Some sharply transitive partially ordered sets

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Abstract . A partially ordered set $(X, \leq)$ is called sharply transitive if its automorphism group is sharply transitive on X, that is, it is transitive and the stabilizer of every element is trivial. It is shown that every free group is the automorphism group of a sharply transitive partially ordered set. It is also shown that there exists a sharply transitive partially ordered set $(X, \leq)$ having some maximal chains isomorphic to the rationals and automorphism group isomorphic to the additive group of a vector space of dimension two over the rationals.

The automorphism group $Aut(X, \leq)$ of a partially ordered set $(X, \leq)$ is the group of all permutations g of X such that $x \leq y$ if and only if $xg \leq yg$ for all $x, y \in X$. The partially ordered set $(X, \leq)$ is called sharply transitive if $Aut(X, \leq)$ is sharply transitive on X, that is, it is transitive and the stabilizer of every element is trivial. Sharply transitive linearly ordered sets were first studied by Tadashi Ohkuma [5], [6], and later by A.M.W. Glass, Yuri Gurevich, W. Charles Holland and Saharon Shelah [4] (see also [3],[7]). The author gave some constructions and non-existence results for sharply transitive partially ordered sets in [1] and [2].

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If G is the (full) automorphism group of a sharply transitive partially ordered set then either G has order at most 2 or G contains an element of infinite order. However, this condition is not sufficient (Prop. 2.1 in [1]). All examples of non-trivial sharply transitive partially ordered sets constructed in [1] and [2] contain an infinite cyclic group in their centre. We shall show in this paper that this is not a necessary property. Indeed every countable free group (which has a trivial centre if it has more than one free generator) is isomorphic to the automorphism group of a sharply transitive partially ordered set. Another common feature of the partially ordered sets in [1] and [2] is that maximal chains are order-isomorphic to the integers. We shall construct a countable sharply transitive partially ordered set having maximal chains order-isomorphic to the integers and to the rationals (and to some other countable order types) whose automorphism group is isomorphic to the additive group of a vector space of dimension two over the rationals.

Theorem 1. Let F be a free group on finitely or countably many generators. Then there exists a partial order on F such that $Aut(F, \leq)$ is sharply transitive on F and $Aut(F, \leq) \cong F$ via the right regular representation. All maximal chains in $(F, \leq)$ are order-isomorphic to the integers.

Proof. Let $(F, \cdot)$ be freely generated by $\{a_i | i \in I\}$ where $I = \mathbb{N}$ or $I = \{0, 1, \cdots, n\}$ for some $n \in \mathbb{N}$, and let $I' = I \backslash \{0\}$. The result is obvious for $|I| = 1$. For $x \in F$ and $i \in I'$ define $x < a_o x$, $x < a_i a_o x$ and $x < {a_o}^{i+1} {a_i}^{-1} x$. Let $\leq$ be the reflexive, transitive closure of this relation. In order to show that it is a partial order, we have to show that it is antisymmetric. Suppose $x, y \in F$ with $x < y$ and $y < x$. Then there exist $c_1, \cdots, c_r, c_{r+1}, \cdots, c_s \in \{a_o, a_i a_o, {a_o}^{i+1} {a_i}^{-1} | i \in I'\}$ such that $x = c_1 \cdots c_r y$ and $y = c_{r+1} \cdots c_s x$. Hence

$x = c_1 \cdots c_r c_{r+1} \cdots c_s x$, and thus $c_1 \cdots c_s = 1$. However the sum of the exponents of each c_j written as a word in the free generators is positive, hence so is that of $c_1 \cdots c_s$, which is a contradiction. This proves antisymmetry. Furthermore, it is clear that $(F, \cdot)$ is a subgroup of $Aut(F, \leq)$ via the right regular representation.

In order to show that $(F, \cdot)$ is the whole of $Aut(F, \leq)$, it remains to prove that the stabilizer in $Aut(F, \leq)$ of an element of F is trivial. Note that maximal chains in $(F, \leq)$ are order-isomorphic to the integers, and $(F, \leq)$ is connected, which follows from $\{a_o, a_i a_o, {a_o}^{i+1} {a_i}^{-1} \mid i \in I'\}$ being a generating system for $(F, \cdot)$. It is therefore sufficient to show that an automorphism that stabilizes an element also stabilizes all elements which cover it and all elements covered by it.

Let $\alpha \in Aut(F, \leq)$ and $x \in F$ with $x\alpha = x$. Now $A = \{a_o x, a_i a_o x, {a_o}^{i+1} {a_i}^{-1} x \mid i \in I'\}$ is the set of all elements covering x, and is thus setwise fixed by α. Then also the set B of elements which cover some element of A is setwise fixed by α. Note that

$$B = \{{a_o}^2 x\} \cup \{a_i a_o a_j a_o x \mid i, j \in I'\}$$

$$\cup \{{a_o}^{i+1} {a_i}^{-1} {a_o}^{j+1} {a_j}^{-1} x \mid i, j \in I'\} \cup \{a_j {a_o}^{i+2} {a_i}^{-1} x \mid i, j \in I'\}$$

$$\cup \{a_0 a_i a_0 x,\ a_0^{i+2} a_i^{-1} x,\ a_i a_0^2 x,\ a_0^{i+1} a_i^{-1} a_0 x \mid i \in I'\}$$

$$\cup \{{a_o}^{j+1} {a_j}^{-1} a_i a_o x \mid i, j \in I', i \neq j\}$$

$$\cup \{{a_o}^{i+2} x = {a_o}^{i+1} {a_i}^{-1} a_i a_o x \mid i \in I'\}.$$

Let C be the set of elements which cover some element of B. Then C is also setwise fixed by α, and it is not hard to see that $B \cap C = \{{a_o}^{i+2} x \mid i \in I'\}$. The maximal cardinality of a chain in $\{z \in F \mid x \leq z \leq {a_o}^{i+2} x\}$ for $i \in I'$ is $i + 3$, which has to be invariant under α, thus α fixes each element of $B \cap C$. Furthermore, $a_i a_o x$ is the unique element covering x and covered by $a_o{}^{i+2}x$, thus α also fixes $a_i a_o x$. As $\{x, a_o x, a_o{}^2 x, a_o{}^3 x\}$ is the only 4-element chain in $\{z \in F| x \leq z \leq a_o{}^3 x\}$ it is clear that α also fixes $a_o x$ and $a_o{}^2 x$. By the dual argument, it follows that α fixes $a_o{}^{-1}x$, $a_o{}^{-2}x$, $a_o{}^{-(i+2)}x$ and $(a_o{}^{i+1}a_i{}^{-1})^{-1}x$ for all $i \in I'$.

Now consider the remaining elements covered by x and covering x, namely $(a_i a_o)^{-1}x = a_o{}^{-1}a_i{}^{-1}x$ and $a_o{}^{i+1}a_i{}^{-1}x$ for $i \in I'$. By the same arguments as above, it follows that for $i, j \in I'$ there exists a 4-element maximal chain in $\{z \in X|\ a_o{}^{-1}a_j{}^{-1}x \leq z \leq a_o{}^{i+2}a_i{}^{-1}x\}$ if and only if $i = j$, and for $i \in I'$ the maximal cardinality of a chain in $\{z \in X|a_o{}^{-1}a_i{}^{-1}x \leq z \leq a_o{}^{i+1}a_i{}^{-1}x\}$ is $i+3$. Thus α has to fix all elements $a_o{}^{-1}a_i{}^{-1}x$ and $a_o{}^{i+1}a_i{}^{-1}x$ for $i \in I'$, which concludes the proof.

Theorem 2. There exists a partial order $\leq$ on $\mathbb{Q}^2$ with the following properties:

(1) $(\mathbb{Q}^2, +, \leq)$ is a partially ordered group.

(2) $(\mathbb{Q}^2, \leq)$ is sharply transitive.

(3) $Aut(\mathbb{Q}^2, \leq) \cong (\mathbb{Q}^2, +)$ via the right regular representation.

(4) The orbits of $H_1 = \{(0, x)|x \in \mathbb{Q}\}$ are maximal chains order-isomorphic to the rationals.

(5) The orbits of $H_2 = \{(x, 0)|x \in \mathbb{Q}\}$ are maximal antichains.

(6) The orbits of $D = \{(z, z/2)|z \in \mathbb{Z}\}$ are maximal chains order-isomorphic to the integers.

Proof. We define the partial order as follows. If (x, y),$(x', y') \in \mathbb{Q}^2$ then let $(x, y) \leq (x', y')$ if and only if there exist $k \in \mathbb{N}$, $n_1, \cdots, n_k$

$\in \mathbb{N} \setminus \{0\}$ and $\delta, \varepsilon \in \mathbb{Q}$ with $\delta, \varepsilon \geq 0$ such that

$$x' = x + \sum_{j=1}^{k} n_j^{-1} - \delta$$

and

$$y' = y + k - \sum_{j=1}^{k} (2n_j)^{-1} + \delta + \varepsilon.$$

It is not hard to check that this defines a partial order relation on $\mathbb{Q}^2$, and it is clear that addition of any element of $\mathbb{Q}^2$ induces an automorphism of this partial order. Thus $(\mathbb{Q}^2, +, \leq)$ is a partially ordered group. In Figure 1 we indicate the set $\{z \in \mathbb{Q}^2 | z \geq (0,0)\}$. It is not hard to see that the orbits of H_1, H_2 and D are as described in the statement of the theorem. In order to show that $Aut(\mathbb{Q}^2, \leq)$ is isomorphic to $(\mathbb{Q}^2, +)$ and sharply transitive on $\mathbb{Q}^2$, it thus remains to prove that the stabilizer of an element of $\mathbb{Q}^2$ in $Aut(\mathbb{Q}^2, \leq)$ is trivial.

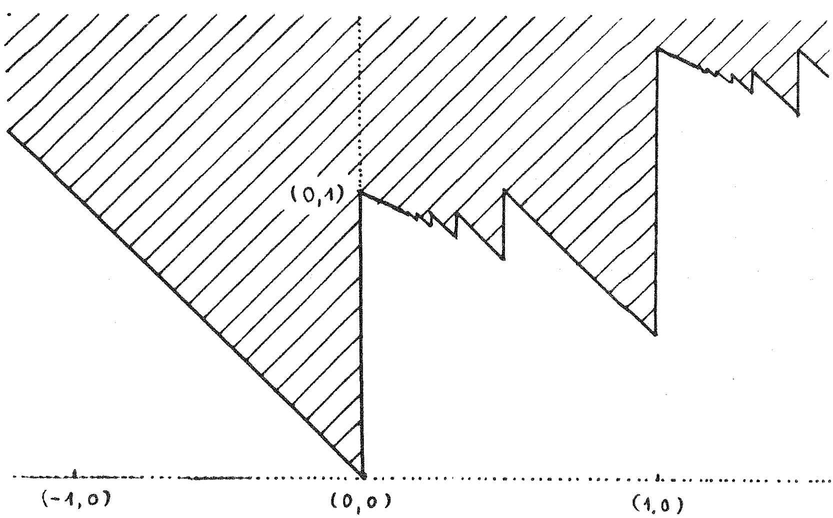

Figure 1

The elements covering $(x,y) \in \mathbb{Q}^2$ are just the elements $(x+n^{-1}, y+1-(2n)^{-1})$ for all $n \in \mathbb{N}\setminus\{0\}$. For $(x,y) \in \mathbb{Q}^2$ we define $D(x,y) = \{(x',y') \in \mathbb{Q}^2 \mid (x,y) \leq (x',y')$ and (x,y) and (x',y') lie in a dense maximal chain of $(\mathbb{Q}^2, \leq)\}$. Thus if $\alpha \in Aut(\mathbb{Q}^2, \leq)$ maps (x,y) to (x', y') it follows that α also maps $D(x,y)$ onto $D(x',y')$. Also, it is not hard to see that $D(x,y) = \{(x',y') \in \mathbb{Q}^2 \mid x' \leq x$ and $y' \geq y + x - x'\}$.

Let $(x,y) \in \mathbb{Q}^2$ and $\alpha \in Aut(\mathbb{Q}^2, \leq)$ such that α fixes (x,y). Then α has to fix $A = \{(x+n^{-1}, y+1-(2n)^{-1}) \mid n \in \mathbb{N}\setminus\{0\}\}$ setwise. The set $D(x,y) \cap D(x+n^{-1}, y+1-(2n)^{-1})$ contains a smallest element, namely $(x, y+1+(2n)^{-1})$. Therefore α also has to fix $B = \{(x, y+1+(2n)^{-1}) \mid n \in \mathbb{N}\setminus\{0\}\}$ setwise. But the order induced on this set is just isomorphic to the ordered set of negative integers, thus α fixes B pointwise, and hence it also fixes A pointwise. Now the set $\{z \in D(x,y) \mid z \leq b$ for all $b \in B\}$ has a greatest element, namely $(x, y+1)$. Thus α also fixes $(x, y+1)$. By symmetry, α also fixes $(x, y-1)$ and $(x, y-(1+(2n)^{-1}))$ for all $n \in \mathbb{N}\setminus\{0\}$, and also $(x-n^{-1}, y-1+(2n)^{-1})$ for all $n \in \mathbb{N}\setminus\{0\}$.

Let $\alpha \in Aut(\mathbb{Q}^2, \leq)$ be such that α fixes $(0,0)$. As $\{\pm 1, \pm(1+(2n)^{-1}) \mid n \in \mathbb{N}\setminus\{0\}\}$ generates the additive group of $\mathbb{Q}$, the results of the preceding paragraph imply that α fixes all elements $(0,q)$ for $q \in \mathbb{Q}$. As the additive group of $\mathbb{Q}$ is also generated by $\{\pm n^{-1} \mid n \in \mathbb{N}\setminus\{0\}\}$, it follows that for every $p \in \mathbb{Q}$, the automorphism α fixes an element of the form (p,q) for some $q \in \mathbb{Q}$. Using the same arguments again, it then follows that α fixes the whole of $\mathbb{Q}^2$, which concludes the proof.

References

1. Gerhard Behrendt, Sharply transitive partially ordered sets, Discrete Math. 61(1986) 133-139.
2. Gerhard Behrendt, Sharply transitive posets with free abelian automorphism group, Math. Proc. Cambridge Philos. Soc. (to appear).
3. A.M.W. Glass, Ordered Permutation Groups, London Math.Soc. Lecture Notes Series 55, Cambridge Univ. Press, Cambridge, 1981.
4. A.M.W. Glass, Yuri Gurevich, W. Charles Holland and Saharon Shelah, Rigid homogeneous chains, Math. Proc. Cambridge Philos. Soc. 89 (1981) 7-17.
5. Tadashi Ohkuma, Sur quelques ensembles ordonnés linéairement, Proc. Japan Acad. 30(1954) 805-808.
6. Tadashi Ohkuma, Sur quelques ensembles ordonnés linéairement, Fund. Math. 43(1956) 326-337.
7. Joseph G. Rosenstein, Linear Orderings, Academic Press, New York, 1982.