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Orthogonal double covers of complete graphs by certain spanning subgraphs

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Abstract

An orthogonal double cover (ODC) of a complete graph is a collection of graphs such that every two of them share exactly one edge and every edge of the complete graph belongs to exactly two of the graphs. In this paper, we construct ODCs of K_n where all graphs are isomorphic to certain spanning subgraphs G (such as co-triangles, a star with co-triangles, and two other graph classes).

1 Introduction

Let $G = \{G_0, G_1, \ldots, G_{n-1}\}$ be a collection of spanning subgraphs (called pages) of K_n. G is an orthogonal double cover (ODC) of K_n if

1. every edge of K_n belongs to exactly two members of G;
2. any two distinct members intersect in exactly one edge.

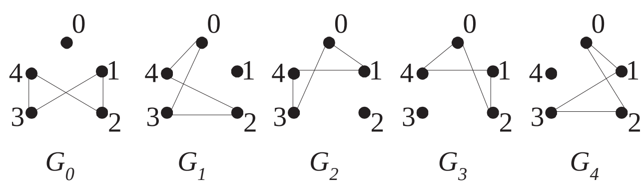
If all pages in G are isomorphic to a graph G then G is said to be an ODC by G. Clearly, G must have exactly $n-1$ edges. The original motivation for investigating ODCs comes from a problem of Demetrovics et al. [6] on minimal databases, and a problem of Hering and Rosenfeld [3] on the organization of statistical test planning. ODCs by G have been considered for several families of graphs: short cycles [1], clique graphs [2], trees [5], small graphs [7, 8]. A survey on the topic is given in [4].

We will label the vertices of K_n by the elements of $\mathbb{Z}_n$, the residue group of integers modulo n.

Let G be a spanning subgraph of K_n and $a \in \mathbb{Z}_n$. Then the graph G with $E(G+a) = \{(u+a, v+a) : (u,v) \in G\}$ is called the a-translate of G. Recall that an ODC, G, of K_n is group-generated by $\mathbb{Z}_n$ if we derive the page G_i from a given page G by adding a fixed element $i \in \mathbb{Z}_n$ to every vertex of G.

We define the length $l(\{a,b\})$ of an edge $\{a,b\}$ to be the set $\{a-b, b-a\}$. Let $e_1 = \{a,b\}$ and $e_2 = \{s,t\}$ to be two edges with the same lengths. Without loss of generality, let $a-t = b-s$. The distance dist (e_1, e_2) of the edges e_1 and e_2 is the set $\{t-a, a-t\}$, that is, the distance is the set of the two translations (with respect to $\mathbb{Z}_n$) that move one edge onto the other.

Figure 1: An ODC of K_5 by C_5 with respect to $\mathbb{Z}_5$.



We call G an ODC-generating graph (or ODC-generator) with respect to the group $\mathbb{Z}_n$ if the following conditions are satisfied:

1. Length condition: For every element e of $\mathbb{Z}_n$ of order greater than two, there are exactly two edges of length $\{e, -e\}$ in G. If there is an element e of order two, then there is exactly one edge of length $\{e\}$ in G.
2. Distance condition: The union of the distances of all pairs of distinct edges of the same lengths in G consists of all elements from $\mathbb{Z}_n$ of order greater than two.

The collection $G + \mathbb{Z}_n$ is an ODC, if and only if G is an ODC-generating graph. For more details see [4]. The graph in Figure 1 is an ODC-generator with respect to $\mathbb{Z}_5$, since $l(\{1,2\}) = l(\{3,4\}) = \{\pm 1\}$, $l(\{1,3\}) = l(\{2,4\}) = \{\pm 2\}$ and $\text{dist}(\{1,2\},\{3,4\}) = \{\pm 2\}$, $\text{dist}(\{1,3\},\{2,4\}) = \{\pm 1\}$. The ODC presented in Figure 1 is $G_0 + \mathbb{Z}_5$.

Our main aim in this paper is to construct the ODC-generating graphs of K_n by the following graph classes.

$G_{1,n}$ $n \equiv 0 \bmod 2$, graphs consisting of $(\frac{n}{2} - 1)$ triangles which all share one common edge and $(\frac{n}{2} - 1)$ isolated points (we call them co-triangles, see Figure 2).

$G_{2,n}$ $n \equiv 1 \bmod 2$, graphs consisting of modified stars whose central vertices are glued together, where a modified star can be described as a star (possibly with only one edge) in which all leaves are connected by a path (possibly of length zero) and one further edge attached to one end vertex of the path, and if $n = 2m+1$ then the graphs have $(m-1)/2$ isolated vertices for odd m, or $m/2$ isolated vertices for even m, as shown in Figure 3.

Figure 2: ODC-generating graph of K_n, $n = 2m$, by $G_{1,n}$ with respect to $\mathbb{Z}_n$.

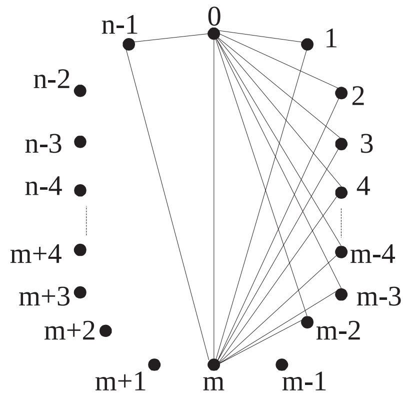


Figure 3: ODC-generating graph of K_n, $n = 2m + 1$, by $G_{2,n}$ with respect to $\mathbb{Z}_n$.

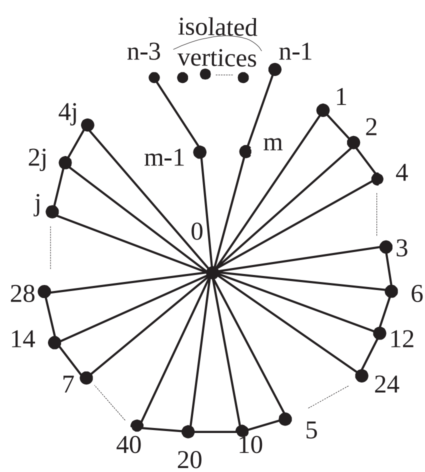


Figure 4: ODC-generating graph of K_n, $n = 2m$, by $G_{3,n}$ with respect to $\mathbb{Z}_n$.

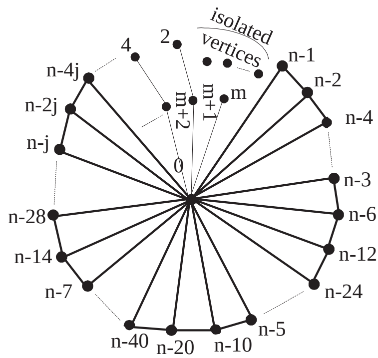


Figure 5: ODC-generating graph of K_n, by $G_{4,n}$ with respect to $\mathbb{Z}_n$.

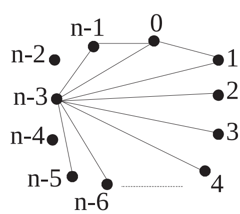
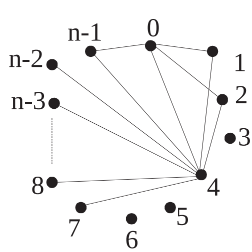


Figure 6: ODC-generating graph of K_n, by $G_{5,n}$ with respect to $\mathbb{Z}_n$.



$G_{3,n}$ $n \equiv 0 \bmod 2$, graphs consisting of modified stars whose central vertices are glued together, as described above; see Figure 4.

$G_{4,n}$ graphs consisting of 2 triangles which share one common edge, $n-6$ further edges which are attached to one end vertex of the common edge of the triangles, and 2 isolated points, as shown in Figure 5.

$G_{5,n}$ graphs consisting of 3 triangles which share one common edge, $n-8$ further edges which are attached to one end vertex of the common edge of the triangle, and 3 isolated points, as shown in Figure 6.

The construction of ODCs with pages of a given shape is of particular interest, and our results lend themselves to this purpose: It is natural to construct ODCs by such new infinite graph classes. All of these results will be considered in the following section.

2 The main results

In this section, we construct the ODC-generating graphs which are isomorphic to the graphs $G_{i,n}$, $i = 1, \ldots, 5$. In each proof of the following theorems it is easy to check that the given edges satisfy both the length and the distance condition.

Theorem 1 *Let n be even, then there is an ODC of K_n by $G_{1,n}$.*

Proof. Let us define a graph on $n = 2m$ vertices labelled by the elements of $\mathbb{Z}_{2m}$, where the edge set is given by $E(G) = \{0, m\} \cup \{\{0, j\}, \{m, j\} : j = 1, 2, \ldots, m-2,$ or $j = n-1\}$. ■

Theorem 2 *Let n be odd, then there is an ODC of K_n by $G_{2,n}$.*

Proof. Let us define a graph G on $n = 2m+1$ vertices labelled by the elements of $\mathbb{Z}_{2m+1}$, where the edge set is given by $E(G) = \{\{0, j\}, \{j, 2j\} : j = 1, 2, 3, \ldots, m\}$. as shown in Figure 3. ■

Theorem 3 *Let n be even, then there is an ODC of K_n by $G_{3,n}$.*

Proof. Let us define a graph G on $n = 2m$ vertices labelled by the elements of $\mathbb{Z}_{2m}$, where the edge set is given by $E(G) = \{\{0, j\} : j = m, m+1, m+2, \ldots, n-1\} \cup \{\{n-j, n-2j\} : j = 1, 2, 3, \ldots, m-1\}$ as shown in Figure 4. ■

Theorem 4 *For all positive integers $n \geq 6$, then there is an ODC of K_n by $G_{4,n}$.*

Proof. Let us define a graph G on n vertices labelled by the elements of $\mathbb{Z}_n$ and $n \geq 6$, where the edge set is given by $E(G) = \{\{n-3, j\} : 0 \leq j \leq n-5$, or $j = n-1\} \cup \{\{0, j\} : j = 1, n-1\}$ as shown in Figure 5. ■

Theorem 5 *For all positive integers $n \geq 8$, then there is an ODC of K_n by $G_{5,n}$.*

Proof. Let us define a graph on n vertices labelled by the elements of $\mathbb{Z}_n$ and $n \geq 8$, where the edge set is given by $E(G) = \{\{0, 4\}, \{0, 1\}, \{1, 4\}, \{0, 2\}, \{2, 4\}, \{0, n-1\}, \{n-1, 4\} \cup \{\{4, j\} : 7 \leq j \leq n-2\}$ as shown in Figure 6. ■

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