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The Tutte polynomial and the generalized Petersen graph

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Abstract

Graphs G and H are T-equivalent if they have the same Tutte polynomial. G is T-unique if any arbitrary graph H being T-equivalent to G implies that H is isomorphic to G. We show that the generalized Petersen graph $P(m,2)$ and the line graph of $P(m,2)$ are T-unique.

1 The Tutte Polynomial

Let G be a graph with vertex set V and edge set E, and let $G[A]$ denote the subgraph of G induced by the edge set $A \subseteq E$. The following two variable polynomial is referred to as the Tutte polynomial of a graph G:

$$T_G(x,y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

The rank of $G[A]$, denoted $r(A)$, is defined as $|V(G[A])| - \omega(G[A])$ where $\omega(G[A])$ is the number of components of $G[A]$. Graphs G and H are T-equivalent if $T_G(x,y) = T_H(x,y)$ and G is T-unique if any graph H being T-equivalent to G implies that H is isomorphic to G. We will always assume that G does not contain isolated vertices since these do not effect the rank or number of edges of G.

The Tutte polynomial, introduced by Tutte in 1954, is a powerful tool containing much of a graph's structural information. The question we wish to consider is related to the amount of information received from the Tutte polynomial. Does $T_G(x,y)$ give enough information so that it uniquely determines G? In general, the answer to this is no. Indeed, it can easily be argued that trees are not uniquely determined. For graphs with higher connectivity, this is more difficult as shown by Tutte [11] and Brylawski [3] who did find such graphs. However, we wish to find classes of graphs that are determined by their Tutte polynomial, or are T-unique. In [8], de Mier and Noy show that wheels, squares of cycles, ladders, Möbius Ladders, complete multipartite graphs, and hypercubes are T-unique graphs.

T-unique graphs have also been studied indirectly under a less general object. The chromatic polynomial of a graph G, denoted $p(G,k)$, is defined as the number of k-colorings of G where a k-coloring is a proper vertex coloring of k colors. A graph G is χ-unique if a graph H having the same chromatic polynomial implies that H is isomorphic to G. Several classes of graphs have been shown to be χ-unique and many of these can be found in [5] and [6]. It is a well known fact that the Tutte polynomial generalizes the chromatic polynomial [1]. Clearly then, every χ-unique graph is T-unique. However, the converse is not true [8]. Thus [5] and [6] also give a list of T-unique graphs. We also mention that the authors in [2] have conjectured that almost every graph is T-unique meaning that as $|V|$ approaches infinity, the probability that a graph is T-unique approaches one.

The following theorem [8] gives information concerning G that is determined by $T_G(x,y)$.

Theorem 1 *Let G be a 2-connected graph. Then the following parameters of G are determined by its Tutte polynomial:*

1. *The number of vertices and the number of edges.*
2. *For every k, the number of edges with multiplicity k.*
3. *The number of cycles of shortest length.*
4. *The edge-connectivity $\lambda(G)$. In particular, a lower bound for the minimum degree $\delta(G)$.*
5. *If G is simple, the number of cliques of each size.*
6. *If G is simple, the number of cycles of length three, four, and five. For cycles of length four, it is also possible to know how many of them have exactly one chord.*

Using the Tutte polynomial as defined above is somewhat awkward in the proofs to follow, therefore we will use the rank-size generating polynomial defined as

$$F_G(x,y) = \sum_{A \subseteq E} x^{r(A)} y^{|A|}.$$

The coefficients count the number of subgraphs of G with rank i and j edges. Also, note that

$$T_G(x,y) = (x-1)^{r(E)} \sum_{A \subseteq E} \left((x-1)(y-1)\right)^{-r(A)} (y-1)^{|A|}.$$

Therefore $T_G(x,y)$ also counts the number of subgraphs with rank i and j edges. Hence $T_G(x,y)$ and $F_G(x,y)$ contain the same structural information concerning G. Throughout this paper $[x^i y^j]$ will denote the coefficient of $x^i y^j$ in $F_G(x,y)$.

2 The generalized Petersen graph

Perhaps the most familiar graph is the Petersen graph. The structure of the Petersen graph can be generalized in terms of the following graph $P(m, n)$ called the generalized Petersen graph. $P(m, n)$ has $2m$ vertices which we name $\{v_1, \ldots, v_{2m}\}$ and $3m$ edges. The edges are defined as follows:

1. $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{m-1}, v_m), (v_m, v_1)\}$,
2. $\{(v_i, v_{i+m}) : i \leq m\}$, and
3. $\{(v_{i+m}, v_{j+m}) : i, j \leq m$ and $|j - i| = n\}$.

In this notation the Petersen graph would be denoted as $P(5, 2)$.

Note that for $m \notin \{3n, 4n\}$, $P(m, n)$ does not contain 3 or 4-cycles, and $P(m, n)$ is 3-regular for $m, n \in \mathbb{N}$. If $n = 2$ and $m \notin \{5, 10\}$, then $P(m, 2)$ contains m 5-cycles. These facts will be readily used.

The entirety of this paper concerns $n = 2$. The following theorem [10] yields structural information concerning the cycle spectra of $P(m, 2)$, denoted $cs(P(m, 2))$.

Theorem 2 *For m odd and $k = 2$, the cycle spectra is as follows:*

$$cs(P(5,2)) = \{5, 6, 8, 9\},$$
$$cs(P(7,2)) = [5, 14], \text{ and}$$
$$cs(P(9,2)) = \{5\} \cup \{7\} \cup [8, 18].$$

$$\text{For } m \geq 11,\ cs(P(m,2)) = \begin{cases} 5 \cup [8, 2m-1], & \text{for } m \equiv 5 \bmod 6 \\ 5 \cup [8, 2m], & \text{for } m \not\equiv 5 \bmod 6. \end{cases}$$

Another fact that will be readily used is the following lemma.

Lemma 1 *$P(m, 2)$ has no 6-cycles for $m \geq 13$.*

Proof. Theorem 2 shows this for m being odd. For m being even, the edges from (3) in the definition of $P(m, n)$ can make a cycle of size no less than seven. However it is quite easy to see that if we must utilize the other edges of $P(m, 2)$, there cannot be a 6-cycle.□

3 The T-uniqueness of $P(m, 2)$

In [8], the authors show that $C_m \times K_2$ is T-unique for $m \geq 3$. Since $C_m \times K_2$ is isomorphic to $P(m, n)$ for $n = 1$, $P(m, 1)$ is T-unique. Therefore it seems natural to consider $n = 2$. The strategy for showing the T-uniqueness of $P(m, 2)$ is to find a subgraph that uniquely contributes to $[x^i y^j]$ for some i and j, show that this subgraph uniquely contributes to $[x^i y^j]$ for a graph T-equivalent to $P(m, 2)$, and finally show that $P(m, 2)$ must necessarily follow from this information. In the process of executing this strategy, we will readily use the following theorem [8] and lemmas.

Theorem 3 *If G is a 2-connected graph and H is T-equivalent to G, then H is a 2-connected graph.*

Theorem 3 allows us to use information given in Theorem 1. Let H be T-equivalent to $P(m,2)$ for $m \geq 13$. Then H has $2m$ vertices and $3m$ edges, H does not have 3 or 4-cycles, and H has m 5-cycles. Furthermore H is a simple graph with the minimum degree being at least three.

Lemma 2 *Suppose H is T-equivalent to $P(m,2)$ for $m \geq 13$. Then H has no 6-cycles.*

Proof. $[x^5y^6]$ counts the number of 6-cycles and the number of 5-cycles with one additional edge. Note that this additional edge can not be a chord of a 5-cycle. Then the number of 6-cycles in H is $[x^5y^6] - m(3m-5)$ as Theorem 1 guarantees that H has m 5-cycles. However, from Lemma 1, $[x^5y^6] = m(3m-5)$. Therefore H has no 6-cycles.□

Lemma 3 *Suppose H is T-equivalent to $P(m,2)$ for $m \geq 13$. Then the only subgraph contributing to $[x^7y^9]$ in H is an 8-cycle with a chord. In particular, this chord is an edge of two 5-cycles.*

Proof. Consider $[x^7y^9]$. H could contain either of the following subgraphs which have rank 7 and 9 edges:

1. A subgraph consisting of an 8-cycle with a chord.
2. A subgraph containing a cycle of size less than 8 with a chord.
3. A subgraph containing two 5-cycles sharing a path of length 2.
4. A subgraph consisting of a 5 and 6-cycle sharing a path of length 2.
5. A subgraph consisting of two 6-cycles sharing a path of length 3.

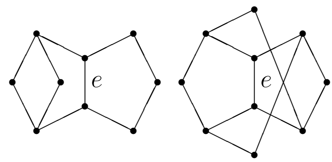
(2)-(5) can not occur since this would imply that H has either a 3, 4, or 6-cycle. This leaves (1) and therefore we can have an 8-cycle with a chord only if this chord is an edge of two 5-cycles.□

Theorem 4 $P(m,2)$ *is T-unique for $m \geq 13$.*

Proof. Let H be a graph so that $T_H(x,y) = T_{P(m,2)}(x,y)$. Since $\delta(H) \geq 3$ and $\sum_{v \in V(H)} d(v) = 6m$, H must be 3-regular. Lemma 2 guarantees that H contains no 6-cycles and Lemma 3 guarantees that H contains $2m$ 8-cycles with one chord. We will show that from this information, it necessarily follows that H is isomorphic to $P(m,2)$.

Claim 1: If e is a chord of an 8-cycle, then it can not be a chord of two 8-cycles.

Since H is 3-regular, there are only two scenarios in which e is a chord of two 8-cycles. One can check that both scenarios, given below, give 4 and 6-cycles.



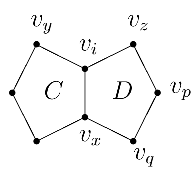
Let $E_1(H)$ be the set of edges which are chords of 8-cycles in H and let $E_2(H) = E(H) \setminus E_1(H)$. Then from the previous claim we must have $|E_1| = 2m$ and so $|E_2| = m$. For $v_i \in V(H)$ where $1 \le i \le 2m$, set

$$\begin{aligned} \delta_i &= |\{e : e \in E_2 \text{ and } e \text{ is incident to } v_i\}|, \text{ and} \\ \delta_i' &= |\{e : e \in E_1 \text{ and } e \text{ is incident to } v_i\}|. \end{aligned}$$

As H is 3-regular, $\delta_i + \delta_i' = 3$ for all i. Hence there are four ways in which we can describe the degree of v_i.

Claim 2: (1) $\delta_i = 1$ and $\delta_i' = 2$ and (2) $\delta_i = 3$ and $\delta_i' = 0$ are not possible in H.

First consider (1). Then for some i, v_i is incident to two chords $e_1 = (v_i, v_x)$ and $e_2 = (v_i, v_y)$ and one non-chord edge $e_3 = (v_i, v_z)$. Consider the 8-cycle with the chord e_1 and let C and D be the two 5-cycles sharing e_1. Without loss of generality, let e_2 and e_3 be edges on C and D as shown in the following graph.



Since e_2 is a chord, there must be a path from v_y to a vertex on D yielding a 5-cycle. If there is a path of length 1 between v_y and v_p or v_q, then H will contain a 4-cycle. If there is a path of length 2 between v_y and v_p or v_q, then H will contain a 6-cycle. If there is a path of length 3 between v_y and v_z, then e_3 is a chord.

Next, consider (2). Then for some i, v_i is incident to three non-chord edges. If every edge in H is an edge of a 5-cycle, then it follows that (2) is not possible since every pair of edges incident to v_i are in a 5-cycle and so one of these edges is a chord of an 8-cycle. Otherwise we get a subgraph with forbidden cycles. Let (e_i, C_5) denote the number of 5-cycles that contain the edge e_i for $1 \le i \le 3m$. Since there are m 5-cycles,

$$\sum_{i=1}^{3m} (e_i, C_5) = 5m.$$

We know $|E_1| = 2m$, and so, without loss of generality, we set $(e_1, C_5) = (e_2, C_5) = \ldots = (e_{2m}, C_5) = 2$. This leaves

$$\sum_{i=2m+1}^{3m} (e_i, C_5) = m$$

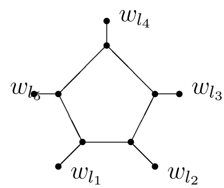
and so it must follow that for $2m+1 \leq i \leq 3m$, $(e_i, C_5) = 1$ as an edge can not be in two 5-cycles unless it is a chord of an 8-cycle. Therefore we have the possibilities of $\delta_i = 2$ and $\delta_i' = 1$ and of $\delta_i = 0$ and $\delta_i' = 3$.

Let F_i be the subgraph of H induced by the edges E_i for $i = 1, 2$.

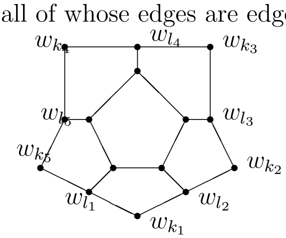
Claim 3: F_1 is composed of an m-cycle with vertices $v_1, \ldots, v_m$ along with the edges (v_i, v_{m+i}) for $i = 1, 2, \ldots, m$.

We start by showing that each 5-cycle of H contains exactly one edge from F_2. Consider a 5-cycle of H with vertices $w_1, w_2, \ldots, w_5$. Suppose that $(w_1, w_2), (w_2, w_3)$ are edges of F_2. Then $\delta_i'(w_1) = \delta_i'(w_2) = \delta_i'(w_3) = 1$ and let $(w_2, w_x) \in E(F_1)$. Since (w_2, w_x) must be an edge of a 5-cycle, then either (w_1, w_2) or (w_2, w_3) must be an edge of this 5-cycle. However, in order to avoid 4 and 6-cycles, either (w_1, w_2) or (w_2, w_3) will be a chord of an 8-cycle. Hence (w_1, w_2) and (w_2, w_3) can not both be edges of F_2. Therefore each 5-cycle of H can have at most 2 edges from F_2.

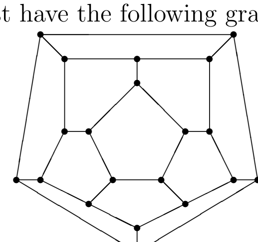
If (w_1, w_2) and (w_3, w_4) are edges of F_2, then (w_2, w_3) is an edge of F_1 and thus must be a component in F_1. Since there are m 5-cycles in H and m edges in F_2, F_1 must then contain a 5-cycle. Therefore each vertex of this 5-cycle must be a vertex of degree 3 in F_1 as shown below.



Now each edge of this 5-cycle is a chord of an 8-cycle in H. Therefore it must be that w_{l_i} and $w_{l_{i+1}}$ are adjacent to a common vertex for $i = 1, 2, 3, 4, 5$ where we set $w_{l_6} = w_{l_1}$. This common vertex can not be w_{l_j} for $j \neq i$ and $j \neq i+1$, otherwise we have included 4 and 6-cycles. Since no 5-cycle can have two incident edges from F_2, we have the following graph, all of whose edges are edges of F_1.



Now w_{k_i} can not be adjacent to w_{k_j} for any i or j, otherwise we include 3 and 6-cycles. Furthermore, w_{k_i} and w_{k_j} can not be adjacent to a common vertex other than the ones given in the graph above. Again, each of the edges in this graph are edges of F_1. Therefore we must have the following graph.



The vertices in the graph above all have degree 3 and we still have the component (w_2, w_3). But Theorem 1 guarantees that H is connected. So it must be that every 5-cycle of H contains exactly one edge from F_2. Therefore, as H is connected, F_1 is connected since the intersection of a cycle and an edge-cut can not have size one. Also, since F_1 has $2m$ vertices and $2m$ edges, it has only one cycle as a subgraph. Suppose that this cycle, C, has vertex set $\{v_1, \ldots, v_p\}$ where $7 \leq p \leq m$ and edge set $E(C) = \{(v_i, v_{i+1}) \cup (v_p, v_1) : \ 1 \leq i \leq p-1\}$. Let $\{v_q, \ldots, v_{2m}\}$ be the leaves of F_1 where $q \geq p$ and note that if $p < m$, then $q > p+1$. Furthermore, let T_i be the tree of $F_1 \setminus E(C)$ including v_i as a leaf for $1 \leq i \leq p$. Consider an edge (v_i, v_{i+1}) of C and let v_x be a leaf belonging to T_i so that $q \leq x \leq 2m$. Suppose that T_i has at least three leaves. In order for (v_i, v_{i+1}) to be a chord of an 8-cycle, we may assume that v_x is adjacent to a leaf, v_y, in T_j for $j \neq i$ where (v_x, v_y) is an edge of F_2. But (v_i, v_{i+1}) is an edge of a 5-cycle only if $j = i+1$, and T_{i+1} consists of one edge. Therefore (v_i, v_{i+1}) can not be a chord of an 8-cycle unless we include 4 or 6-cycles. Hence T_i contains two leaves and so consists of exactly one edge for all i which necessarily implies that $p = m$ and $q = p+1$. This proves claim 3.

In order to ensure that each edge in F_2 is in exactly one 5-cycle and that each edge of F_1 is a chord of an 8-cycle, H must be isomorphic to $P(m, 2)$.□

The reason for restricting m to values larger than 12 is because for m smaller, $P(m, 2)$ has 3,4, and 6-cycles. We will investigate the remaining cases individually. For each case, H is T-equivalent to $P(m, 2)$ for $m \leq 12$ and so H is 2-connected and 3-regular.

Theorem 5 *$P(m, 2)$ is T-unique for $m \leq 12$.*

Case 1.

One can easily show that for $m \leq 5$, these are T-unique. For $m = 9, 11$, $P(m, 2)$ does not contain 3,4, or 6-cycles so is T-unique from above.

Case 2. $m = 6$.

H has two triangles, no 4-cycles, and six 5-cycles which do not contain a chord. Furthermore, H has six 6-cycles with a chord since the number of 6-cycles with a chord is $[x^5 y^7] - 12$.

We begin with the two triangles. These two triangles must be vertex and edge disjoint. Also note that the edges in the two triangles must be chords of 6-cycles. Consider the subgraph induced by the remaining six vertices. This subgraph must contain at least six edges. Therefore, it must contain a cycle. Clearly this cycle cannot have size three or four. It cannot have size five either since H has exactly six 5-cycles of which these must appear in the six 6-cycles with a chord. Hence this cycle must be a 6-cycle. We are left then with H being isomorphic to $P(6, 2)$ since each edge in the two triangles must be a chord of a 6-cycle.

Case 3. $m = 7$.

We begin by showing that H has sixteen 7-cycles. Consider $[x^6 y^7]$. This coefficient counts the number of 7-cycles, the number 6-cycles with an additional edge, and the number of 5-cycles with two additional edges. Note that in H, 5,6, and 7-cycles do not contain chords. Since we know that $P(7,2)$ and H have seven 5-cycles and seven 6-cycles, H has the same number of 7-cycles as $P(7,2)$. Hence H has sixteen 7-cycles.

Let F, a subgraph of H, be a 7-cycle in which two of the vertices are adjacent to a vertex outside of the cycle. Then F must consist of a 5 and 6-cycle sharing a path of length two. Thus for each subgraph F of H, we get a 5 and 6-cycle. If each seven cycle of H is an F subgraph, then H would contain at least eight 5-cycles since at most two F subgraphs can share a 5-cycle. Therefore H must contain a 7-cycle so that each vertex of the cycle is adjacent to distinct vertices outside the cycle. The remaining edges of H must form a 7-cycle. To guarantee that H has seven 5-cycles H must be isomorphic to $P(7,2)$ or $P(7,3)$ but $P(7,3)$ has fourteen 6-cycles.

Case 4. $m = 8$.

H has no triangles, two disjoint 4-cycles, eight 5-cycles, no 6-cycles and H has eight 7-cycles with a chord which forms a 4-cycle and a 5-cycle joined at an edge. Furthermore, an edge is a chord of only one 7-cycle otherwise we would have a subgraph contributing to $[x^5y^7]$, but this coefficient must be zero. Therefore we are forced to construct H as $P(8,2)$ following the same argument as in case 2.

Case 5. $m = 10$.

H has no triangles, no 4-cycles, twelve 5-cycles, no 6-cycles and so H has thirty 8-cycles with a chord. As in Theorem 4, an edge can be a chord of only one 8-cycle. Therefore every edge in H is a chord of an 8-cycle. Then the edges of two disjoint 5-cycles is an edge of another 5-cycle. We are then forced, as before, to construct $P(10,2)$.

Case 6. $m = 12$.

This will follow from an argument similar to case 2,4, or 5. □

Using a similar argument to the one given in Theorem 4, one could show that $P(m,3)$ is T-unique for $m \geq 22$. All that is needed is to show that $P(m,3)$ does not contain 7-cycles for $m \geq 22$ and that H being T-equivalent to $P(m,3)$ has the same number of 10-cycles with a chord. However, for values of n larger than 3, this argument would be difficult to generalize because $P(m,n)$ contains 8-cycles for all values of m. Despite this, it should still be true that $P(m,n)$ is T-unique for all values of n and m. In addition, it is not hard to conceive that there is an argument for $P(m,n)$ being T-unique with no bound on m.

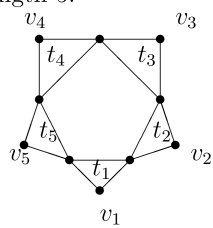
4 The T-uniqueness of $L\left(P(m,2)\right)$

Let G be a graph. The line graph of G, denoted $L(G)$, is the graph obtained from G by replacing the edges with vertices and (e_i, e_j) is an edge of $L(G)$ if and only if e_i and e_j are incident in G. The following theorem characterizes all line graphs [4].

Theorem 6 *A graph G is a line graph if and only if the edges of G can be partitioned into cliques such that no vertex of G lies in more than two of the cliques. Furthermore, if $M_1, \ldots, M_r$ are cliques of G such that every vertex belongs to exactly* *two of them, then* $G = L(G_0)$*, where* $V(G_0) = \{m_1, \ldots, m_r\}$ *and an edge between* m_i *and* m_j *if* M_i *meets* M_j*.*

In this section we will investigate the T-uniqueness of the line graph of $P(m, 2)$. The T-uniqueness of line graphs has recently been studied in [9]. Mier and Noy show that the line graph of a regular complete t-partite graph, the line graph of the complete bipartite graph, and the line graph of the complete graph are T-unique [9].

We define a cycle of triangles of length j to be the cycle C_j with each vertex replaced by a triangle and each edge replaced by two triangles joined at a vertex. Furthermore, let the vertices of degree 2 be labeled as $v_1, \ldots, v_j$ and let the triangles be labeled as $t_1, \ldots, t_j$ where t_i is the triangle containing vertex v_i. The following graph is a cycle of triangles of length 5.



The line graph of $P(m, 2)$ contains many lengths of cycles of triangles. Indeed, each vertex in $P(m, 2)$ becomes a triangle in $L(P(m, 2))$, and two adjacent vertices become two triangles joined at a vertex. So the pattern in $P(m, 2)$ is repeated in $L(P(m, 2))$ in the context of triangles. Note that $L(P(m, 2))$ is 4-regular with $3m$ vertices and $6m$ edges. It has $2m$ triangles, no 4-cycles with a chord but has a 4-cycle if $m \in \{4, 8\}$, m 5-cycles, and has $5m$ 6-cycles with a chord since each 5-cycle is contained in a cycle of triangles of length 5.

We continue with the following theorem [9].

Theorem 7 *Let* G *be a* d*-regular* d*-edge connected graph on* n *vertices, and assume that* $d \geq 3$ *and, if* $d = 3$*, then* G *is triangle-free. If a graph* H *is* T*-equivalent to* $L(G)$*, then* $H = L(G_0)$ *where* G_0 *is a* d*-regular connected graph on* n *vertices.*

Corollary 1 *If* H *is* T*-equivalent to* $L(P(m, 2))$ *for* $m \notin \{3, 6\}$*, then* $H = L(G)$*, where* G *is 3-regular, connected, and has* $2m$ *vertices.*

*P*roof. Since $P(m, 2)$ is 3-regular, 3-edge connected, and triangle-free on $2m$ vertices for $m \notin \{3, 6\}$, then by Theorem 7, $H = L(G)$ where G is a 3-regular connected graph on $2m$ vertices. □

Lemma 4 *If* H *is* T*-equivalent to* $L(P(m, 2))$ *for* $m \geq 13$*, then* $H = L(G)$ *where* G *has no 6-cycles.*

Proof. Let H be T-equivalent to $L(P(m, 2))$ for $m \geq 13$. Then H is 4-regular with no 4-cycles and $H = L(G)$ where G is 3-regular. Suppose that G does contain a 6-cycle. Note that a 6-cycle in G can not have a chord. Otherwise, H contains a cycle of triangles of length 3 or 4 but H can not contain 4-cycles. This then implies that H contains a cycle of triangles of length 6. However if we can show that every 6-cycle in H must contain a chord, then G can not contain a 6-cycle.

Claim: Every 6-cycle in H has a chord. Two triangles in H are either edge and vertex disjoint or they share a vertex. Therefore $(e_i, C_3) \leq 1$ for $1 \leq i \leq 6m$ and since $\sum_{i=1}^{6m}(e_i, C_3) = 6m$, $(e_i, C_3) = 1$ for each i. This implies that each of the $3m$ vertices are in precisely two triangles and so each edge of a 5-cycle is also an edge of a triangle. Since Theorem 1 guarantees that H has m 5-cycles, then H has $5m$ 6-cycles with a chord. Consider $[x^5y^6]$. The subgraphs that contribute to this coefficient are the 6-cycles and the 5-cycles with one additional edge. By examining $L(P(m,2))$ for $m \geq 13$, one can show that $[x^5y^6] = 6m^2$. So the number of 6-cycles in H is $[x^5y^6] - (6m^2 - 5m) = 5m$. Hence each 6-cycle in H has a chord. □

Lemma 5 *If H is T-equivalent to $L(P(m,2))$ for $m \geq 13$, then $H = L(G)$ where G has $2m$ 8-cycles with a chord. In particular, this chord is an edge of two 5-cycles.*

Proof. Let H be T-equivalent to $L(P(m,2)$ for $m \geq 13$. Then H has $3m$ vertices and $6m$ edges. Consider two 5-cycles, C and D, in H.

Claim: Either C and D share exactly one vertex or they are vertex and edge disjoint. Since every edge of H must be an edge of a triangle and no 5-cycle can contain a chord, this guarantees that C and D can not share more than one edge. This also includes C and D sharing more than one vertex but being edge disjoint. Suppose that C and D share one edge, and let this edge be denoted (u, v). Then as each edge is an edge of a triangle, there must be a vertex w outside the vertices of C and D so that (u, w) and (v, w) are edges of H. Let (u, x_1) be an edge of C and (u, y_1) an edge of D. In order for (u, x_1) to be an edge of a triangle, x_1 must be adjacent to w, v, or y_1. If x_1 is adjacent to w then we obtain a 4-cycle and since there are four edges incident to v, it must be that x_1 is adjacent y_1. We argue the same for (v, x_2) and (v, y_2) being edges of C and D respectively. Therefore there is a 6-cycle including the edges (x_1, y_1) and (x_2, y_2) without a chord. Hence (u, v) can not be an edge of a triangle and so we have proved our claim.

Furthermore, since H has 3m vertices and is 4-regular, there must be $2m$ vertices each belonging to two 5-cycles. For every pair of 5-cycles sharing a vertex we obtain a cycle of triangles of length 8 so that for t_i and t_{i+4}, $v_i = v_{i+4}$ for some i. Therefore, by Theorem 7 and Theorem 6, G contains $2m$ 8-cycles with a chord and this chord is an edge of two 5-cycles. □

Theorem 8 *For $m \geq 13$, $L(P(m,2))$ is T-unique.*

Proof. Let H be a graph T-equivalent to $L(P(m,2))$. Then $H = L(G)$ where G is 3-regular on $2m$ vertices and so G has $3m$ edges. Furthermore G has no triangles, no 4-cycles, and no 6-cycles and G has $2m$ 8-cycles with a chord. Then from this information, it necessarily follows from Theorem 4 that G is isomorphic to $P(m,2)$. Hence $L(P(m,2))$ is T-unique.□

Given the previous conjecture that $P(m,n)$ is T-unique for all values of n and m, we additionally conjecture that $L(P(m,n))$ is T-unique. For a graph G in general, the authors show in [9] that $L(G)$ need not be T-unique. In fact they show that non-isomorphic T-equivalent graphs can arise from d-regular graphs. However, we ask the following less general question. If G is T-unique, then is $L(G)$ T-unique?

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