AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **40** (2008), Pages 27–36

A note on small on-line Ramsey numbers for paths and their generalization

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Abstract

In this note, we consider the on-line Ramsey numbers $\overline{\mathcal{R}}(P_n)$ for paths and their generalization. The standard on-line Ramsey game is played on an unbounded set of vertices, whereas the new variant of the game we consider is the game where the number of vertices is bounded.

Using a computer cluster of 80 processors, we 'calculated' some new values for short paths, both for the generalized on-line Ramsey numbers and the classical ones. In particular, we showed that $\overline{\mathcal{R}}(P_7) = 12$, $\overline{\mathcal{R}}(P_8) = 15$, and $\overline{\mathcal{R}}(P_9) \leq 17$.

1 Introduction and definitions

In this paper, we consider the following variant of the on-line Ramsey game introduced independently by Beck [1] and Friedgut et al. [2]. Let H be a fixed graph. The game between two players, called Builder and Painter, is played on an unbounded set of vertices. In each of her moves the Builder draws a new edge which is immediately coloured red or blue by the Painter. The goal of the Builder is to force the Painter to create a monochromatic copy of H; the goal of the Painter is the opposite, he is trying to avoid it for as long as possible. The payoff to the Painter is the number of moves until this happens. The Painter seeks the highest possible payoff. Since this is a two-person, full information game with no ties, one of the players must have a winning strategy. *The on-line Ramsey number* $\overline{\mathcal{R}}(H)$ is the smallest payoff over all possible strategies of the Builder, assuming the Painter uses an optimal strategy.

Similar to the classical Ramsey numbers (see a dynamic survey of Radziszowski [7] which includes all known nontrivial values and bounds for Ramsey numbers), it is extremely hard to compute the exact value of $\overline{\mathcal{R}}(H)$ unless H is trivial. It seems

* Research partially supported by grants from NSERC, MITACS, AceNet, and CFI.

that paths are the easiest graphs to consider but even in this case only a few first values are known.

Kurek and Ruciński considered in [5] the case where $H = K_n$, but besides the trivial $\overline{\mathcal{R}}(K_2) = 1$, they were able to determine only one more value, namely $\overline{\mathcal{R}}(K_3) = 8$, by mimicking the proof of the upper bound for $R(K_3)$.

Recently, Grytczuk et al. [4], dealing with many labourious subcases, determined the on-line Ramsey numbers for a few short paths ($\overline{\mathcal{R}}(P_2) = 1$, $\overline{\mathcal{R}}(P_3) = 3$, $\overline{\mathcal{R}}(P_4) = 5$, $\overline{\mathcal{R}}(P_5) = 7$, $\overline{\mathcal{R}}(P_6) = 10$). It is clear that $\overline{\mathcal{R}}(P_n) \geq 2n - 3$ for $n \geq 2$ since the Painter may color safely the first $n - 2$ edges red, and the next $n - 2$ edges blue. Also it is not hard to prove that $\overline{\mathcal{R}}(P_n) \leq 4n - 7$ for $n \geq 2$ (see [4] for more details) but it seems that determining the exact values for paths of length more than 5 requires computer support.

In this note, we also consider a new version of the on-line Ramsey numbers, related to a game similar to the one described before but where the number of vertices is not unbounded anymore. The Builder starts with an empty graph with k vertices. *The generalized on-line Ramsey number* $\overline{\mathcal{R}}_k(H)$ is defined as the minimum number of rounds in such a game if the Builder wins, otherwise $\overline{\mathcal{R}}_k(H) = \infty$ (that is, after $\binom{k}{2}$ moves the game is still not finished but the Builder has no more edges to present). Note that $\overline{\mathcal{R}}(H)$ moves are enough to win a game on unbounded set of vertices but it does mean the Builder does not use more than $2\overline{\mathcal{R}}(H)$ vertices in this game (in fact, this number is much smaller; see also Conjecture 4.3). Thus, $\overline{\mathcal{R}}_{2\overline{\mathcal{R}}(H)}(H) = \overline{\mathcal{R}}(H)$.

2 Main results and theory

The main results of the paper are the following:

Theorem 2.1. $\overline{\mathcal{R}}(P_7) = 12$

Theorem 2.2. $\overline{\mathcal{R}}(P_8) = 15$

Theorem 2.3. $\overline{\mathcal{R}}(P_9) \leq 17$

All results are collected in the table below. The first five classical on-line Ramsey numbers follow from [4].

$\overline{\mathcal{R}}(P_2) = 1$ $\overline{\mathcal{R}}_1(P_2) = \infty$ $\overline{\mathcal{R}}_k(P_2) = 1, k \geq 2$	$\overline{\mathcal{R}}(P_3) = 3$ (see [4]) $\overline{\mathcal{R}}_k(P_3) = \infty, k \leq 2$ $\overline{\mathcal{R}}_k(P_3) = 3, k \geq 3$	$\overline{\mathcal{R}}(P_4) = 5$ (see [4]) $\overline{\mathcal{R}}_k(P_4) = \infty, k \leq 4$ $\overline{\mathcal{R}}_k(P_4) = 5, k \geq 5$	$\overline{\mathcal{R}}(P_5) = 7$ (see [4]) $\overline{\mathcal{R}}_k(P_5) = \infty, k \leq 5$ $\overline{\mathcal{R}}_6(P_5) = 8$ $\overline{\mathcal{R}}_k(P_5) = 7, k \geq 7$
$\overline{\mathcal{R}}(P_6) = 10$ (see [4]) $\overline{\mathcal{R}}_k(P_6) = \infty, k \leq 7$ $\overline{\mathcal{R}}_k(P_6) = 10, k \geq 8$	$\overline{\mathcal{R}}(P_7) = 12$ $\overline{\mathcal{R}}_k(P_7) = \infty, k \leq 8$ $\overline{\mathcal{R}}_k(P_7) = 12, k \geq 9$	$\overline{\mathcal{R}}(P_8) = 15$ $\overline{\mathcal{R}}_k(P_8) = \infty, k \leq 10$ $\overline{\mathcal{R}}_k(P_8) = 15, k \geq 11$	$\overline{\mathcal{R}}(P_9) \leq 17$ $\overline{\mathcal{R}}_k(P_9) = \infty, k \leq 11$

Note that, for any graph H and $k, l \in \mathbb{N}$, $k < l$

$$\overline{\mathcal{R}}_k(H) \geq \overline{\mathcal{R}}_l(H) \geq \overline{\mathcal{R}}(H) \tag{1}$$

since in the generalized version of the game the Builder has more restrictions to follow. Thus, if one can show that $\overline{\mathcal{R}}_{k_0}(H) = \overline{\mathcal{R}}(H)$, then $\overline{\mathcal{R}}_k(H) = \overline{\mathcal{R}}(H)$ for all $k \geq k_0$.

The next theorem gives us a lower bound (as a function of k) for a winning strategy of the Builder.

Fact 2.4. $\overline{\mathcal{R}}_k(P_n) = \infty, k \leq \lfloor \frac{3n-4}{2} \rfloor$.

This simple observation follows from the fact that $R(P_n) = \lfloor \frac{3n-2}{2} \rfloor$ (see [3]) since, in general, for arbitrary H, $\overline{\mathcal{R}}_k(H) = \infty$ if $k \leq R(H) - 1$ (one can fix a properly coloured labeled copy of K_k and follow the predetermined colouring). For completeness of exposition, we give a proof of (weaker) Fact 2.4.

Proof. We have to present a winning strategy of the Painter. It is clear that the Builder has no chance to win if $k \leq n-1$ since at least n vertices are needed to construct a path of length $n-1$. So we can assume that $k \geq n$. Now consider the complete graph H on $k \leq \lfloor \frac{3n-4}{2} \rfloor$ vertices containing a red copy of K_{n-1} and all other edges are coloured blue. This graph does not contain a monochromatic path of length $n-1$; the longest red path has length $n-2$ and any blue path has length at most

$$2\,(k-(n-1)) \leq 2\left(\left\lfloor \frac{3n-4}{2} \right\rfloor - n + 1\right) = 2\left\lfloor \frac{n-2}{2} \right\rfloor \leq n-2\,,$$

since each edge in the path must be adjacent to one of $k-(n-1)$ vertices outside the red clique.

Now every time the Builder presents an edge to a previously isolated vertex, the Painter associates it with a previously unassociated vertex of the clique H. Thus the Painter can avoid a monochromatic path P_n by using his predetermined colours. □

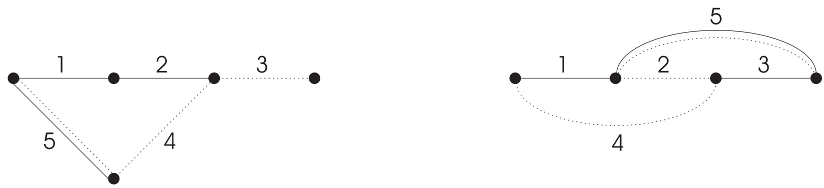
Using Fact 2.4 and inequalities (1), it is enough to show that the following hold: $\overline{\mathcal{R}}_2(P_2) \leq 1$, $\overline{\mathcal{R}}_3(P_3) \leq 3$, $\overline{\mathcal{R}}_5(P_4) \leq 5$, $\overline{\mathcal{R}}_6(P_5) = 8$, $\overline{\mathcal{R}}_7(P_5) \leq 7$ in order to finish the first part of the main results for paths of length at most 4. The first two cases are trivial; the Builder just shows one edge or three edges of the triangle, respectively.

<u>Paths of length 3</u>

The inequality $\overline{\mathcal{R}}_5(P_4) \leq 5$ is also easy to prove. After presenting three edges of a path P_4, there are only two possible patterns (up to symmetry): *bbr* and *brb*. Then the Builder creates a monochromatic path P_4 in the next two moves, as depicted in Figure 1. (The final edge is drawn in two colors.)

<u>Paths of length 4</u>

Theorem 2.5. $\overline{\mathcal{R}}_6(P_5) = 8$.



(a) Case 1 (b) Case 2

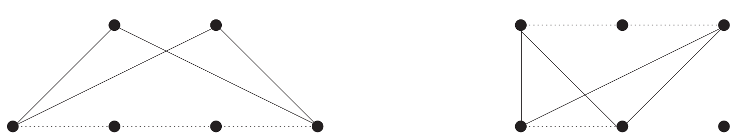
Figure 1: Forcing a path P_4 on 5 vertices

Proof. In order to prove the equality $\overline{\mathcal{R}}_6(P_5) = 8$ we have to present two strategies of the game on 6 vertices:

- the strategy of the Painter to avoid creating a monochromatic P_5 in 7 moves (that is, $\overline{\mathcal{R}}_6(P_5) > 7$),
- the Builder's strategy to force the Painter to create a monochromatic P_5 in 8 moves (that is, $\overline{\mathcal{R}}_6(P_5) \le 8$).

$\underline{\overline{\mathcal{R}}_6(P_5) > 7}$: The Painter can use the following strategy: 'use red if this does not create a red P_5; otherwise use blue'. Using this strategy, the first three edges will be always coloured red. Moreover, if the Painter can use red one more time, then a monochromatic P_5 cannot be forced in seven moves.

For a contradiction, suppose that $\overline{\mathcal{R}}_6(P_5) \le 7$. This means that the Builder has to present a path P_4 (or two paths P_3 and P_2) during the first three moves, since otherwise the Painter can use red colour in the very next move. Then, in order to prevent the Painter from using red, the Builder must present edges with one endpoint at the end of the path and the other outside the path (at the end of the other path, respectively). But this gives us a contradiction since, in both cases, it is not possible for the Builder to force the Painter to construct a blue path of length 4 on 6 vertices (see Figure 2).



(a) Case P_4 (b) Case P_3 and P_2

Figure 2: Avoiding a path P_5 on 6 vertices in 7 moves

$\underline{\overline{\mathcal{R}}_6(P_5) \le 8}$: In the first four moves, the Builder constructs two paths P_3 so that one of the four possible colour patterns: *bb bb*, *bb br*, *bb rr*, *br br*, or equivalent appears. Then she obtains a monochromatic P_5 in at most four additional moves, as shown in Figure 3. (A circled number means that the Painter had a choice in that move, which led to a branching into subcases; note also that in the last three cases we can assume, without loss of generality, that the first edge is blue.)

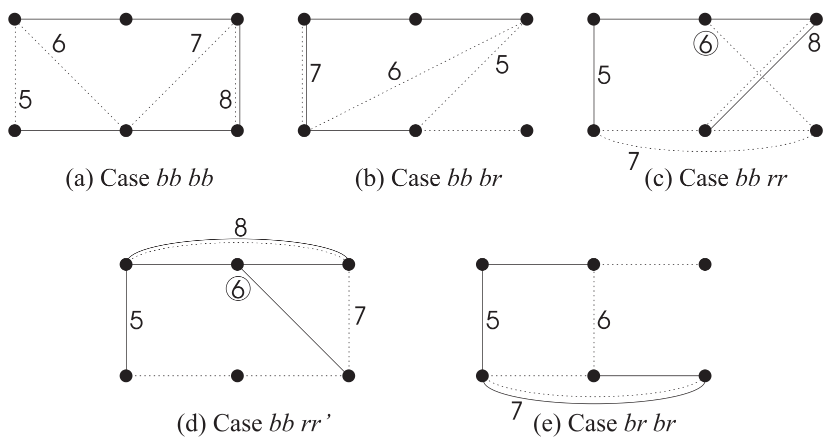


Figure 3: Forcing a path P_5 on 6 vertices in 8 moves

□

Finally, to show that $\overline{\mathcal{R}}_7(P_5) \leq 7$ we can use the same construction as in the proof that $\overline{\mathcal{R}}(P_5) = 7$ (see [4] for more detailes).

3 Simulations

In this section, we describe the simulations we made in order to show the main results for paths of length at least 5. Again, using Fact 2.4 and inequalities (1), it is enough to show that: $\overline{\mathcal{R}}_8(P_6) \leq 10$, $\overline{\mathcal{R}}(P_7) = 12$, $\overline{\mathcal{R}}_9(P_7) \leq 12$, $\overline{\mathcal{R}}(P_8) = 15$, $\overline{\mathcal{R}}_{11}(P_8) \leq 15$, $\overline{\mathcal{R}}(P_9) \leq 17$.

We implemented and ran programs written in C/C++ using backtracking algorithms. (The programs can be downloaded from [6].) Backtracking is a refinement of the brute force approach, which systematically searches for a solution to a problem among all available options. Since it is not possible to examine all possibilities, we used many advanced validity criteria to determine which portion of the solution space needed to be searched. For example, one can look at the coloured graph in every round and try to estimate the number of red (and blue) edges needed to create desired structure. This knowledge can be used to avoid considering the whole branch in the searching tree. If the Painter can use red colour and 'survive' additional k rounds, then there is no point to check whether using blue colour forces him to finish the game earlier.

In this work, we used a cluster-optimised server of 40 dual-processor Mac G5 Xserves. Each 64-bit PowerPC G5 processor at speed of 2.3GHz features an optimized Velocity Engine unit, two floating-point units, and robust branch prediction

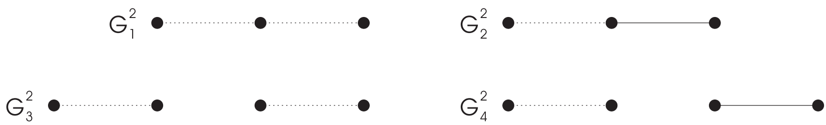


Figure 4: Coloured graphs with two edges

logic. To get more work done faster, its superpipelined, superscalar architecture can handle large numbers of complex operations in parallel. The ultrafast frontside bus, running at 1.15GHz, maximizes processor performance by transferring instructions and data at rates of up to 9.2GB/s. Each PowerPC G5 has a dedicated frontside bus for a combined throughput of up to 18.4GB/s. A 128-bit memory controller speeds data in and out of main memory at up to 6.4GB/s.

Using a cluster, we were able to run (independently) the program from different initial graphs with a given colouring of edges. In the table below we present the numbers of nonisomorphic coloured graphs with k edges that have been found by computer.

k	# of graphs
1	1
2	4
3	12
4	51
5	203
6	$1,004$
7	$5,117$
8	$29,153$
9	$176,778$

Having results from simulations starting from different initial graphs (even partial ones!) we determined the exact value of the on-line Ramsey numbers. The relations between the partial results in different levels are complicated but can be found using a computer. The relations between levels $1-2$, and $2-3$ are described below.

There is only one possible coloured graph G_1^1 with one edge (up to isomorphism). Graphs with two and three edges are presented in Figure 4 and Figure 5, respectively.

Let x_i^k denote the number of moves of winning strategy in the on-line Ramsey game, provided that after k moves the obtained coloured graph is isomorphic to G_i^k. Using the notation

$$\begin{aligned} x_1 \vee x_2 &= \max\{x_1, x_2\} \\ x_1 \wedge x_2 \wedge \cdots \wedge x_k &= \min\{x_1, x_2, \ldots, x_k\}, \end{aligned}$$

it is not hard to see that

$$x_1^1 = (x_1^2 \vee x_2^2) \wedge (x_3^2 \vee x_4^2),$$

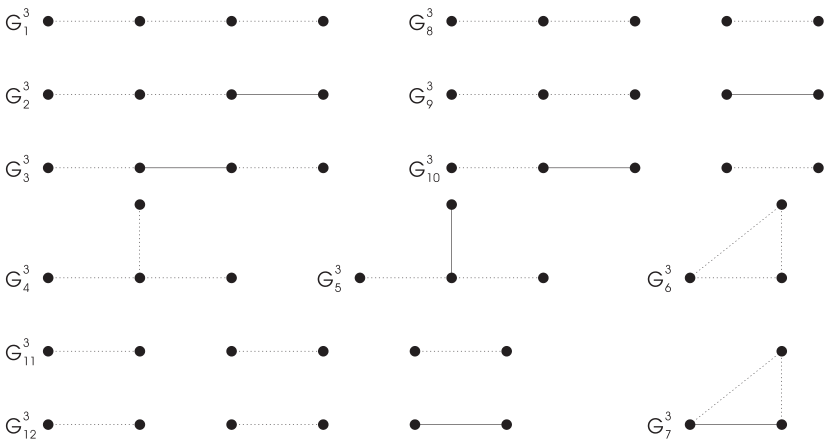


Figure 5: Coloured graphs with three edges

and

$$\begin{aligned} x_1^2 &= (x_1^3 \vee x_2^3) \wedge (x_8^3 \vee x_9^3) \wedge (x_4^3 \vee x_5^3) \wedge (x_6^3 \vee x_7^3) \\ x_2^2 &= (x_3^3 \vee x_2^3) \wedge x_{10}^3 \wedge x_5^3 \wedge x_7^3 \\ x_3^2 &= (x_1^3 \vee x_3^3) \wedge (x_8^3 \vee x_{10}^3) \wedge (x_{11}^3 \vee x_{12}^3) \\ x_4^2 &= x_2^3 \wedge (x_9^3 \vee x_{10}^3) \wedge x_{12}^3 \,. \end{aligned}$$

Each "$\vee$" sign corresponds to the Painter's move, "$\wedge$" corresponds to the Builder's one. He tries to play as long as possible, choosing the maximum value, but she would like to win as soon as possible.

Paths of length 5

In order to show that $\overline{\mathcal{R}}_8(P_6) \leq 10$ we examined 203 initial configurations with 5 edges. Exactly one graph $G^5_{i_0}$ contains a monochromatic P_6 (in fact, G_{i_0} is a monochromatic path of length 5) so we put $x^5_{i_0} = 5$. 9 graphs contain more than 8 vertices; we put $x^5_i = \infty$ for these graphs. For the rest, we run the simulation to check whether $x^5_i \leq 10$. The results are presented below.

	# of initial configurations
$x^5_i = 5$	1
$6 \leq x^5_i \leq 10$	154
$11 \leq x^5_i < \infty$	39
$x^5_i = \infty$	9
total	203

Next we verified that the Builder has a strategy to reach one of the 'good' configurations that ensures her a win in the next five moves.

The total running time was 43 seconds (0.54 per processor).

Paths of length 6

In order to show that $\overline{\mathcal{R}}(P_7) = 12$ and $\overline{\mathcal{R}}_9(P_7) \leq 12$ we examined $5,117$ initial configurations with 7 edges. 28 graphs contain a monochromatic P_7 for which we know that $x_i^7 \leq 7$. 536 graphs contain more than 9 vertices; we put $x_i^7 = \infty$ for these graphs (in the case of generalized numbers). For the rest, we run the simulation to check whether $x_i^7 \leq 12$. The results are presented below.

	# of initial configurations	
	$\overline{\mathcal{R}}(P_7) = 12$	$\overline{\mathcal{R}}_9(P_7) \leq 12$
$x_i^7 \leq 7$	28	28
$8 \leq x_i^7 \leq 12$	$2,832$	$1,758$
$13 \leq x_i^7 < \infty$	$2,257$	$2,795$
$x_i^7 = \infty$	-	536
total	$5,117$	$5,117$

The total running time was 4.58 hours (3.36 minutes per processor) for classical numbers and 2.41 hours (1.8 minutes per processor) for generalized values.

Paths of length 7

The process of verification that $\overline{\mathcal{R}}(P_8) \leq 15$ is relatively easy. We examined 194 (out of 203) initial configurations with 5 edges. Unfortunately, with these partial results we were not able to show that $\overline{\mathcal{R}}(P_8) = 15$ (after a few weeks of running we decided to stop the simulation).

In order to show that $\overline{\mathcal{R}}(P_8) > 14$ and $\overline{\mathcal{R}}_{11}(P_8) \leq 15$ we examined initial configurations with 8 edges. We did check all possibilities for the classical version, but only $20,113$ (out of $29,153$) for the generalized case. We do not need complete results to find $\overline{\mathcal{R}}_{11}(P_8)$. 34 graphs contain a monochromatic P_8 for which we know that $x_i^8 \leq 8$. 762 graphs contain more than 11 vertices; we put $x_i^8 = \infty$ for these graphs (in the case of generalized numbers). The results are presented below.

$\overline{\mathcal{R}}(P_8) \leq 15$		$\overline{\mathcal{R}}(P_8) > 14$		$\overline{\mathcal{R}}_{11}(P_8) \leq 15$	
$x_i^8 = 14$	56	$x_i^8 \leq 8$	34	$x_i^8 \leq 8$	34
$x_i^8 \geq 15$	136	$9 \leq x_i^8 \leq 14$	$5,205$	$9 \leq x_i^8 \leq 15$	$13,828$
unknown value	11	$15 \leq x_i^8 < \infty$	$23,914$	$16 \leq x_i^8 < \infty$	$5,489$
				$x_i^8 = \infty$	762
				unknown value	$9,040$
total	203		$29,153$		$29,153$

The total running time was 93.9 days (1.16 per processor) for classical numbers and $2,244$ days (28 per processor) for generalized values.

Paths of length 8

In order to show that $\overline{\mathcal{R}}(P_9) \leq 17$ we examined all 272 coloured paths with 10 edges as an initial configuration. It is probably not possible to verify all non-path

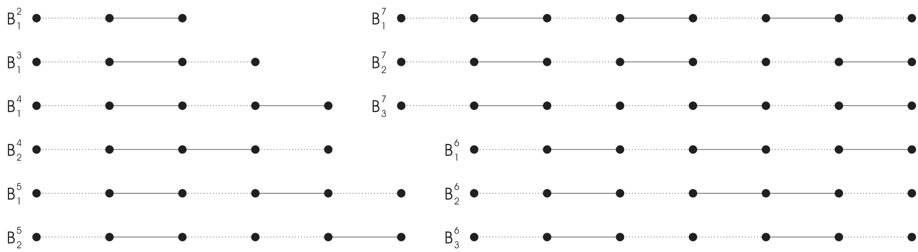


Figure 6: 'Bad' path configurations

configurations (with facility we used), but the results for paths give us the upper bound for $\overline{\mathcal{R}}(P_9)$. 5 paths contain a monochromatic P_9 for which we know that $x_i^{10} \leq 10$. The results are presented below.

	# of initial configurations
$x_i^{10} \leq 10$	5
$11 \leq x_i^{10} \leq 17$	258
$18 \leq x_i^{10} < \infty$	9
total	272

Next we checked all possible strategies of the Builder for the first 10 moves, providing that in each step the Builder constructs a path. Unfortunately, with this restriction the Builder is unable to win in 17 moves, but we found 'safe' configurations with k edges which guarantee that from these configurations $17 - k$ rounds are enough to win. All 'bad' configurations (that is, configurations that are not 'safe') are presented in Figure 6.

Now we present a Builder's strategy to avoid 'bad' configurations on 7 edges which finishes the proof of the upper bound for $\overline{\mathcal{R}}(P_9)$. In the first two moves, the Builder presents a path of length 2, and the Painter must use two colours to create B_1^2 (see Figure 6). In the next two moves, she presents an independent path, and again he cannot use the same colour twice (the Builder can connect a vertex adjacent to a red edge to a vertex adjacent to a blue edge and then the Painter must create a 'safe' configuration with 5 edges). After presenting an edge between vertices adjacent to edges of different colours, the Builder can force the Painter to draw B_2^5 (path $rbrrb$). In the next step, an isolated edge is presented and coloured. It does not matter which colour is used by the Painter, since the Builder can connect this edge to a vertex adjacent to an edge of opposite colour to create a 'safe' configuration with 7 edges.

The total running time was 33 days (9.9 hours per processor).

4 Open problems

In this section we state a few open problems. The first two are related to the classical on-line Ramsey numbers, but the third one is associated with the generalized numbers.

According to the definition, the Builder has a strategy to produce a monochromatic path of length $n-1$ in $\overline{\mathcal{R}}(P_n)$ moves. It seems that she needs to present at least two more edges to force the Painter to create a monochromatic path of length n. But no proof of the following conjecture is known.

Conjecture 4.1. $\overline{\mathcal{R}}(P_n) \geq \overline{\mathcal{R}}(P_{n-1}) + 2$, for $n \geq 3$.

If the Conjecture 4.1 is true, then the following holds too.

Conjecture 4.2. $\overline{\mathcal{R}}(P_9) = 17$.

We observed that $\overline{\mathcal{R}}_k(P_n) = \infty$, for $k \leq \lfloor \frac{3n-4}{2} \rfloor$. But the only known value of n for which $\overline{\mathcal{R}}_{\lfloor \frac{3n-4}{2} \rfloor + 1}(P_n)$ is different than $\overline{\mathcal{R}}(P_n)$ is 5. Is this the only such a case?

Conjecture 4.3. For $n \geq 6$

$$\overline{\mathcal{R}}_{\lfloor \frac{3n-2}{2} \rfloor}(P_n) = \overline{\mathcal{R}}(P_n)\,.$$

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(Received 8 Nov 2006; revised 7 June 2007)