1&0&1&1&0&1&1&1&0&0&1&1&1&1\\
1&0&1&1&1&1&0&1&1&0&1&1&0&1&1
\end{array}\right)
$$

$$P_6 = \left(\begin{array}{ccccc|ccccc|ccccc}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&0&0&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&0&0&0&1&1\\
1&0&1&1&0&1&1&0&1&1&0&1&1&1&1\\
1&0&1&1&1&0&1&1&0&1&1&0&1&1&1\\
1&1&0&1&0&1&1&1&0&1&1&1&0&1&1\\
1&1&0&1&1&0&1&1&1&0&0&1&1&1&1\\
1&1&0&1&1&1&0&0&1&1&1&0&1&1&1
\end{array}\right)$$

$$P_7 = \left(\begin{array}{ccccc|ccccc|ccccc}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&0&0&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&0&0&0&1&1\\
1&0&1&1&0&1&1&0&1&1&0&1&1&1&1\\
1&0&1&1&1&0&1&1&0&1&1&0&1&1&1\\
1&1&0&1&0&1&1&1&0&1&1&1&0&1&1\\
1&1&0&1&1&0&1&1&1&0&0&1&1&1&1\\
1&1&1&0&0&1&1&1&1&0&1&0&1&1&1
\end{array}\right)$$

$$P_8 = \left(\begin{array}{ccccc|ccccc|ccccc}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&0&0&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&0&0&0&1&1\\
1&0&1&1&0&1&1&0&1&1&0&1&1&1&1\\
1&0&1&1&1&0&1&1&0&1&1&0&1&1&1\\
1&1&0&1&0&1&1&1&0&1&1&1&0&1&1\\
1&1&0&1&1&0&1&1&1&0&0&1&1&1&1\\
1&1&1&0&1&0&1&0&1&1&1&1&0&1&1
\end{array}\right)$$

$$P_9 = \left(\begin{array}{ccccc|ccccc|ccccc}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&0&0&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&0&0&0&1&1\\
1&0&1&1&0&1&1&0&1&1&0&1&1&1&1\\
1&0&1&1&1&0&1&1&0&1&1&0&1&1&1\\
1&1&0&1&0&1&1&1&0&1&1&1&0&1&1\\
1&1&0&1&1&0&1&1&1&0&0&1&1&1&1\\
1&1&1&0&1&1&0&1&0&1&0&1&1&1&1
\end{array}\right)$$

$$P_{10} = \left(\begin{array}{ccccc|ccccc|ccccc}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&0&0&1&1&1&1&1\\
1&0&1&1&0&1&1&0&1&1&0&1&1&1&1\\
1&0&1&1&1&0&1&1&0&1&1&0&1&1&1\\
1&1&0&1&0&1&1&1&1&0&1&0&1&1&1\\
1&1&0&1&1&1&0&1&0&1&0&1&1&1&1\\
1&1&1&0&1&0&1&1&1&0&0&1&1&1&1\\
1&0&1&1&1&1&0&1&1&0&1&1&0&1&1
\end{array}\right)$$

$$P_{11} = \left(\begin{array}{ccccc|ccccc|ccccc}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&0&0&1&1&1&1&1\\
1&0&1&1&0&1&1&0&1&1&0&1&1&1&1\\
1&0&1&1&1&0&1&1&0&1&1&0&1&1&1\\
1&1&0&1&0&1&1&1&1&0&1&0&1&1&1\\
1&1&0&1&1&1&0&1&0&1&0&1&1&1&1\\
1&1&1&0&1&0&1&1&1&0&0&1&1&1&1\\
1&1&0&1&1&0&1&0&1&1&1&1&0&1&1
\end{array}\right)$$

$$P_{12} = \left(\begin{array}{ccccc|ccccc|ccccc}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&0&0&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&0&0&1&1&1&1&1\\
1&0&1&1&0&1&1&0&1&1&0&1&1&1&1\\
1&0&1&1&1&0&1&1&0&1&1&0&1&1&1\\
1&1&0&1&0&1&1&1&1&0&1&0&1&1&1\\
1&1&0&1&1&1&0&1&0&1&0&1&1&1&1\\
1&1&1&0&1&0&1&1&1&0&0&1&1&1&1\\
1&1&1&0&0&1&1&1&0&1&1&1&0&1&1
\end{array}\right)$$

$$P_{13} = \left(\begin{array}{ccccc|ccccc|ccccc}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1
\end{array}\right)$$

References

[1] K. Akiyama and C. Suetake, The nonexistence of projective planes of order 12 with a collineation group of order 8, *J. Combin. Designs* **16** (2008), 411–430.

[2] K. Horvatic-Baldasar, E. Kramer and I. Matulic-Bedenic, Projective planes of order 12 do not have an abelian group of order 6 as a collineation group, *Punime Mat.* **1** (1986), 75–81.

[3] K. Horvatic-Baldasar, E. Kramer and I. Matulic-Bedenic, On the full collineation group of projective planes of order 12, *Punime Mat.* **2** (1987), 9–11.

[4] Y. Hiramine and C. Suetake, A contraction of square transversal designs, *Discrete Math.* **308** (2008), 3257–3264.

[5] Z. Janko and T. van Trung, On projective planes of order 12 which have a subplane of order three I, *J. Combin. Theory Ser. A* **29** (1980), 254–256.

[6] Z. Janko and T. van Trung, Projective planes of order 12 do not have a non-abelian group of order 6 as a collineation group, *J. Reine Angew. Math.* **326** (1981), 152–157.

[7] Z. Janko and T. van Trung, Projective planes of order 12 do not possess an elation of order 3, *Stud. Sci. Math.* **16** (1981), 115–118.

[8] Z. Janko and T. van Trung, On projective planes of order 12 with an automorphism of order 13. Part I. Kirkman designs of order 27, *Geom. Dedicata* **11** (1981), 257–284.

[9] Z. Janko and T. van Trung, On projective planes of order 12 with an automorphism of order 13. Part II. Orbit matrices and conclusion, *Geom. Dedicata* **12** (1982), 87–99.

[10] Z. Janko and T. van Trung, The full collineation group of any projective plane of order 12 is a $\{2,3\}$ group, *Geom. Dedicata* **12** (1982), 101–110.

[11] Z. Janko and T. van Trung, A generalization of a result of L. Baumert and M. Hall about projective planes of order 12, *J. Combin. Theory Ser. A* **32** (1982), 378–385.

[12] Z. Janko and T. van Trung, Projective planes of order 12 do not have a four group as a collineation group, *J. Combin. Theory Ser. A* **32** (1982), 401–404.

[13] M. J. Kallaher, *Affine planes with transitive collineation groups*, North Holland, New York, Amsterdam, Oxford, (1982).

[14] C. Suetake, The nonexistence of projective planes of order 12 with a collineation group of order 16, *J. Combin. Theory Ser. A* **107** (2004), 21–48.

(Received 11 Nov 2007)