

# EXTREMAL PROBLEMS FOR LOCAL PROPERTIES OF GRAPHS

L.H. Clark, R.C. Entringer, J.E. McCanna and L.A. Székely\*

University of New Mexico, Albuquerque, NM 87131

We consider some instances of the general problem of determining the maximum size of a graph of given order in which all vertex neighbourhoods are required to have a specified property.

The concept of local property has been used in two senses in the literature. In the restricted sense, a graph has a local property if all the neighbourhoods are isomorphic to a particular graph. (By the neighbourhood of a vertex  $v$  we mean the subgraph induced by the set of vertices adjacent to  $v$ .) In this case there is no extremal problem since the number of vertices determines the number of edges, if there is a graph with that local property at all. Indeed, the major problem is the problem of existence. Specific instances are discussed by Doyen, Hubaut and Reynaert [2]. For good survey papers see Hell [3] and Sedlacek [6].

In the general sense, a graph has the local property  $P$  if all the neighbourhoods have the property. From now on we use the concept of local property only in the general sense. In

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\*Permanent address: Department of Computer Science  
Eötvös Lorand University  
H-1088 Budapest  
Hungary

this case the existence problem is usually simpler and we focus on the extremal problem: What is the maximum number of edges,  $e = e(n)$ , in a graph with order  $n$  and with a given local property? By *graph* we will always mean a finite simple graph. The main objective of the present paper is to raise interest in this fascinating and hard problem.

Consider the following instances.

**I.** All neighbourhoods are independent sets. In other words, the underlying graph is triangle-free. This particular case of Turán's theorem was already solved in 1906 by Mantel [4]:  $e \leq n^2/4$ ; the bound is tight.

**II.** All neighbourhoods are paths - for 3-connected planar graphs. Zelinka [7] proved  $e \leq 2n + 3\lfloor n/4 \rfloor - 6$  for  $n \geq 8$ ; the bound is tight.

**III.** All neighbourhoods are 1-regular (= perfect matching) graphs - for 3-connected planar graphs. Zelinka [8] proved  $e \leq \left(\frac{12}{5}\right)(n - 2)$ ; the bound is tight.

**IV.** All neighbourhoods are 1-regular graphs. As we learned from Zelinka [8], this class of graphs was introduced in an unpublished lecture of D. Froncek in 1986. However, the corresponding extremal problem was implicitly solved by Ruzsa and Szemerédi [5] in 1976. We are indebted to Professor Paul Erdős for pointing out this reference.

The result is  $e = o(n^2)$ , and  $e$  can reach  $cr_3(n)n$ , where  $r_3(n)$  denotes the largest number  $m$  such that there are  $m$  positive integers less than  $n$  no three of which form an arithmetic progression. According to a result of F. Behrend [1],  $r_3(n) \geq n^{1 - \frac{c}{\sqrt{\log n}}}$ .

In order to prove the result  $e = o(n^2)$  for IV, we recall the Ruzsa-Szemerédi theorem and derive from it an upper bound  $e = o(n^2)$  for a large collection of problems.

**Theorem.** (Ruzsa-Szemerédi [5]) If there is a triplet system on  $n$  vertices such that no six vertices carry three or more triplets then the number of triplets is  $o(n^2)$ .

**Theorem 1.** Suppose  $G$  is a graph with  $n$  vertices and  $e$  edges, such that each edge belongs to at least one triangle, but at most  $k$ . Then  $e = o(n^2)$ .

**Proof.** We are going to decompose  $G = \cup G_i$  such that

- each  $G_i$  has the same vertex set as  $G$
- each triangle of  $G$  is contained in some  $G_i$
- the edge set of  $G_i$  is the union of some triangles of  $G$ , such that no edge belongs to two of the triangles and no three triangles make a 3-cycle (using one edge from each of the triangles)
- the number of  $G_i$ 's is sufficiently small.

Then we can apply the Ruzsa-Szemerédi theorem to the triangle set of each  $G_i$  separately and get the theorem.

Introduce a graph  $T$  whose vertex set is the triangle set of  $G$  and two triangles make an edge in  $T$  if, in  $G$ , they intersect in an edge. In  $T$  the degrees are bounded by  $3(k-1)$ . Introduce a graph  $S$  whose vertex set is that of  $T$  and two different triangles make an edge iff their distance is at most 2 in  $T$ . In  $S$  the degrees are bounded by  $\Delta = 9(k-1)^2$ , while  $S$  is  $(\Delta+1)$ -vertex-colourable. Let  $G_i$  be the spanning subgraph of  $G$  whose edge set is the edge set of triangles of colour  $i$ , for  $i = 1, 2, \dots, \Delta+1$ . The second and third claims for  $G_i$  ensure the applicability of the Ruzsa-Szemerédi theorem. The fourth claim ensures that the sum of the numbers of edges in the  $G_i$ 's is still  $o(n^2)$ . The second and third claims hold because of the definition of  $S$  and the existence of its  $(\Delta+1)$ -vertex colouring. []

Notice, that for graphs having 1-regular neighbourhoods,  $k = 1$  and the Ruzsa-Szemerédi theorem applies directly to the entire triangle set of the graph. Also, making the  $e = o(n^2)$  explicit in the theorem of Ruzsa and Szemerédi allows  $k = k(n)$  to approach infinity (in a sufficiently slow manner).

Ruzsa and Szemerédi gave a construction for a triplet system on  $n$  vertices with  $(1/100)r_3(n)n$  triplets with no six vertices spanning three triplets. Fortunately, no two triplets intersect in two vertices in their construction so that, taking all pairs covered by triplets as the edge set for a graph, we get a graph with  $cr_3(n)n$  edges, in which the neighbourhoods are 1-regular.

We also give a construction for such a graph here since we think this construction is better motivated geometrically.

**Construction 2.** Suppose  $1 \leq a_1 < a_2 < \dots < a_m < n/2$ , where  $m = r_3(n/2)$  and no three term arithmetic progression occurs among the  $a$ 's. Consider the following vertex set:  $\{(i, j): 1 \leq j \leq n, 1 \leq i \leq 3\}$ . Make triplets of those three vertices which lie on a straight line of the type  $y = a_k x + b$ . Define a graph  $G$  by making edges from those pairs of vertices which are covered by a triplet. Drop from  $G$  the isolated vertices. Notice that three edges coming from different triplets can make a triangle if and only if  $a_u + a_v = 2a_w$  for the corresponding straight lines. Also, no two triplets - being parts of straight lines - intersect in two vertices. Finally, for  $c < 1$ ,  $r_3(cm) \geq c'r_3(m)$  for some  $c'$ . []

V. All neighbourhoods are paths (of possibly different lengths at least 1). Theorem 1 applies with  $k = 2$ ,  $e = o(n^2)$ . We have such graphs with  $cn \log n$  edges for some values of  $n$ .

**Construction 3.** Let  $Q_k$  denote the  $k$ -dimensional cube, whose vertices are the  $k$ -digit 0 – 1 sequences so that  $n = 2^k$ . For  $i = 1, 2, \dots, k - 1$  set

$$A_i^0 = \{(a_1 a_2 \dots a_{i-1} 0 1 a_{i+2} \dots a_k, a_1 a_2 \dots a_{i-1} 1 0 a_{i+2} \dots a_k):$$

$$a_m = 0 \text{ or } 1, \text{ for } m < i \text{ and } m > i + 1\}$$

and

$$A_i^1 = \{(a_1 a_2 \dots a_{i-1} 0 0 a_{i+2} \dots a_k, a_1 a_2 \dots a_{i-1} 1 1 a_{i+2} \dots a_k):$$

$$a_m = 0 \text{ or } 1, \text{ for } m < i \text{ and } m > i + 1\}.$$

Define  $Q'_k$  by adding to the edge set of  $Q_k$  the set  $\bigcup_{j=1}^{k-1} A_j^t$ , where  $t_j = 0$  or 1 (so the definition of  $Q'_k$  depends on the choice of  $t_j$ 's). In  $Q'_k$  all neighbourhoods are paths. Without loss of generality we consider only the neighborhood of 00...0. List the vertices:

- |         |           |
|---------|-----------|
| 1)      | 1000...0  |
| 2)      | 1100...0  |
| 3)      | 0100...0  |
| 4)      | 0110...0  |
| 5)      | 0010...0  |
| ⋮       | ⋮         |
| ⋮       | ⋮         |
| ⋮       | ⋮         |
| 2n – 1) | 000...01. |

Out of this sequence the odd-numbered vertices always belong to the neighbourhood of 00...0, the even-numbered ones may or may not. If they do, they are connected to the two neighbouring (in the list) vertices by edges from  $Q_k$ . If they do not, their neighbours on the list above make an edge, since if  $(00...0, v) \in A_j^t$ , then the pair made from the two (list) neighbours of  $v$  belong to  $A_j^{1-t}$ . []

**VI.** All neighbourhoods are cycles, possibly of different length.

Theorem 1 applies with  $k = 2$ ,  $e = o(n^2)$ . There are such graphs with  $cn \log n$  edges for some values of  $n$ . Construction 3 goes through with slight modifications.

**Construction 4.** Set

$$A_n^0 = \{ \{0a_2 \dots a_{n-1}1, 1a_2 \dots a_{n-1}0\} : a_m = 0 \text{ or } 1, \quad 1 < m < n \},$$

$$A_n^1 = \{ \{0a_2 \dots a_{n-1}0, 1a_2 \dots a_{n-1}1\} : a_m = 0 \text{ or } 1, \quad 1 < m < n \}$$

and set  $E(Q_k) = E(Q_k) \cup \bigcup_{j=1}^n A_n^j$ , where  $t_j = 0$  or  $1$ . The definition of  $Q_k$  depends again on the choice of  $t_j$ 's. In the course of proving Construction 4, it is easy to check that the neighbourhood of  $00\dots 0$  is a cycle. []

**VII.** All neighbourhoods are unions of disjoint copies of the same graph  $H$ . Theorem 1 holds with  $k = \max$  degree in  $H$ ,  $e = o(n^2)$ .

The following theorem gives a good lower bound in the case of some interesting  $H$ 's:

**Theorem 5.** Suppose there is a graph  $L$  such that every neighbourhood in  $L$  is the bipartite graph  $H$ . Then we have infinitely many graphs  $G_n$  with  $|V(G_n)| = n$ ,  $|e(G_n)| \geq cr_3(n)n$ , such that each neighbourhood in  $G_n$  is a disjoint union of copies of  $H$ .

**Proof.** Set  $G_n = G \times L$ , where  $G$  is the graph from Construction 2. By  $\times$  we denote the graph product in which two points in the product of the vertex sets make an edge iff their projections both are edges. Denote by  $N_H(v)$  the subgraph of  $H$  induced by the neighbours of  $v$ . We recall the well-known fact, that for a bipartite graph  $M$ ,  $K_2 \times M$  is the juxtaposition of two copies of  $M$ . Now observe, that for  $g \in V(G)$  and  $l \in V(L)$ ,  $N_{G \times L}(g, l) = N_G(g) \times N_L(l)$ . Since  $N_G(g)$  is a juxtaposition of some edges and  $N_L(l)$  is the bipartite graph  $H$ ,  $N_G(g) \times N_L(l)$  is the disjoint union of some copies of  $H$ . We leave the enumeration of edges and vertices in  $G_n$  to the reader. []

**Applications.** For  $H$  an even cycle or a path with at least three edges we have a bipartite graph  $H$  for which  $L$  exists (see [2]), therefore Theorem 5 applies.

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