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Some V(12,t) vectors and designs from difference and quasi-difference matrices

R. Julian R. Abel

School of Mathematics and Statistics
University of New South Wales
Sydney, 2052
Australia
rjabel@unsw.edu.au

Abstract

In this paper we provide examples of $\mathrm{V}(12, t)$ vectors for $800 \leq 12t+1 \leq 5000$, and the 3 unknown $\mathrm{V}(m, t)$ vectors with $m = 8, 9$. We also provide other examples of transversal and incomplete designs, coming from difference and quasi-difference matrices. These include a $\mathrm{TD}(6, 34)$, a $\mathrm{TD}(7, v)$ for $v = 28, 42, 44, 52, 54, 62$, an updated list of unknown $TD_\lambda(9, v)$ for $\lambda > 1$ and 2 ITDs which give $(v, 6, 1)$ BIBDs for $v \in \{496, 526\}$.

1 Introduction

A *transversal design* $TD_\lambda(k, v)$ consists of a set V of kv elements partitioned into *groups* of size v plus a collection of k-element subsets of V called *blocks* such that (1) each block contains exactly one element from each group and (2) any two elements in different groups appear in exactly λ blocks. The parameter λ can be omitted if it equals 1.

An ITD or incomplete transversal design, $TD_\lambda(k, v) - TD_\lambda(k, u)$ is a $TD_\lambda(k, v)$ missing a subdesign $TD_\lambda(k, u)$. Such a design can exist even if the missing $TD_\lambda(k, u)$ does not; however, if it does, it can be used in conjunction with the incomplete TD to obtain a $TD_\lambda(k, v)$.

Let $v, k, h, \lambda_1, \lambda_2$ be integers with $v, k > 0$ and $h, \lambda_1, \lambda_2 \geq 0$. A $(v-h, h, \lambda_1, \lambda_2; k)$ quasi-difference matrix (or QDM) over an abelian group G of size $v-h$ is an array A with k rows, $c = \lambda_1(v+h-1) + \lambda_2$ columns such that (1) each entry of A is either an element of G or blank; (2) no column of A has more than one blank entry (3) for any two rows i, j of A, each non-zero element of G occurs λ_1 times and zero occurs λ_2 among the differences $A_{i,t} - A_{j,t} : t = 1 \ldots c$, $A_{i,t}, A_{j,t}$ not blank. If $h = 0$ and $\lambda_1 = \lambda_2$, then the simpler notation (v, k, λ_1) difference matrix (or DM) can be is used. Also, it is well known that if $\lambda_2 \le \lambda_1$, then existence of a $(v-h, h, \lambda_1, \lambda_2; k)$ QDM implies existence of a $TD_{\lambda_1}(k,v) - TD_{\lambda_1}(k,h)$.

In this paper, we obtain a number of TDs and ITDs using difference or quasi-difference matrices. Two of them also give GDDs which were used to obtain $(v, 6, 1)$ BIBDs for $v = 496, 526$ in [6].

Our first new QDMs will come from $\mathrm{V}(m,t)$ vectors. Let $q = mt+1$ be a prime power, and let x be a primitive element in $\mathrm{GF}(q)$. One can define multiplicative cosets C_s in $\mathrm{GF}(q)$ by $C_s = (y : y$ is of the form x^n with $n \equiv s \; mod \; m)$.

In [23], Wilson defined a vector $V = (v_1, v_2, \ldots, v_{m+1})$ with entries from $\mathrm{GF}(mt+1)$ to be a $V(m,t)$ vector if for every s, $1 \le s \le (m+2)/2$, the set of m differences $\{v_i - v_j : 1 \le i, j \le m+1, i-j \equiv s \pmod{m+2}\}$ contains exactly one element from each of the cosets C_s. For instance, $(0,1,3,6)$ is a $\mathrm{V}(3,2)$ vector over $\mathrm{GF}(7)$. Here, using $x = 5$ as a primitive element in $\mathrm{GF}(7)$, the sets $\{1-0 \in C_0, 3-1 \in C_1, 6-3 \in C_2\}$ and $\{3-0 \in C_2, 6-1 \in C_1, 0-6 \in C_0\}$ both contain 1 element from each C_s.

If such a vector exists, then a $(q, m+2; 1, 0; t)$-QDM can be obtained as follows: Start with a single column whose first entry is blank, and whose other entries are $v_1, v_2, \ldots, v_{m+1}$ (in that order). Multiply this column by the m'th roots of unity in $\mathrm{GF}(q)$ (i.e., the elements of C_0) and form $m(m+2)$ columns by taking the $m+2$ cyclic shifts of each of these columns.

2 QDMs from V(12,t) vectors

Recent results ([1], [12], [15], [14], [17], [18], [24]) have established that a $\mathrm{V}(m,t)$ vector exists when m, t are not both even in the following two cases:

1. $2 \le m \le 8$, $q = mt+1$ is a prime power, except for $(m,t) \in \{(2,1),(3,5),(7,9)\}$ and possibly for $m = 8$, $mt+1 \in \{3^6, 3^{10}\}$ or $m = 9$, $mt+1 = 5^6$;
2. $10 \le m \le 11$, $q = mt+1$ is prime, $t \ge m-1$, $q < 5000$ and $(m,t) \notin \{(9,8),(11,18)\}$.

However increasing m by even 1 considerably increases the amount of computer time required to obtain $\mathrm{V}(m,t)$ vectors. In addition for fixed m, provided t is not too large, the amount of time required to find $\mathrm{V}(m,t)$ vectors increases considerably as t decreases; for several values of $t < 140$ in the table below, the given $\mathrm{V}(12,t)$s took more than $20,000$ hours of CPU time to produce. By contrast, none of the $\mathrm{V}(11,t)$s in [1] for $t \ge 30$ took more than $2,000$ hours of CPU.

Below, we give $\mathrm{V}(12,t)$ vectors for odd t such that $q = 12t+1$ is a prime in the range $[800, 5000]$ (i.e. for $66 < t < 416$). We have also managed to establish that there is no $\mathrm{V}(12,t)$ vector for all odd values of $t \le 15$ such that $12t+1$ is prime, i.e.

$t = 1, 3, 5, 9, 13$ or 15. The existence of V(12, t) vectors for t odd and $15 < t < 66$ remains an open problem.

These vectors are given in the table below. In each case, x is a primitive element in GF(q).

t	x	q	V(12, t) vector
69	2	829	(0 1 527 449 471 497 677 20 778 88 366 721 753)
71	2	853	(0 1 645 446 813 543 413 7 55 177 468 503 646)
73	2	877	(0 1 607 719 837 496 240 645 184 829 451 830 770)
83	7	997	(0 1 627 898 836 939 742 42 847 531 173 607 361)
85	10	1021	(0 1 778 1000 913 819 961 456 507 186 509 495 300)
89	6	1069	(0 1 602 894 827 661 350 647 304 47 430 533 550)
91	5	1093	(0 1 777 1054 855 892 792 134 224 740 240 898 631)
93	2	1117	(0 1 601 1004 872 557 599 819 381 248 270 1091 49)
101	2	1213	(0 1 787 1049 818 1064 288 346 464 958 1188 340 1192)
103	2	1237	(0 1 770 1027 806 1082 515 436 1096 1060 57 1135 1144)
115	2	1381	(0 1 747 1179 873 484 969 692 679 153 1237 1110 616)
119	6	1429	(0 1 701 1225 834 515 367 727 1349 407 891 1189 153)
121	2	1453	(0 1 713 1265 848 421 998 69 874 1126 693 467 1164)
129	2	1549	(0 1 623 1170 824 450 1099 418 948 177 207 797 59)
133	11	1597	(0 1 648 1157 822 371 407 180 1120 898 342 548 117)
135	2	1621	(0 1 712 1253 844 623 943 992 191 845 299 1381 611)
139	2	1669	(0 1 627 1216 711 489 642 904 733 1246 96 1617 12)
141	2	1693	(0 1 447 522 967 763 1035 344 93 561 1137 523 828)
145	2	1741	(0 1 426 582 937 534 1538 1606 1148 1436 191 1406 823)
149	6	1789	(0 1 420 509 957 593 835 1031 1502 319 1552 1047 993)
155	2	1861	(0 1 300 482 962 638 1207 1682 885 211 1838 1244 531)
161	5	1933	(0 1 455 318 952 400 470 584 1368 292 678 1138 383)
169	2	2029	(0 1 425 326 951 1211 1881 1063 1631 1363 1554 665 1600)
171	2	2053	(0 1 432 319 933 688 549 63 2002 1702 653 1081 1813)
185	2	2221	(0 1 404 324 935 605 366 360 178 221 533 1940 30)
189	2	2269	(0 1 303 329 957 866 2180 1899 597 2209 1186 994 1301)
191	2	2293	(0 1 491 527 939 377 1685 1735 1967 1176 391 2192 681)
195	7	2341	(0 1 331 313 934 384 2105 479 1546 86 184 1127 1822)
199	2	2389	(0 1 377 524 946 560 316 1591 2036 273 1841 2091 713)
203	2	2437	(0 1 324 312 933 341 547 68 39 1008 561 1372 1300)
213	2	2557	(0 1 343 312 933 378 229 60 1179 1781 1960 66 536)
223	2	2677	(0 1 463 316 933 413 970 1083 2322 491 1226 1809 560)
229	6	2749	(0 1 338 312 933 380 401 2398 612 1279 1514 268 528)
233	5	2797	(0 1 405 314 934 398 1053 310 2254 2250 2652 1300 1079)
243	7	2917	(0 1 486 314 933 375 697 151 1964 1623 1590 1756 1152)
253	2	3037	(0 1 322 312 933 395 1047 12 176 1859 881 1220 2465)
255	2	3061	(0 1 463 316 938 345 360 2537 2648 2270 789 2959 2796)

t	x	q	V$(12,t)$ vector
259	11	3109	(0 1 486 314 933 350 575 1962 2347 750 3054 2719 1841)
265	7	3181	(0 1 333 312 933 343 759 1754 2650 1633 2479 2718 1164)
269	6	3229	(0 1 432 312 938 345 567 2441 966 1935 470 2105 3043)
271	2	3253	(0 1 463 313 933 356 453 2869 793 748 2116 3126 2839)
275	6	3301	(0 1 477 313 943 358 474 2312 1258 52 1452 2370 260)
281	5	3373	(0 1 483 313 933 387 418 961 1586 766 2937 275 2569)
289	2	3469	(0 1 474 313 943 367 963 3147 2157 238 12 1610 2189)
293	2	3517	(0 1 423 335 945 397 235 2878 1793 2484 2440 503 1609)
295	7	3541	(0 1 428 337 931 406 360 1978 68 375 721 2390 2465)
301	2	3613	(0 1 436 351 924 367 1196 265 2527 720 664 105 250)
303	2	3637	(0 1 487 572 946 462 2646 2616 1249 3143 21 2537 2128)
309	2	3709	(0 1 417 327 944 341 1924 1975 2308 1234 1658 1829 1606)
311	2	3733	(0 1 435 557 937 371 267 428 1289 3355 2948 3030 861)
321	2	3853	(0 1 319 325 952 364 674 2128 643 393 1025 619 868)
323	2	3877	(0 1 445 344 920 365 567 3483 3364 1240 344 2683 3070)
335	2	4021	(0 1 478 557 969 462 1587 1457 2552 2575 2420 168 924)
341	2	4093	(0 1 498 362 954 440 584 421 3867 3964 404 664 2233)
355	2	4261	(0 1 415 329 927 512 615 2336 127 2245 2250 2272 1888)
363	2	4357	(0 1 541 368 971 370 297 555 148 4195 1197 1527 211)
379	6	4549	(0 1 424 545 948 415 378 1181 2984 3458 3288 3888 74)
383	5	4597	(0 1 477 534 964 441 246 972 2504 3957 3101 4366 2168)
385	2	4621	(0 1 543 334 943 531 793 1852 538 4231 4492 580 3816)
399	2	4789	(0 1 487 571 964 391 300 4515 2211 3063 2771 2586 1056)
401	2	4813	(0 1 442 543 964 514 567 763 3816 3621 2124 1092 1456)
405	11	4861	(0 1 433 552 963 385 684 63 4243 3494 3500 560 4611)
409	6	4909	(0 1 426 541 954 411 708 1875 2058 2443 1913 2924 3673)
411	2	4933	(0 1 430 558 963 397 372 492 2502 3948 18 1191 3761)
413	2	4957	(0 1 436 546 977 467 242 3695 682 483 3026 461 1334)

We conclude this section by giving examples of the unknown V(m,t)s mentioned at the start of this section for $m = 8, 9$. For $m = 8$, $8t + 1 = 3^6$, if we take x to be a primitive element of GF(3^6) satisfying $x^6 = 2x + 1$, then we have $V(8, 91) = (0, 1, x^{163}, x^{376}, x^{53}, x^{140}, x^{122}, x^{553}, x^{378})$. For $m = 8$, $8t+1 = 3^{10}$, if we take x to be a primitive element of GF(3^{10}) satisfying $x^{10} = x^7+x^6+x^5+x^4+x^3+x^2+2x+1$, then we have $V(8, 7381) = (0, 1, x^{255}, x^{619}, x^{51}, x^{52}, x^{197}, x^{1290}, x^{2383})$. For $m = 9$, $9t + 1 = 5^6$, if we take x to be a primitive element of GF(5^6) satisfying $x^6 = 4x^4+4x^3+4x^2+4x+2$, then we have $V(9, 1736) = (0, 1, x^{202}, x^{123}, x^{22}, x^{21}, x^{47}, x^{4181}, x^{607}, x^{9408})$. So we now have:

Theorem 2.1 *If $2 \le m \le 9$, $mt + 1$ is a prime power, $t \ge m - 1$ and m, t are not both even, then a V(m,t) vector exists, except for $(m,t) \in \{(3,5), (7,9)\}$.*

3 Other QDMs which cyclically permute the rows

ITDs that come from V(m,t) vectors have a very large automorphism group. For most other known ITDs the automorphism group is notably smaller, but finding such ITDs is often still feasible when this automorphism group is large enough. The next lemma provides some examples of ITDs with $k=7$ and an automorphism that cyclically permutes the groups of the ITD. Other examples of this can be found in [2].

Lemma 3.1 *There exists a* $(v-h,\ h,\ 1,\ 1;\ 7)$*-QDM and hence also a* $TD(7,v)-TD(7,h)$ *in the following cases:*

1. $(v,h)\in\{(45,5),\ (50,6),\ (52,4),\ (55,8),\ (55,9),\ (59,5),\ (62,8)\}$.
2. $v=6h$ *and* $5\le h\le 15$.

These QDMs are over Z_{v-h}. For $(v,h)\neq(90,15)$ or $(30,5)$, we give two matrices A_1, A_2; the required QDMs are obtained by replacing each column of $[A_1|-A_1|A_2]$ by its 7 cyclic shifts. Also, in 3 cases ($(v,h)=(45,5),(55,9)$ and (59,5)) a column of zeros should be added.

$$(v,h)\ =\ (45,5):\quad A_1=\begin{pmatrix}0&-&-\\22&0&0\\7&26&2\\34&17&38\\18&29&33\\21&35&32\\32&5&25\end{pmatrix}\qquad A_2=\begin{pmatrix}-\\0\\23\\4\\24\\3\\20\end{pmatrix}.$$

$$(v,h)\ =\ (50,6):\quad A_1=\begin{pmatrix}0&-&-\\3&0&0\\36&1&23\\28&17&35\\4&35&37\\39&20&12\\5&7&6\end{pmatrix}\qquad A_2=\begin{pmatrix}-&-\\0&0\\17&14\\24&18\\2&18\\39&14\\22&0\end{pmatrix}.$$

$$(v,h)\ =\ (52,4):\quad A_1=\begin{pmatrix}0&0&-\\41&12&0\\39&26&45\\6&9&18\\1&25&29\\43&29&30\\18&19&2\end{pmatrix}\qquad A_2=\begin{pmatrix}-&-\\0&0\\13&26\\4&34\\28&34\\37&26\\24&0\end{pmatrix}.$$

$$(v,h) \;=\; (55,8): \quad A_1 = \begin{pmatrix} - & - & - & - \\ 0 & 0 & 0 & 0 \\ 1 & 7 & 12 & 18 \\ 3 & 45 & 18 & 4 \\ 6 & 17 & 3 & 31 \\ 43 & 9 & 46 & 10 \\ 26 & 25 & 33 & 32 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 \\ 5 \\ 16 \\ 40 \\ 40 \\ 16 \\ 5 \end{pmatrix}.$$

$$(v,h) \;=\; (55,9): \quad A_1 = \begin{pmatrix} - & - & - & - \\ 0 & 0 & 0 & 0 \\ 21 & 28 & 44 & 38 \\ 22 & 17 & 15 & 12 \\ 37 & 39 & 12 & 18 \\ 41 & 25 & 2 & 30 \\ 36 & 41 & 9 & 39 \end{pmatrix} \qquad A_2 = \begin{pmatrix} - \\ 0 \\ 33 \\ 6 \\ 29 \\ 10 \\ 23 \end{pmatrix}.$$

$$(v,h) \;=\; (59,5): \quad A_1 = \begin{pmatrix} 0 & 0 & - & - \\ 36 & 48 & 0 & 0 \\ 2 & 45 & 47 & 39 \\ 48 & 40 & 21 & 20 \\ 38 & 28 & 12 & 45 \\ 1 & 15 & 10 & 7 \\ 33 & 4 & 50 & 31 \end{pmatrix} \qquad A_2 = \begin{pmatrix} - \\ 0 \\ 23 \\ 22 \\ 49 \\ 50 \\ 27 \end{pmatrix}.$$

$$(v,h) \;=\; (62,8): \quad A_1 = \begin{pmatrix} 0 & - & - & - \\ 17 & 0 & 0 & 0 \\ 29 & 28 & 35 & 23 \\ 36 & 50 & 5 & 33 \\ 31 & 2 & 43 & 30 \\ 16 & 47 & 44 & 51 \\ 41 & 11 & 1 & 17 \end{pmatrix} \qquad A_2 = \begin{pmatrix} - & - \\ 0 & 0 \\ 2 & 8 \\ 6 & 22 \\ 33 & 22 \\ 29 & 8 \\ 27 & 0 \end{pmatrix}.$$

$$(v,h) \;=\; (36,6): \quad A_1 = \begin{pmatrix} - & - \\ 0 & 0 \\ 22 & 24 \\ 11 & 6 \\ 21 & 20 \\ 4 & 16 \\ 9 & 17 \end{pmatrix} \qquad A_2 = \begin{pmatrix} - & - \\ 0 & 0 \\ 9 & 3 \\ 16 & 5 \\ 1 & 5 \\ 24 & 3 \\ 15 & 0 \end{pmatrix}.$$

$$(v,h) \;=\; (42,7): \quad A_1 = \begin{pmatrix} - & - & - \\ 0 & 0 & 0 \\ 18 & 11 & 5 \\ 26 & 10 & 30 \\ 20 & 3 & 33 \\ 5 & 25 & 24 \\ 17 & 4 & 22 \end{pmatrix} \qquad A_2 = \begin{pmatrix} - \\ 0 \\ 4 \\ 23 \\ 23 \\ 4 \\ 0 \end{pmatrix}.$$

$$(v,h) \;=\; (48,8): \quad A_1 = \begin{pmatrix} - & - & - \\ 0 & 0 & 0 \\ 22 & 29 & 31 \\ 10 & 37 & 18 \\ 34 & 38 & 12 \\ 9 & 8 & 5 \\ 32 & 4 & 7 \end{pmatrix} \qquad A_2 = \begin{pmatrix} - & - \\ 0 & 0 \\ 19 & 3 \\ 24 & 17 \\ 4 & 17 \\ 39 & 3 \\ 20 & 0 \end{pmatrix}.$$

$$(v,h) \;=\; (54,9): \quad A_1 = \begin{pmatrix} - & - & - & - \\ 0 & 0 & 0 & 0 \\ 1 & 27 & 16 & 7 \\ 24 & 40 & 1 & 35 \\ 10 & 30 & 22 & 44 \\ 5 & 18 & 14 & 33 \\ 30 & 16 & 33 & 27 \end{pmatrix} \qquad A_2 = \begin{pmatrix} - \\ 0 \\ 3 \\ 7 \\ 7 \\ 3 \\ 0 \end{pmatrix}.$$

$$(v,h) \;=\; (60,10): \quad A_1 = \begin{pmatrix} - & - & - & - \\ 0 & 0 & 0 & 0 \\ 17 & 40 & 46 & 44 \\ 48 & 5 & 47 & 32 \\ 37 & 33 & 42 & 2 \\ 11 & 31 & 28 & 20 \\ 27 & 39 & 1 & 17 \end{pmatrix} \qquad A_2 = \begin{pmatrix} - & - \\ 0 & 0 \\ 37 & 43 \\ 28 & 14 \\ 3 & 14 \\ 12 & 43 \\ 25 & 0 \end{pmatrix}.$$

$$(v,h) \;=\; (66,11): \quad A_1 = \begin{pmatrix} - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 \\ 45 & 35 & 51 & 46 & 28 \\ 30 & 17 & 34 & 3 & 9 \\ 5 & 6 & 35 & 36 & 1 \\ 11 & 22 & 12 & 38 & 53 \\ 4 & 8 & 41 & 43 & 32 \end{pmatrix} \qquad A_2 = \begin{pmatrix} - \\ 0 \\ 24 \\ 37 \\ 37 \\ 24 \\ 0 \end{pmatrix}.$$

$$(v,h) = (72,12): \quad A_1 = \begin{pmatrix} - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 \\ 25 & 53 & 54 & 46 & 57 \\ 4 & 15 & 18 & 44 & 25 \\ 45 & 20 & 49 & 52 & 5 \\ 35 & 16 & 32 & 34 & 6 \\ 23 & 32 & 47 & 43 & 39 \end{pmatrix} \qquad A_2 = \begin{pmatrix} - & - \\ 0 & 0 \\ 34 & 47 \\ 57 & 36 \\ 27 & 36 \\ 4 & 47 \\ 30 & 0 \end{pmatrix}.$$

$$(v,h) = (78,13): \quad A_1 = \begin{pmatrix} - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 25 & 33 & 48 & 61 & 55 & 56 \\ 1 & 15 & 63 & 58 & 11 & 49 \\ 20 & 10 & 57 & 4 & 49 & 37 \\ 40 & 12 & 29 & 26 & 62 & 2 \\ 24 & 20 & 30 & 52 & 28 & 25 \end{pmatrix} \qquad A_2 = \begin{pmatrix} - \\ 0 \\ 14 \\ 43 \\ 43 \\ 14 \\ 0 \end{pmatrix}.$$

$$(v,h) = (84,14): \quad A_1 = \begin{pmatrix} - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 15 & 44 & 45 & 40 & 65 & 68 \\ 2 & 61 & 11 & 26 & 55 & 21 \\ 39 & 2 & 42 & 34 & 5 & 49 \\ 1 & 24 & 15 & 58 & 8 & 61 \\ 5 & 18 & 22 & 7 & 17 & 62 \end{pmatrix} \qquad A_2 = \begin{pmatrix} - & - \\ 0 & 0 \\ 18 & 16 \\ 39 & 45 \\ 4 & 45 \\ 53 & 16 \\ 35 & 0 \end{pmatrix}.$$

For $(v,h) = (90,15)$ our construction is similar, but here we cyclically permute the rows of $[A_1|49\cdot A_1|A_2]$ (instead of $[A_1|-A_1|A_2]$). The reason for using 49 (instead of -1) as a multiplier is that it equals 1 (mod 3). These two arrays were found by computer after prespecifying the (mod 3) values of their entries; when prespecifying values (mod t), our experience suggests it is more efficient to use a multiplier $\equiv 1$ (mod t) if possible. We also point out that each column of A_2 remains invariant if we first multiply that column by -49, and then reverse the order of the non blank entries in that column.

$$A_1 = \begin{pmatrix} - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 30 & 19 & 23 & 39 & 44 & 15 \\ 19 & 28 & 5 & 47 & 15 & 74 \\ 7 & 48 & 37 & 1 & 41 & 33 \\ 20 & 51 & 9 & 59 & 47 & 44 \\ 58 & 9 & 31 & 35 & 21 & 17 \end{pmatrix} \qquad A_2 = \begin{pmatrix} - & - & - \\ 0 & 25 & 50 \\ 54 & 5 & 10 \\ 64 & 12 & 8 \\ 14 & 12 & 58 \\ 54 & 55 & 35 \\ 0 & 50 & 25 \end{pmatrix}.$$

Finally, for $(v,h) = (30,5)$, the required QDM is obtained by cyclically permuting the 7 rows of A where

$$A = \begin{pmatrix} - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 22 & 11 & 13 \\ 18 & 15 & 5 & 21 & 19 \\ 0 & 7 & 14 & 8 & 24 \\ 20 & 3 & 15 & 2 & 22 \\ 24 & 5 & 18 & 17 & 11 \end{pmatrix}.$$

Filling in the size h hole of a $\mathrm{TD}(k,v) - \mathrm{TD}(k,h)$ with a $\mathrm{TD}(k,h)$ gives a $\mathrm{TD}(k,v)$. A $\mathrm{TD}(7,h)$ exists for $h = 7,8,9$; therefore applying this result to 3 of the ITDs in the previous lemma gives a $\mathrm{TD}(7,v)$ (or equivalently 5 mutually orthogonal Latin squares) of orders $v = 42$, 54 and 62. For $v = 42,\ 54,\ 62$ these are the best known results so far.

A $\mathrm{TD}(k,v)$ is called idempotent if it possesses at least one parallel class, i.e. a set of blocks containing each point exactly once. We note that deleting one group of the above $\mathrm{TD}(7,30)-\mathrm{TD}(7,5)$ gives a $\mathrm{TD}(6,30)-\mathrm{TD}(6,5)$ with 25 disjoint parallel classes; these correspond to the sets of blocks containing each of the 25 non-holey points in the deleted group. Filling in the size 5 hole with a $\mathrm{TD}(6,5)$ therefore gives an idempotent $\mathrm{TD}(6,30)$. In 1996, 30 and 60 were the only values ≥ 26 for which an idempotent $\mathrm{TD}(6,v)$ was unknown [7]; further, R.S. Rees [21] solved $v = 60$. In addition, $v = 15$ is solved later in Lemma 4.1; now $v = 10, 14, 18, 22, 26$ remain the only unsolved cases.

The other $\mathrm{TD}(7,v)-\mathrm{TD}(7,h)$s in the previous lemma all appear to be new even though they do not yield new TDs. However those with $v = 6h$ have another application: Every block in a $\mathrm{TD}(7,6h)-\mathrm{TD}(7,h)$ contains exactly one holey point, and therefore deleting all the holey points in this ITD gives a (frame resolvable) $(6,1)$-GDD of type $5h^7$. (The blocks containing any holey point form a partial parallel class missing the $5h$ non-holey points in the group containing it). In particular for $h = 14$ and 15, this gives $(6,1)$-GDDs of types 70^7 and 75^7; as noted in [6], the groups of these GDDs can be filled (using 6 or 1 extra points and a $(76,6,1)$ BIBD) to produce new $(496,6,1)$ and $(526,6,1)$ BIBDs.

4 Some TDs from difference matrices and QDMs with $h = 1$

A number of known general constructions for $\mathrm{TD}(k,v)$s are known; however for $v \leq 80$ and the largest known k, constructions for these TDs tend to be of a somewhat miscellaneous nature when they come from difference or quasi-difference matrices and v is not a prime power. In this section, we provide a few of these.

In [19], Mills gave a couple of $\mathrm{TD}(k,v)$s with $v = pq$ where p, q were prime powers and $q \equiv 1 \pmod{p}$; these TDs all possessed an automorphhsm group of order p which interchanged the first p groups of the TD and mapped each other group onto itself. Other examples of incomplete TDs with a similar structure can be found in Lemma

3.18 of [8]. The TDs for $v = 34, 44$ in the next lemma both possess a similar property; although here and in [8], the order of this automorphism group is 5, and does not divide v or $v - 1$.

Lemma 4.1 *The following TDs exist: an idempotent* $TD(6, 15)$*, an idempotent* $TD(6, 34)$ *and a* $TD(7, 44)$*.*

Proof: For $v = 15$, 34, these TDs are obtainable from a $(v - 1, 6; 1, 0; 1)$ QDM. The first, for $v = 15$ is obtainable from an orthogonal BIBD, OBIBD$(15, 2, 1; 3)$ found by by D.H. Rees [20]. For more information and the definition of OBIBDs see [16]. Not all OBIBDs give QDMs; D.H. Rees found 24 OBIBD$(15, 2, 1; 3)$s, and fortunately in one of them we were able to rearrange the order of elements in each base block so that the rearranged base blocks could be taken as suitable generating columns for our QDM. Let A_1, A_3 be the arrays below, and let A_2 be the array obtained by interchanging rows i, $i + 3$ in A_1 (for $i = 1, 2, 3$). Then $[A_1|A_2|A_3]$ is the required QDM.

$$A_1 = \begin{pmatrix} - & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & - & 8 & 10 & 6 & 12 & 9 \\ 8 & 8 & - & 4 & 10 & 9 & 11 \\ 0 & 11 & 2 & 1 & 4 & 5 & 6 \\ 5 & 13 & 5 & 12 & 11 & 4 & 10 \\ 3 & 12 & 1 & 7 & 2 & 10 & 13 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 2 \\ 0 & 7 \\ 0 & 8 \\ 0 & 9 \end{pmatrix}.$$

For $v = 34$, consider the following matrices over Z_{33}:

$$A_1 = \begin{pmatrix} - & 0 & 0 & 0 & 0 & 0 \\ 30 & 17 & 10 & 25 & 23 & 8 \\ 22 & 4 & 32 & 29 & 28 & 22 \\ 25 & 10 & 20 & 15 & 21 & 16 \\ 0 & 12 & 15 & 16 & 32 & 23 \\ 6 & 11 & 18 & 14 & 9 & 20 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 1 & 3 & 10 & 5 \\ 0 & 4 & 12 & 7 & 20 \\ 0 & 16 & 15 & 28 & 14 \\ 0 & 31 & 27 & 13 & 23 \\ 0 & 25 & 9 & 19 & 26 \\ 0 & 11 & 11 & 0 & - \end{pmatrix}.$$

We replace each column $C = (a, b, c, d, e, f)^T$ of A_1 by the five columns $t^i(C)$, $0 \le i \le 4$, where $t(C) = (4e, 4a, 4b, 4c, 4d, 4f)^T$. Then append the columns of A_2. We then have a (33,6;1,1;1) quasi-difference matrix, and hence also an idempotent TD$(6, 34)$.

For $v = 44$, let A_1 and A_2 be the folllowing arrays over $Z_2 \times Z_2 \times Z_{11}$:

$$A_1 = \begin{pmatrix} (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\ (1,1,4) & (0,1,4) & (1,1,7) & (1,0,6) & (1,1,9) & (0,1,2) & (0,1,5) & (0,1,1) \\ (1,0,6) & (0,1,3) & (1,0,0) & (0,1,9) & (1,1,1) & (0,1,4) & (1,1,9) & (1,0,9) \\ (1,1,6) & (1,1,9) & (0,1,2) & (1,1,0) & (0,1,0) & (1,1,5) & (0,0,4) & (0,0,9) \\ (1,0,9) & (0,0,2) & (0,0,1) & (1,0,2) & (0,0,7) & (1,1,6) & (1,1,0) & (1,0,7) \\ (1,0,1) & (1,0,6) & (1,1,3) & (0,1,5) & (0,0,5) & (0,1,3) & (0,1,0) & (1,1,0) \end{pmatrix}$$

$$A_2 = \begin{pmatrix} (0,0,0) & (1,0,1) & (1,1,2) & (0,0,8) \\ (0,0,0) & (1,0,5) & (1,1,10) & (0,0,7) \\ (0,0,0) & (1,0,3) & (1,1,6) & (0,0,2) \\ (0,0,0) & (1,0,4) & (1,1,8) & (0,0,10) \\ (0,0,0) & (1,0,9) & (1,1,7) & (0,0,6) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \end{pmatrix}.$$

Here, we replace each column $C = [(x_1, y_1, z_1), (x_2, y_2, z_2), ..., (x_5, y_5, z_5), (x_6, y_6, z_6)]^T$ by the five columns $t^i(C)$, where $t(C) = [(x_5, y_5, 5z_5), (x_1, y_1, 5z_1), (x_2, y_2, 5z_2), (x_3, y_3, 5z_3), (x_4, y_4, 5z_4), (x_6, y_6, 5z_6)]^T$. Then append the four columns of A_2, each of which remains invariant under t. This gives us a $(44, 6, 1)$ difference matrix over $Z_2 \times Z_2 \times Z_{11}$ and hence also a TD$(7, 44)$.

In [10] and [22], TD$(6, v)$s for $v = 20, 38, 44$ were given; these TDs all had an automorphism group of order order $18(v-1)$. The two TDs in the next lemma possess a similar automorphism group of order $9v$; one order 2 automorphism used in [10] and [22] is not present here.

Lemma 4.2 *There exists a TD$(7, v)$ for $v \in \{28, 52\}$.*

Proof: *For $v = 52$, consider the following arrays over $GF(4, z^2 = z+1) \times Z_{13}$:*

$$A_1 = \begin{pmatrix} (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (z^2,10) & (0,7) & (1,10) & (z,10) & (z^2,3) \\ (z,10) & (z^2,2) & (1,11) & (z,2) & (z^2,7) \\ (z,8) & (z^2,12) & (0,10) & (z^2,11) & (z^2,6) \\ (1,2) & (0,2) & (z^2,8) & (z,3) & (z,7) \\ (1,6) & (z,12) & (0,7) & (z^2,6) & (z,2) \end{pmatrix}$$

$$A_2 = \begin{pmatrix} (1,1) & (z^2,11) \\ (z,3) & (1,7) \\ (z^2,9) & (z,8) \\ (1,4) & (z^2,3) \\ (z,12) & (1,9) \\ (z^2,10) & (z,1) \end{pmatrix}.$$

For each column $C = ((x_1, y_1), (x_2, y_2) \dots (x_6, y_6))^T$, let $t_1(C) = ((z \cdot x_3, 3 \cdot y_3), (z \cdot x_1, 3 \cdot y_1), (z \cdot x_2, 3 \cdot y_2), (z \cdot x_6, 3 \cdot y_6), (z \cdot x_4, 3 \cdot y_4), (z \cdot x_5, 3 \cdot y_5))^T$, and $t_2(C) = ((x_3, y_3), (x_1, y_1), (x_2, y_2), (x_5, y_5), (x_6, y_6), (x_4, y_4))^T$. Applying the group of order 9 generated by t_1, t_2 to the columns of A_1 give 45 columns. Six more columns are obtained by applying the group of order 3 generated by t_2 to the two columns of A_2; these two columns both remain invariant under t_1. Finally add one extra column whose entries all equal (0,0); the resulting 52 columns then give us a $(52, 6, 1)$ difference matrix and hence also a TD$(7, 52)$.

Similarly, for $v = 28$ we use the array B below over $GF(4, z^2 = z+1) \times Z_7$, and for each column $C = ((x_1, y_1), (x_2, y_2) \dots (x_6, y_6))^T$, let $t_1(C) = ((z \cdot x_3,\ 2 \cdot y_3),$

$(z{\cdot}x_1, 2{\cdot}y_1)$, $(z{\cdot}x_2, 2{\cdot}y_2)$, $(z{\cdot}x_6, 2{\cdot}y_6)$, $(z{\cdot}x_4, 2{\cdot}y_4)$, $(z{\cdot}x_5, 2{\cdot}y_5))^T$, and $t_2(C) = ((x_3, y_3)$, (x_1, y_1), (x_2, y_2), (x_5, y_5), (x_6, y_6), $(x_4, y_4))^T$. We apply the group of order 9 generated by t_1, t_2 to the columns of B, and add one extra column whose entries all equal $(0, 0)$. This gives us a $(28, 6, 1)$ difference matrix and hence also a TD$(7, 28)$.

$$B = \begin{pmatrix} (0,0) & (1,1) & (0,0) \\ (z,2) & (z,1) & (z^2,3) \\ (z,3) & (z^2,5) & (z^2,2) \\ (0,5) & (0,6) & (0,4) \\ (0,3) & (0,0) & (1,1) \\ (1,3) & (1,5) & (z^2,6) \end{pmatrix}.$$

5 More examples of Incomplete TDs

In Lemmas 3.5 and 2.4 of [8] and [9] respectively, a $TD(6, v) - TD(6, 2)$ for $v = 15$ and 19 was obtained by multiplying the columns of an initial array by 1 and -1. The ITD in the next lemma is obtained in this manner.

Theorem 5.1 *There exists a* $TD(6, 17) - TD(6, 2)$.

Proof: A $(15, 2; 1, 0; 6)$-QDM is obtained by multiplying the columns of the following array by 1 and -1:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - \\ 5 & 7 & 1 & 3 & 2 & 4 & 6 & - & 0 \\ 12 & 10 & 6 & 7 & 1 & 2 & - & 4 & 6 \\ 9 & 12 & 13 & 1 & 11 & - & 5 & 8 & 7 \\ 10 & 4 & 9 & 14 & - & 13 & 8 & 3 & 1 \\ 7 & 11 & 10 & - & 3 & 9 & 14 & 13 & 3 \end{pmatrix}.$$

Any $(v-h, h; 1, 0; k)$-QDM (but not a $(v-h, h; 1, 1; k)$-QDM) which is obtained by cyclically permuting the rows in an initial array gives an incomplete perfect Mendelsohn design, or more precisely, a k-IPMD(v, h). For more information on these designs, see for instance, [2] or [3]. The next lemma gives 3 new 6-IPMD$(v, 8)$s, although in only one of these cases, $v = 53$, was the corresponding $TD(6, v) - TD(6, 8)$ previously unknown. Other 6-IPMD$(39 + h, h)$s with $1 \le h \le 7$, $h \ne 2$ can be found in [3].

Theorem 5.2 *There exists a* $TD(6, v) - TD(6, 8)$ *and a* 6*-IPMD*$(v, 8)$ *for* $v = 47$, 53, *and* 59.

Proof: Cyclically permute the the 6 rows in the arrays below. In each case, this gives a $(v - 8, 8; 1, 0; 6)$-QDM.

$$v = 47 : \begin{pmatrix} 0 & - & - & - & - & - & - & - & - \\ 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 25 & 4 & 31 & 10 & 20 & 27 & 17 & 26 & 25 \\ 20 & 32 & 5 & 24 & 23 & 18 & 16 & 19 & 34 \\ 27 & 14 & 13 & 4 & 34 & 33 & 18 & 16 & 28 \\ 17 & 19 & 9 & 22 & 11 & 17 & 24 & 28 & 26 \end{pmatrix}.$$

$$v = 53 : \begin{pmatrix} 0 & 0 & - & - & - & - & - & - & - & - \\ 37 & 28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 34 & 4 & 22 & 1 & 23 & 21 & 41 & 35 & 10 \\ 19 & 23 & 35 & 17 & 33 & 32 & 18 & 22 & 10 & 13 \\ 1 & 30 & 29 & 31 & 7 & 37 & 42 & 15 & 40 & 1 \\ 30 & 2 & 37 & 33 & 25 & 17 & 41 & 6 & 7 & 14 \end{pmatrix}.$$

$$v = 59 : \begin{pmatrix} 0 & 0 & 0 & - & - & - & - & - & - & - & - \\ 40 & 28 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 43 & 38 & 4 & 10 & 19 & 38 & 42 & 35 & 2 & 34 \\ 19 & 41 & 26 & 34 & 35 & 18 & 8 & 45 & 16 & 10 & 28 \\ 4 & 36 & 49 & 35 & 49 & 40 & 28 & 18 & 21 & 7 & 37 \\ 45 & 14 & 25 & 31 & 9 & 22 & 44 & 11 & 39 & 19 & 29 \end{pmatrix}.$$

Theorem 5.3 *There exists a $TD(7, 43) - TD(7, 7)$.*

Proof: To obtain a $(36, 7; 1, 0; 7)$-QDM we cyclically permute the the first 6 rows in the array A_1 below while leaving the 7th row unaltered. Then append one extra column from the array A_2.

$$A_1 = \begin{pmatrix} - & - & - & - & - & - & - & 0 \\ 28 & 24 & 9 & 17 & 12 & 14 & 30 & 15 \\ 8 & 32 & 27 & 4 & 6 & 7 & 16 & 17 \\ 34 & 23 & 19 & 15 & 26 & 2 & 25 & 20 \\ 11 & 22 & 31 & 13 & 33 & 3 & 29 & 25 \\ 21 & 5 & 20 & 1 & 18 & 35 & 10 & 22 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & - \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 \\ 6 \\ 12 \\ 18 \\ 24 \\ 30 \\ - \end{pmatrix}.$$

The next theorem gives three new ITDs. Existence of the first two was mentioned in [11], but they appear to have not been published before.

Theorem 5.4 *A $TD(6, 70) - TD(6, 3)$, a $TD(6, 77) - TD(6, 4)$ and a $TD(6, 41) - TD(6, 4)$ all exist.*

Proof These are obtainable from $(67, 3, 1, 0; 6)$, $(73, 4, 1, 1; 6)$ and $(37, 4, 1, 1; 6)$-QDMs, (using the same method as in [22, 10] for TD$(6, v)$ with $v \in \{20, 38, 44\}$). In each case, let w be a given cube root of unity in Z_{v-h}; we take $w = 29$, 8 and 10 respectively for $v - h = 67$, 73 and 37. Below we give some generating columns for these QDMs. We then define 3 automorphisms T_1, T_2 and T_3 (of orders 3, 3 and 2) on these columns as follows:

$$T_1(a,b,c,d,e,f)^T = (w\cdot c, w\cdot a, w\cdot b, w\cdot f, w\cdot d, w\cdot e)^T,$$
$$T_2(a,b,c,d,e,f)^T = (b,c,a,f,d,e)^T, \ \ T_3(a,b,c,d,e,f)^T = (d,e,f,a,b,c)^T.$$

We then apply the group of order 18 generated by T_1, T_2, T_3 to the columns below. Each column of A_1 generates 18 distinct columns in the required QDM. For the second ITD, the column of A_2 remains invariant under T_2T_1 and generates 6 columns in the QDM. The column of A_3 remains invariant under both T_1 and T_3 and generates 3 columns in the QDM.

$$(67,\ 3,\ 1,\ 0;\ 6)-QDM:\ A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 18 & 54 & 52 & 42 & 15 \\ 45 & 51 & 60 & 46 & 17 \\ 11 & 8 & 42 & 18 & 20 \\ 31 & 44 & 47 & 16 & 25 \\ - & 45 & 23 & 61 & 22 \end{pmatrix}.$$

$$(73,\ 4,\ 1,\ 1;\ 6)-QDM:\ A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 18 & 60 & 58 & 50 \\ 59 & 61 & 60 & 24 \\ 7 & 10 & 32 & 44 \\ 53 & 44 & 57 & 36 \\ - & 3 & 27 & 69 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 \\ 0 \\ - \\ 1 \\ 64 \\ 8 \end{pmatrix} \quad A_3 = \begin{pmatrix} 3 \\ 24 \\ 46 \\ 3 \\ 24 \\ 46 \end{pmatrix}.$$

$$(37,\ 4,\ 1,\ 1;\ 6)-QDM:\ A_1 = \begin{pmatrix} 0 & 0 \\ 19 & 1 \\ 22 & 25 \\ 35 & 21 \\ 5 & 7 \\ - & 29 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 \\ 0 \\ - \\ 1 \\ 26 \\ 10 \end{pmatrix} \quad A_3 = \begin{pmatrix} 3 \\ 30 \\ 4 \\ 3 \\ 30 \\ 4 \end{pmatrix}.$$

6 A few resolvable TDs with index greater than 1

For $\lambda = 1$, existence of a resolvable $TD_\lambda(k, v)$ is equivalent to that of a $TD_\lambda(k + 1, v)$; however for $\lambda > 1$, having a resolvable $TD_\lambda(k, v)$ is a stronger result, since a $TD_\lambda(k+1, v)$ only gives a $TD_\lambda(k, v)$ with v λ-parallel classes, instead of λv parallel classes.

Any (v, k, λ) difference matrix gives a resolvable $TD_\lambda(k, v)$ (since the blocks of the TD corresponding to any column in the DM form a parallel class). We now give two new resolvable TDs using difference matrices:

Theorem 6.1 *There exists a resolvable $TD_2(8,22)$ and a resolvable $TD_2(7,34)$.*

Proof: For $v = 34$, consider the following arrays over Z_{34}:

$$A_1 = \begin{pmatrix} 0 & 0 & 10 & 7 & 11 & 30 & 19 & 15 & 13 & 9 & 20 \\ 1 & 15 & 32 & 14 & 21 & 31 & 22 & 17 & 27 & 13 & 28 \\ 33 & 10 & 22 & 23 & 32 & 20 & 6 & 26 & 25 & 3 & 27 \\ 2 & 7 & 6 & 17 & 23 & 16 & 11 & 33 & 12 & 30 & 9 \\ 4 & 21 & 19 & 3 & 16 & 28 & 5 & 5 & 29 & 8 & 24 \\ 8 & 29 & 4 & 24 & 12 & 14 & 31 & 2 & 18 & 18 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 25 & 26 \\ 25 & 26 \\ 25 & 26 \\ 25 & 26 \\ 25 & 26 \\ 25 & 26 \end{pmatrix}.$$

Cyclically permute the six rows of the array of the array A_1, then append the two columns of A_2 plus an extra row of zeros. This gives a $(34,7,2)$ difference matrix and hence also a resolvable $TD_2(7,34)$ and a $TD_2(8,34)$.

Similarlarly for a $(22,8,2)$ difference matrix, we cyclically permute the seven rows of the array A_1 below, and again add 2 extra columns from A_2 plus a column of zeros. This gives a $(22,8,2)$ difference matrix and a new resolvable $TD_2(8,22)$; however we point out that a $TD_2(9,22)$ is already known [4].

$$A_1 = \begin{pmatrix} 1 & 14 & 10 & 16 & 1 & 11 \\ 19 & 6 & 4 & 0 & 18 & 5 \\ 9 & 20 & 15 & 8 & 17 & 12 \\ 17 & 0 & 2 & 4 & 7 & 16 \\ 5 & 7 & 21 & 13 & 20 & 15 \\ 10 & 13 & 14 & 6 & 21 & 3 \\ 12 & 11 & 9 & 3 & 19 & 8 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 2 & 18 \\ 2 & 18 \\ 2 & 18 \\ 2 & 18 \\ 2 & 18 \\ 2 & 18 \end{pmatrix}.$$

In [13], the existence of $TD_\lambda(k,v)$ with $\lambda > 1$ and $k = 8,9$ was investigated; later a few improvements on these results were given in [4, 5] as well as some results for $k = 10$. For completeness, we summarize these results (together with the new $TD_2(8,34)$ above) in the 3 theorems below.

Theorem 6.2 *If $\lambda > 1$, then a $TD_\lambda(8,n)$ exists except for $\lambda = 2$, $v \in \{2,3\}$ and possibly for $\lambda = 2$, $v = 6$.*

Theorem 6.3 *If $\lambda > 1$, then a $TD_\lambda(9,n)$ exists except for $\lambda = 2$, $v \in \{2,3\}$ and possibly in the following cases:*

1. $\lambda = 2$ *and* $v \in \{6,14,34,38,39,50,51,54,62\}$;
2. $\lambda = 3$ *and* $v \in \{5,45,60\}$;
3. $\lambda = 5$ *and* $v \in \{6,14\}$.

Theorem 6.4 *A $TD_3(10,v)$ exists except possibly for $v \in \{5, 6, 14, 20, 35, 45, 55, 56, 60, 78, 84, 85, 102\}$. Also a $TD_9(10,v)$ exists, except possibly for $v = 35$.*

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