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Rainbow regular order of graphs

Zdzisław Skupień   Andrzej Żak

Faculty of Applied Mathematics
AGH University of Science and Technology
al. Mickiewicza 30, 30–059 Kraków
Poland
skupien@uci.agh.edu.pl   zakandrz@uci.agh.edu.pl

Abstract

Assume that the vertex set of the complete graph K_t is $\mathbb{Z}_t$ if t is odd and $\mathbb{Z}_{t-1} \cup \{\infty\}$ otherwise, with convention that $x + \infty = 2x$. If the color of any edge xy is defined to be $x + y$ then $\mathcal{G}K_t$ stands for K_t together with the resulting edge coloring. Hence color classes are maximum matchings rotationally/cyclically generated if t is even/odd. A rainbow subgraph of $\mathcal{G}K_t$ has all edges with distinct colors. Given a graph H, the optimization problem we deal with is to determine the rainbow regular order, $\rho(H)$, which is the smallest possible t such that $\mathcal{G}K_t$ contains a rainbow copy of H. We have determined ρ for cycles, wheels, complete bipartite graphs and some families of trees. We have also obtained an upper bound for ρ for all trees. Furthermore, we have contributed to a related problem by Hartman [*Discrete Math.* 62 (1986), 183–196], which is to determine rainbow subgraphs of all minimally edge colored copies of a graph G. We have solved this problem for cycles, wheels and complete bipartite graphs in case G is a complete graph. Relations between our version of modular sum labelings and the known labelings, especially harmonious and elegant ones, are presented.

1 Introduction

A *rainbow subgraph* (also called a heterochromatic subgraph) of an edge colored graph is defined to be the subgraph all of whose edges have distinct colors. In what follows a *coloring* is an edge coloring. Moreover, all colorings we deal with are *proper*, i.e., colors of adjacent edges are distinct. A coloring of a graph G is called *minimum* if the number of colors which are actually used is equal to the chromatic index $q(G)$ of G. In particular, a coloring of the complete t-vertex graph K_t is minimum if all color classes are maximum matchings.

The *rainbow order* of a graph H is defined to be the smallest integer t such that there exists a minimum coloring of the complete graph K_t with a rainbow copy of H included in K_t. At the beginning of our investigations into rainbow subgraphs we have considered [19] the following specification of the rainbow order. To this end, we specify coloring of K_t to be a so-called regular coloring. For even t, the *regular coloring* of K_t, designated $\mathcal{G}K_t$ (after [2]), is the well-known classical 1-*rotational* 1-*factorization* of K_t. Then, for odd order $t-1$, the regular coloring of K_{t-1} is denoted [16] by $\mathcal{G}K_{t-1}$ and is a special *near*-1-*factorization* of K_{t-1}. Namely, $\mathcal{G}K_{t-1}$ results from $\mathcal{G}K_t$ on removing the vertex of K_t, which is fixed under rotation. Thus $\mathcal{G}K_{t-1}$ is the known cyclic coloring of K_{t-1}. The construction of $\mathcal{G}K_t$ dates back to Kirkman [14], Reiss [18] and Walecki (see [15]). The symbol $\mathrm{R_rO}(H)$ (*rainbow regular order* of H) denotes the smallest integer t (odd or even) such that $\mathcal{G}K_t$ contains a rainbow copy of H. For results concerning $\mathrm{R_rO}(K_n)$, see [19].

There are several reasons for justifying the choice of regular coloring. Firstly, it makes possible to reformulate our problem in the language of additive number theory. On the other hand, it leads to a new graph labeling which can be viewed as a generalization of known graph labelings. Those known related labelings are named elegant, harmonious, indexable, and sequential labelings. Despite the coloring restriction our results solve the following Hartman problem for some graphs. Given a graph G, the problem is to study [13] the family $M(G)$ (denoted by M_t if $G = K_t$) of subgraphs H of G such that $H \in M(G)$ if the coloring of a rainbow copy of H can be extended to a minimum coloring of G. Call M_t to be the *set of general rainbow subgraphs of* K_t. We determine when cycles, wheels and complete bipartite graphs are general rainbow subgraphs of K_t. Our main objective is to determine $\mathrm{R_rO}$ of those special graphs inclusive of some families of trees, and to find an upper bound for $\mathrm{R_rO}$ for all trees.

2 Preliminaries

In our notation $\mathcal{G}K_t$ stands for the complete graph K_t together with its decomposition into maximum matchings, the decomposition being generated from a special matching by a vertex permutation which is 1-rotational if t is even and cyclic if t is odd. Recall that 1-rotational permutation has a cycle which avoids exactly one element. A decomposition of K_t into maximum matchings is called a 1-factorization of K_t if t is even and a near-1-factorization if t is odd, see survey paper [16] of 1985 for interesting historical data on 1-factorizations. The definition of the coloring $\mathcal{G}K_t$ follows. Define the vertex set of K_t to be

$$V(K_t) = \begin{cases} \mathbb{Z}_{t-1} \cup \{\infty\} & \text{for even } t, \\ \mathbb{Z}_t & \text{for odd } t \end{cases}$$

where $\mathbb{Z}_t = \{0, 1, \ldots, t-1\}$ with addition modulo t. Let q be the chromatic index of K_t, i.e., $q = \mathrm{odd}\{t, t-1\}$, the odd integer among $t, t-1$. For $x \in \mathbb{Z}_q$, on using the convention that $x + x =: 2x$ and on assuming that $x + \infty = 2x$, let $x + y$ denote the color of the edge xy of K_t. Thus $\mathcal{G}K_t$ has been defined. Let us see color classes. For $i \in \mathbb{Z}_q$, let F_i denote the class of edges with color $2i$ in K_t. Therefore, for even t, $F_i = \{\{i + j, i - j\} \colon j \in \mathbb{Z}_{t-1},\ 0 < j < t/2\} \cup \{\{i, \infty\}\}$, F_i being a matching of cardinality $t/2$. Note that, for odd t, removing the vertex ∞ (together with incident edges) from $\mathcal{G}K_{t+1}$ results in $\mathcal{G}K_t$ and its color classes. Hence in both cases $F_i = F_0+i$ which is equivalent to rotationally/cyclically permuting the matching F_0, a property claimed above.

We sometimes replace the phrase 'rainbow regular' by the letter ρ. Given $\mathcal{G}K_t$ and a graph H of order t or less, a mapping $\lambda\colon V(H) \to V(K_t)$ is called a *rainbow regular labeling* (*of order* t; t-ρ-*labeling* or ρ-*labeling*) if λ is injective and the induced image of H in $\mathcal{G}K_t$ is a rainbow subgraph. The smallest possible order t is denoted by $\rho(H)$ and is named the ρ-*order*, $\mathrm{R_rO}$, of the graph H. Note that

$$\rho(H) \geq \max\{v(H), \mathrm{odd}\{e(H), e(H)+1\}\} \tag{1}$$

because the number of edge colors in $\mathcal{G}K_t$ is always odd. Let $\rho'(H)$ denote the smallest integer t such that a t'-ρ-labeling of H exists for each integer $t' \geq t$, $\rho'(H)$ being called the *rainbow regular up-order* of H.

3 Results

Since t-ρ-*labeling* coincides, for some large families of graphs, with known other graph labelings, it is necessary to recall some definitions. In this context we refer the reader to an excellent survey of graph labelings due to Gallian [10]. Assume that a graph H, $H = (V, E)$, has e edges and v vertices, $e = |E|$ and $v = |V|$. Vertices are labeled by an injective function f from V into an additive group, edge labels are defined to be sums of vertex labels of endvertices and the edge labeling function $xy \mapsto f(x) + f(y)$ is to be injective, too.

Then H and a required labeling f (only if f exists) are called

- *harmonious* (Graham and Sloane [12]) if $e \geq v$ and $f\colon V \to \mathbb{Z}_e$;
- *strongly c-harmonious* (Chang, Hsu and Rogers [7]) or *sequential* (Grace [11]) if $e \geq v$ and $f\colon V \to \{0, 1, ..., e-1\} \subset \mathbb{Z}$, and the resulting edge labels cover the integer interval $[c, c+e-1]$;
- *elegant* (Chang, Hsu and Rogers [7]) if $e+1 \geq v$, $f\colon V \to \mathbb{Z}_{e+1}$, and the resulting edge labels are nonzero; furthermore
  - *near-elegant* (Deb and Limaye [9]) if the edge label e, instead of 0, is forbidden;
  - *semi-elegant* (Deb and Limaye [8]) if no specific edge label is forbidden;
- *indexable* (Acharya and Hegde [1]) if $f\colon V \to \{0, 1, ..., v-1\} \subset \mathbb{Z}$.

A tree is called *harmonious* or *strongly c-harmonious* if additionally exactly one vertex label can be used on two vertices; Grace, however, calls a tree *sequential* if the vertex label e can be used. Note that strongly c-harmonious (as well as sequential) graphs make up a subclass of harmonious graphs. It is an open problem if the subclass is proper.

As a straightforward consequence one obtains the following result.

Proposition 1 *Let H be a graph with v vertices and e edges.*

(*i*) *If e is odd and H is not any tree, H is harmonious if and only if $\rho(H) = e$.*

(*ii*) *If H is not any tree and H is sequential (i.e. H is strongly c-harmonious), then $\rho(H) = \rho'(H)$ and the common value is the odd number among $e, e+1$.*

(*iii*) *If e is even and H is elegant or near-elegant, then $\rho(H) = e+1$.*

(*iv*) *For even e, H is semi-elegant if and only if $\rho(H) = e+1$.*

(*v*) *If H is indexable, then $\rho'(H) \le 2v - 3$.*

■

Hence the parameters $\rho(H)$ and $\rho'(H)$ can be derived for graphs H which are known to have some of the above labelings. We prove some more results which cannot be derived from Proposition 1. For instance, odd cycles are sequential [11] and even cycles are elegant [5, 17] but not harmonious [12]. Nevertheless, the following result requires a proof.

Theorem 2 *For the cycle C_n, $\rho'(C_n) = \rho(C_n)$ and the common value is the odd integer among $n, n+1$.*

Proof. We represent a cycle by a circular sequence of its vertices. In order to prove the theorem, it is sufficient to present a vertex sequence which represents a rainbow C_k for each possible cycle length k in $\mathcal{G}K_{2n+1}$ only, because $\mathcal{G}K_{2n+1} \subset \mathcal{G}K_{2n+2}$ and both structures have the same number of edge colors. If k is odd, we take $0, 1, ..., k-1, 0$ as a C_k. Otherwise, we consider a few cases where m is an integer, $m \ge 2$. If $k = 2m$ with $1 < m \le n-1$ (whence $k \le 2n-2$), we put $\epsilon = m \bmod 2$ $(= 0,\ 1)$ and take

$$0, 1, 2, ..., m-1, m, m+2, m+1, m+4, m+3, m+6, m+5, m+8, ..., 2m-2\epsilon, 0$$

where the initial (increasing) section of $m+1$ vertices is followed by section comprising all pairs of the form $m+2i, m+2i-1$ where $1 \le i \le (m-1)/2$.

Let $k = 2m = 2n$. For four values 2, 3, 4, and 7 of m the following cycles are required rainbow cycles:

$m = 2$, C_4 in K_5: $1, 3, 4, 2, 1$;

$m = 3$, C_6 in K_7: $1, 5, 4, 6, 2, 3, 1$;

$m = 4$, C_8 in K_9: $1, 2, 3, 4, 6, 5, 8, 7, 1$;

$m = 7$, C_{14} in K_{15}: $1, 2, 3, 4, 5, 6, 8, 11, 10, 7, 9, 14, 13, 12, 1$.

Otherwise $m = 5, 6$, or $m \ge 8$. Assume that nonnegative integers a, b satisfy the equation $m - 2 = 3a + 4b$. One possible solution for the required cycle is the concatenation of the following sequences: $m+3$-sequence

$$2m-2, 1, 2, 3, \ldots, m-1, m+1, m+2, m,$$

next 3-sequences

$$(m+2)+3i, (m+1)+3i, m+3i \qquad \text{for } i = 1, 2, \ldots, a,$$

and finally 4-sequences

$$(m+1)+3a+4j, (m+2)+3a+4j, (m-1)+3a+4j, m+3a+4j \qquad \text{for } j = 1, 2, \ldots, b.$$

■

Note that the labeling of the cycle C_{2n} in $\mathcal{G}K_{2n+1}$ in the above proof is elegant. Thus we have presented a short proof of the following result.

Corollary 3 (Beals et al. [5] or Mollard and Payan [17]) *All even cycles are elegant.* ■

All wheels W_n are harmonious [12] and are sequential if n is odd [11] or if $n \not\equiv 2 \pmod 3$ [7].

Theorem 4 *For the wheel W_n on $n+1$ vertices and $2n$ edges,*

$$\rho'(W_n) = \rho(W_n) = e(W_n) + 1 = 2n + 1.$$

Proof. Because $\mathcal{G}K_t \subset \mathcal{G}K_{t+1}$ for odd t, it is enough to present a rainbow W_n in $\mathcal{G}K_t$ for any odd $t \geq e(W_n)$, $t \geq 2n+1$. Let 0 to be the center (hub) of the wheel. For odd n, $n = 2m+1$, the following cycle can be the rim.

$$1, 2, 5, 6, ..., 4(m-1)+1, 4(m-1)+2, 4m+1, 1 \quad \text{if } t \equiv 3 \pmod 4$$

and

$$2, 3, 6, 7, ..., 4(m-1)+2, 4(m-1)+3, 4m+2, 2 \quad \text{if } t \equiv 1 \pmod 4.$$

For even n, $n = 2m$, the rim can be the following cycle

$$2, 3, 6, 7, ..., 4(m-2)+2, 4(m-2)+3, 1, 4m-2, 2 \quad \text{if } t \equiv 1 \pmod 4$$

and

$$3, 4, 7, 8, ..., 4(m-2)+3, 4(m-1), 2, 4m-1, 3 \quad \text{if } t \equiv 3 \pmod 4.$$

Checking whether edge labels (sums modulo t) are correct is left to the reader. Note, however, that the following observation can be helpful. If vertex labels on the rim are considered as natural numbers then the resulting edge labels on the rim are congruent to t modulo 4 as long as the edge labels are increasing, that is, except for the last edge label if n is odd and for the last three ones if n is even. Moreover, all those edge labels are less than $4n-2 < 2t$, all exceptional ones (even if one is $2n+2$) being less than t. Therefore on passing to edge labels modulo t, the resulting labeling remains injective. ■

The complete bipartite graph $K_{m,n}$ is elegant [4]. It is not harmonious unless it is a star [12].

Theorem 5 *If $K_{m,n}$ is a complete bipartite graph with m white and n black vertices, and with e edges where $e = mn$, then*

$$\rho'(K_{m,n}) = \rho(K_{m,n}) = m \cdot n + 1.$$

Proof. First we find a rainbow $K_{m,n}$ in $\mathcal{G}K_t$ for odd $t > e$. Then the white vertices can be $0, 1, ..., m-1$ and black ones $m, 2m, ..., nm$. On the other hand, there is no rainbow $K_{m,n}$ in $\mathcal{G}K_e$ because the number of edge colors is either to small if e is even or $v = m+n$ is to large, $v > e$, if $K_{m,n}$ is a star. Otherwise e is odd and, due to [12], $K_{m,n}$ is not harmonious. Then $\rho > e$, cf Proposition 1(i). Additionally, for odd e there is a rainbow $K_{m,n}$ in $\mathcal{G}K_{e+1}$; namely with white vertices $\infty, 0, 1, ..., m-2$ and black ones $m-1, 2m-1, ..., nm-1$, which can be checked. ■

Each tree is indexable [3, 6]. Then the edge labels are nonzero and at most $2n-3$ where n is the order of the tree. Hence (cf Proposition 1(v) above)

Corollary 6 *If T_n is a tree of order $n \geq 2$ then $\rho'(T_n) \leq 2n-3$.* ■

This bound for a tree is far from being sharp if n is not small. The lower bound is clearly n. Our next aim is to improve the upper bound.

Theorem 7 *Let T be a tree of order n and diameter d. Let l denote the number of leaves in T. Then $\rho'(T) \leq \mathrm{odd}\{2n+1-d-l, 2n+2-d-l\}$.*

Proof. Let r be an endvertex of a longest path in T. By N_k, $0 \leq k \leq d$, we denote the set of vertices of T which are at distance k from r. Thus, $N_0 = \{r\}$ and N_1 comprises the neighbor of r. Let L_k denote the set of leaves of T which are at distance k from r, $L_k \subset N_k$. Furthermore, let n_k and l_k denote the number of elements of N_k and L_k, respectively. Note that $l_0 = n_0 = 1$, $n_1 = 1$ and $l_d = n_d$. Consider the following vertex labeling (injection) $f\colon V(T) \to \mathbb{Z}$. Let $f(r) = 0$. Let a be a large enough (to ensure that f is injective) positive integer which will be specified later. Label the neighbor of r by a. When all elements of N_k are labeled, let $\max(k) := \max_{v \in N_k} f(v)$.

We label vertices from N_k, $k \geq 2$, in the following way. Let $\{v_1, ..., v_t\} = N_{k-1} \setminus L_{k-1}$ with $f(v_1) < f(v_2) < ... < f(v_t)$. First we label all leaves which are neighbors of v_t, then all leaves which are neighbors of v_{t-1}, and so on till all elements of L_k are labeled. Every leaf incident to v_i gets the label equal to $\max(k-2) + \max(k-1) - f(v_i) + \bar{l}_k + 1$, where $\bar{l}_k$ is the number of already labeled elements of L_k (thus $\bar{l}_k$ ranges from 0 to l_k), see Fig. 1. When all elements of L_k are labeled, we label remaining unlabeled neighbors of v_1, then remaining unlabeled neighbors of v_2 and so on. These vertices, in this order, are labeled by consecutive integers starting from $\max(k-2)+\max(k-1)-f(v_1)+l_k+1$. Then $\max(k) = \max(k-2)+\max(k-1) - f(v_1) + n_k$. Note that vertices $v_1, ..., v_t$ are labeled by consecutive integers. Hence $f(v_t) = f(v_1) + t - 1$, where $f(v_t) = \max(k-1)$ and $t = n_{k-1} - l_{k-1}$. Thus $f(v_1) = \max(k-1) - (n_{k-1} - l_{k-1}) + 1$. Hence $\max(k) = \max(k-2) + n_{k-1} + n_k - l_{k-1} - 1$. Moreover, $\max(0) = 0$ and $\max(1) = a$. Therefore,

$$\max(k) = \sum_{i=2}^{k} n_i - \sum_{i=1}^{(k-1)/2} l_{2i} - \frac{k-1}{2} + a \quad \text{for odd } k$$

and

$$\max(k) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k/2} l_{2i-1} - \frac{k}{2} \quad \text{for even } k.$$

Thus if we choose $a = \max(d-1)+1$ if d is odd, or $a = \max(d)+1$ if d is even, then vertex labels remain different. It follows that if d is odd then

$$\begin{aligned}\max_{v\in V(T)} f(v) = \max(d) &= \sum_{i=2}^{d} n_i - \sum_{i=1}^{(d-1)/2} l_{2i} - \frac{d-1}{2} + \sum_{i=1}^{d-1} n_i - \sum_{i=1}^{(d-1)/2} l_{2i-1} - \frac{d-1}{2} + 1 \\ &= \left(\sum_{i=0}^{d} n_i - n_0 - n_1\right) - \left(\sum_{i=0}^{(d-1)/2} l_{2i} - l_0\right) - \frac{d-1}{2} \\ &+ \left(\sum_{i=0}^{d} n_i - n_0 - n_d\right) - \left(\sum_{i=1}^{(d+1)/2} l_{2i-1} - l_d\right) - \frac{d-1}{2} + 1 \\ &= 2n - 3 - n_d - (l - 1 - l_d) - d + 2 = 2n - d - l \qquad (2)\end{aligned}$$

because $n_d = l_d$. Analogously, $\max_{v\in V(T)} f(v) = \max(d-1) = 2n - d - l$ when d is even. Recall that the label of any edge xy is the sum $f(x) + f(y)$, labels of endvertices. It is easy to see that the N_{k-1}–N_k edges have distinct labels belonging to the interval $\{\max(k-2)+\max(k-1)+1, ..., \max(k-1)+\max(k)\}$. Therefore the edge labels are distinct and belong to $\{a, ..., \max_{v\in V(T)} f(v)+a-1\}$. Hence the edge labels taken *modulo* t for any $t \geq \max_{v\in V(T)} f(v)+1$ are distinct, too. It is so because the edge label if changed is at most $a-2$. Thus we see that there is a rainbow copy of T in $\mathcal{G}K_t$ for any such t if t is odd. Hence the smallest odd integer among those t is the upper bound for $\rho'(T)$ whence, by (2), $\rho'(T) \leq \text{odd}\{2n+1-l-d, 2n+2-l-d\}$. ■

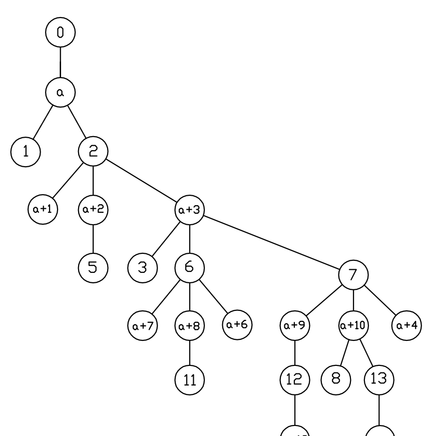

Fig. 1. Rainbow labeling of a tree ($a = 14$): $n = 23$, $d = 7$, $l = 12$, $\rho' \leq 29$.

Due to Theorem 7, the upper bound for the rainbow order has been significantly improved for many families of trees. For example

Corollary 8 *Let T be a tree of order n. Then*

1. *If T is a caterpillar then $\rho'(T) = n$ if n is odd, otherwise $\rho'(T) \in \{n, n+1\}$.*
2. *If every inner vertex of T has degree greater than or equal to $\overline{\delta}$ then $\rho'(T) < n\frac{\overline{\delta}}{\overline{\delta}-1}$.*

Proof. We can assume that $n \geq 3$. Let d, l denote the diameter of T and the number of leaves in T, respectively. If T is a caterpillar then $d + l = n + 1$. Hence, by Theorem 7, $\rho'(T) \leq \mathrm{odd}\{n, n+1\}$ with $\rho'(T) \geq n$ by (1). This result can also be derived from the fact that each caterpillar is sequential [11].

Assume now that each inner vertex of T has degree greater than or equal to $\overline{\delta}$. Then

$$2n - 2 = 2e = \sum_{x \in V(T)} \deg x \geq l + (n-l)\overline{\delta}.$$

Hence

$$l \geq \frac{n(\overline{\delta}-2)+2}{\overline{\delta}-1} > n\frac{\overline{\delta}-2}{\overline{\delta}-1}.$$

Thus, by Theorem 7, $\rho'(T) \leq 2n + 2 - d - l \leq 2n - l$ because $d \geq 2$. Hence

$$\rho'(T) < 2n - n\frac{\overline{\delta}-2}{\overline{\delta}-1} = n\frac{\overline{\delta}}{\overline{\delta}-1}.$$

■

We have also managed to establish an improved upper bound for the rainbow order for all trees. We use the fact that the diameter and the number of leaves cannot be simultaneously small.

Lemma 9 *Let T be a tree of order n and diameter d, $n \geq 3$. Then T has at least $\lceil \frac{n-1-(d \bmod 2)}{\lfloor d/2 \rfloor} \rceil$ leaves, the bound being sharp.*

Proof. Let P be a path of T of order $d + 1$. We visit each vertex of T exactly once using the following procedure. We first visit all vertices of P and next a vertex adjacent to P. If the last of visited vertices is not a leaf, we visit its neighbor and continue visits till we reach a leaf. If the last of visited vertices is a leaf, we visit a neighbor of an already visited vertex. Because at any stage the set of visited vertices induces a subtree, the procedure stops when all vertices are visited. After visiting all vertices of P inclusive of two leaves of P, among any $\lfloor d/2 \rfloor$ consecutively visited vertices, there is at least one leaf, and what is visited at the very end is a leaf. Thus the number of leaves in T is at least $\lceil \frac{n-d-1}{\lfloor d/2 \rfloor} + 2 \rceil$ which agrees with Lemma. The bound is attained if T is either a star (d even) or a double star (d odd) with rays properly subdivided, namely all rays but possibly one are subdivided by $\lfloor \frac{d}{2} \rfloor - 1$ vertices. T may reduce to a path; moreover, a double star T to a star. ■

Corollary 10 *Let T be a tree of order n. Then*

$$\rho'(T) \leq 2n + 2 - 2\sqrt{2n-2}. \tag{3}$$

Proof. Since, by Theorem 7, $\rho'(T) \leq 2n+2-(d+l)$, it suffices to show that $d+l \geq 2\sqrt{2n-2}$. By Lemma 9,

$$d+l \geq d + \left\lceil \frac{n-d-1}{\lfloor d/2 \rfloor} + 2 \right\rceil \geq d + \frac{n-d-1}{d/2} + 2 = d + \frac{2n-2}{d}.$$

It is easy to check that the function $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) := x + \frac{2n-2}{x}$ attains its global minimum at $x = \sqrt{2n-2}$. Thus,

$$d+l \geq \sqrt{2n-2} + \frac{2n-2}{\sqrt{2n-2}} = 2\sqrt{2n-2}.$$

■

For all of the extreme trees T_n (that is, for stars $K_{1,n-1}$ and paths P_n), ρ' is n or $n+1$ due to Theorems 5 and 2 respectively; in fact, $\rho'(P_n) = n$ for each path P_n of odd order n. It is noted by Hartman, that $P_n \notin M_n$ for $n = 4, 6$. Hartman's conjecture that remaining $P_n \in M_n$ is proved by Mollard and Payan [17]. A stronger result follows.

Theorem 11 *For the n-vertex path P_n, $n \geq 2$,*

$$\rho'(P_n) = \rho(P_n) = \begin{cases} n & \text{if } n \neq 4, 6, \\ n+1 & \text{if } n = 4, 6. \end{cases}$$

Proof. It is enough to check that the following vertex sequences represent rainbow Hamiltonian paths P_{2m} in $\mathcal{G}K_{2m}$ for any $m \geq 4$.

$m = 4$, P_8: $1, 0, \infty, 5, 6, 3, 2, 4$;
$m = 6$, P_{12}: $1, 0, \infty, 3, 4, 5, 9, 10, 6, 7, 8, 2$;
$m = 7$, P_{14}: $1, 0, \infty, 3, 4, 5, 6, 11, 12, 9, 7, 8, 10, 2$;
$m = 10$, P_{20}: $1, 0, \infty, 3, 4, 5, 6, 7, 8, 9, 13, 14, 10, 11, 12, 17, 18, 15, 16, 2$.

Otherwise $m = 5, 8, 9$, or $m \geq 11$. Assume that nonnegative integers a, b satisfy the equation $m - 5 = 3a + 4b$. As a required path we take the concatenation of the following sequences: $m+4$-sequence

$$1, 0, \infty, 3, 4, \ldots, m-2, m-1, m+2, m, m+3, m+1,$$

next 3-sequences

$$(m+3)+3i, (m+2)+3i, (m+1)+3i \qquad \text{for } i = 1, 2, \ldots, a,$$

4-sequences

$$(m+2)+3a+4j, (m+3)+3a+4j, m+3a+4j, (m+1)+3a+4j \qquad \text{for } j = 1, 2, \ldots, b,$$

and the single vertex 2 at the end. ■

Theorem 11 includes the related above-mentioned result by Mollard and Payan, and our proof is essentially simpler than theirs.

Conjecture 1 *Let T be a tree. Then $\rho(T) = v(T)$ or $1 + v(T)$.*

A computer program confirmed the conjecture for $v(T) \leq 9$ and found 10 trees T with $\rho = v(T) + 1$. One, four and five of those trees are of orders 4, 6 and 8, respectively. Therefore it is likely that for trees the rainbow regular order and order, if odd, coincide. However, it is not so for each even order $n \geq 4$. Let Ψ_n be a tree which comprises the star $K_{1,n-2}$ and one more vertex joined by an edge to a leaf of the star, e.g., $\Psi_4 = P_4$.

Proposition 12 *For each even order $n = 2m \geq 4$, $\rho'(\Psi_n) = \rho(\Psi_n) = n + 1$.*

Proof. Assume that z, y, x are rainbow labels of respectively the star center, its degree-2 neighbor, and its leaf non-neighbor. Let x_i $(i = 1, 2, \ldots, 2m-3)$ be labels of remaining leaves, neighbors of the center. A rainbow Ψ_{2m} in $\mathcal{G}K_t$ with $t \geq 2m+1$ arises on putting $x = 1$, $y = 2m-1$, $z = 0$, and $x_i = i + 1$ for $i = 1, 2, \ldots, 2m-3$ (which is a sequential labeling of the tree Ψ_{2m}). Hence $\rho' \leq 2m + 1$. To show that $\rho > 2m$, suppose that vertex labels of Ψ_{2m} range over $\mathbb{Z}_{2m-1} \cup \{\infty\}$ and determine a rainbow copy of Ψ_{2m} in $\mathcal{G}K_{2m}$. Then the sum of edge labels, $\sum_e$, as well as that of finite vertex labels, $\sum_v$, are both 0 (modulo $2m-1$) since $0 + 1 + \ldots, 2m - 2 = (m-1)(2m-1) \equiv 0 \pmod{2m-1}$. On the other hand, (still in $\mathbb{Z}_{2m-1}$) if $x = \infty$ then $0 = \sum_e = 2y + \sum_v + (2m-3)z = 2(y-z) = y - z$ because $2m-1$ is odd. Similarly, if $y = \infty$ then $x - z = 0$, if $z = \infty$ then $y - x = 0$, and if $x_i = \infty$ then $y - z = 0$. This is a contradiction in each case. ∎

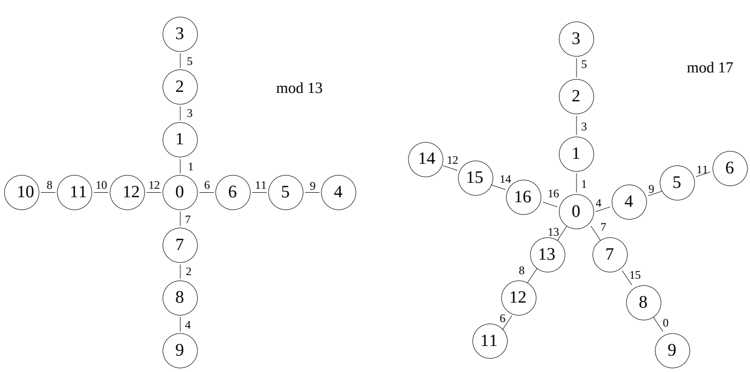

Fig. 2. Rainbow labelings of the subdivided stars $S(3, 4)$ and $S(3, 5)$.

Let $S(k, p)$ denote a tree obtained from a star $K_{1,p}$ by extending each edge to a path of length k. Hence $S(1, p) = K_{1,p}$. Note that for $S(p, 2p)$

$$d + l = 2p + 2p = 4p = 2\sqrt{2 \cdot (2p^2 + 1) - 2} = 2\sqrt{2n - 2},$$

where d, l, n are respectively the diameter, the number of leaves and the order of $S(p, 2p)$. Therefore, in this case the upper bound (3) cannot be improved using Theorem 7. However,

Theorem 13 *Let k and p be positive integers. If p is even then $\rho(S(k,p)) = v(S(k,p))$. If k and p are odd then $\rho(S(k,p)) \le v(S(k,p)) + 1$.*

Proof. Let P_i, $i = \{1, ..., p\}$, be the p paths of order k excluding the center of $S(k,p)$. Let 0 to be the center.
Case 1. p is even.
Label consecutive vertices of P_i by integers

$$(i-1)k+1, (i-1)k+2, ..., ik,$$

starting from a vertex incident to the center, when i is odd, and from a leaf, when i is even. Checking whether edge labels taken *modulo* $n = kp+1$ are correct is left to the reader.
Case 2. k and p are both odd.
Let $p = 2a+1$. For $1 \le i \le a+1$ label consecutive vertices of P_i by integers

$$(i-1)k+1, (i-1)k+2, ..., ik,$$

starting from a vertex incident to the center. For $a+2 \le i \le 2a+1$ label consecutive vertices of P_i by integers

$$(i-1)k+2, (i-1)k+3, ..., ik+1,$$

starting from a leaf. Checking that edge labels taken *modulo* $n+1 = kp+2$ are distinct is left to the reader. ■

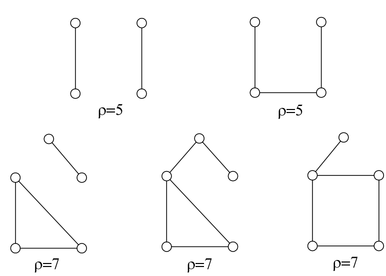

Fig. 3

Five graphs in Fig. 3 are the only small graphs with up to five vertices and with ρ-order exceeding the lower bound in (1).

4 Concluding remarks

We have determined rainbow regular order ρ and related up-order ρ' for complete bipartite graphs, cycles, paths, wheels and stars with a subdivided ray. For all those graphs $\rho = \rho'$. So far we know only two graphs, complete graphs K_{10} and K_{11}, for which $\rho \neq \rho'$ (cf. [19]). Namely, $\rho(K_{10}) = 73$ while $\rho'(K_{10}) = 77$, and $\rho(K_{11}) = 92$ while $\rho'(K_{11}) = 97$. These values have been found by exhaustive computer search. Hence $\mathcal{G}K_t$, a minimally edge colored K_t, can include a rainbow clique which is larger than any one in some $\mathcal{G}K_{t+p}$ with $p \geq 1$.

We have also contributed to the Hartman problem. Note that $H \in M_t$ for all $t \geq \rho'(H)$. Moreover, because the chromatic index of K_t is always odd, the lower bound in (1) holds for any (general, not necessarily regular) rainbow order of H. This bound is attained by ρ' for cycles, most paths (excluding just P_4 and P_6), all wheels and stars as well as for complete bipartite graphs of even size. For the remaining complete bipartite graphs, say B, ρ' exceeds the bound by 1. However, a rainbow copy of any such bipartite graph B is also contained in $\mathcal{G}K_{mn+1} - v$, where v is any vertex of $\mathcal{G}K_{mn+1}$, which is avoided by the copy. Such a vertex v exists because the graph B is not a star, m and n are both odd and ≥ 3, and therefore $v(B) = m + n < mn$.

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