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A computer enumeration of small latin trades

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Abstract

We report the results of a computer enumeration of latin trades and bi-trades of sizes up to 19. The (bi-)trades were counted according to various equivalences and those with special properties (such as being minimal or homogeneous) were identified. Finally we looked at minimal embeddings of the trades in latin squares and of the bi-trades in topological surfaces.

1 Introduction

An $n \times n$ array P with cells that are either empty or contain exactly one symbol from the set $\{1, 2, \ldots, n\}$ is a *partial latin square* (PLS) of order n if no symbol occurs more than once in any row or column. The number of filled cells is the *size* of P.

A partial latin square T_1 is a *latin trade* if there exists a partial latin square T_2 with the properties that (i) a cell is filled in T_1 if and only if the corresponding cell is filled in T_2, (ii) no symbol occurs in the same cell in T_1 and T_2, (iii) in any given row or column T_1 and T_2 contain exactly the same symbols. The pair $T = (T_1, T_2)$ is known as a *latin bi-trade*. We will say that T is *ordered* or *unordered* depending on whether we are thinking of it as an ordered or unordered pair of trades.

The aim of this paper is to report on a computer enumeration of latin trades and bi-trades of all sizes up to 19. Our results greatly extend (and in one case correct an omission in) the enumeration of latin trades of sizes up to 11 by Donovan *et al.* [6].

Readers are refered to [11] for terminology not given here and for a survey of latin trades, particularly in regard to their relationship to critical sets. However, it should be realised that latin trades have many other applications (see [18] for a discussion of some of them). One intriguing application of latin bi-trades is to triangulations of surfaces and in Section 5 we shall examine the topological properties of the bi-trades that we have generated.

2 Further terminology

Let P be a PLS of order n. The *shape* of P is the set of filled cells in P. Let $\{r_1, r_2, \ldots, r_n\}$ and $\{c_1, c_2, \ldots, c_n\}$ be sets of vertices in a bipartite graph in which r_i is adjacent to c_j if and only if the cell in row i and column j of P is filled. We refer to this graph as the *shape graph* of P and use $\{r_1, \ldots, r_n, c_1, \ldots, c_n\}$ to denote its vertices throughout this paper.

Any PLS P can be thought of as an edge-colouring of its shape graph, where each edge $r_i c_j$ is coloured with the symbol that occurs in the cell (i, j). The definition of a PLS ensures that this colouring is proper (i.e. that if edges are incident they receive different colours) and that any proper edge-colouring with colours chosen from $\{1, 2, \ldots, n\}$ corresponds to a PLS. All colourings in the remainder of this paper will be assumed to be proper.

A latin bi-trade corresponds to a pair of edge colourings of the shape graph such that no edge receives the same colour in both colourings but the set of colours on the edges incident with any given vertex is the same in both colourings. Such a pair of colourings is called a 2-*simultaneous edge colouring* [9], [13] [14]. For a graph to have a 2-simultaneous edge colouring it is necessary that it have no bridges, and it is conjectured in [14] that this condition is also sufficient. This conjecture was checked by the authors of [14] for size $s \leq 22$, meaning that it is true in all cases relevant to the current paper.

There are several natural equivalence relations on PLS which should be exploited in any enumeration. By allowing reordering of rows and columns and permutation of the symbols we induce equivalence classes called *isotopy classes*. By also allowing transposition of the matrix we get equivalence classes which we shall call *transposition classes*. (There seems to be no commonly used name for this idea, even though the concept itself is in use. For example, transposition classes are the only equivalence classes considered in [6].) Transposition is an example of the operation called *conjugacy* which allows the roles of rows, columns and symbols to be permuted. If we consider a PLS to be equivalent to every conjugate of everything in its isotopy class then we get equivalence classes called *species* (sometimes also called *main classes*).

The concepts of isotopy classes, transposition classes and species extend naturally to bi-trades, provided that we insist that exactly the same operations must be carried out on both trades in the bi-trade. The equivalence classes will depend on whether we are considering our bi-trades to be ordered or unordered. For unordered trades, (T_1, T_2) is equivalent to (T_2, T_1), while this is not necessarily true for ordered trades.

3 The algorithm

In this section we outline the algorithm used to enumerate all Latin trades of size, say s. The algorithm has a number of stages, with the first 4 stages concerned with the primary goal of compiling a list of species representatives of trades of size s. Once that goal is achieved, further analysis and categorisation is relatively simple.

Since we are seeking species representatives we can safely make the following assumptions (where $\deg(v)$ denotes the degree of a vertex v):

Assumption 1. $\deg(r_1) \geq \deg(r_2) \geq \ldots \geq \deg(r_n)$.

Assumption 2. $\deg(c_i) \leq \deg(r_1)$ for all i.

Assumption 3. No symbol occurs in the trade more than $\max_i \deg(c_i)$ times.

Assumptions 2 and 3 are justified by taking an appropriate conjugate if necessary, while Assumption 1 is justified by applying an appropriate isotopy. There is nothing to be gained by assuming $\deg(c_1) \geq \deg(c_2) \geq \ldots \geq \deg(c_n)$, since we will generate our trades row by row and this condition cannot be used for pruning at an early stage of the search.

3.1 Stage 1: Generating shapes

The first task was to generate all possible shape graphs for trades of size s. By our comments in Section 2 it suffices to enumerate, up to isomorphism, a catalogue of all bridgeless bipartite graphs with s edges. Here "isomorphism" means nearly the same thing as the usual notion of isomorphism for graphs. The only difference is that the bipartition must be preserved. That is, the sets of vertices $\{r_1, \ldots, r_n\}$ and $\{c_1 \ldots, c_n\}$ must be mapped to themselves or completely interchanged. We do not allow some row vertices to map to row vertices while others get mapped to column vertices. Also, we emphasize that bridgeless does not imply connected, but only that each component is 2-edge connected.

This first step was both routine and fast (the computation for $s \leq 21$ was completed in about 5 minutes on a 2GHz PC, and could be made quicker if required). Shape graphs were generated by choosing, in turn, the neighbours for $r_1, r_2, r_3, \ldots, r_n$ with isomorphism testing to remove duplicates after each row was completed. The number of shape graphs was found to be as follows for $s \leq 21$.

s	4	6	7	8	9	10	11	12	13	14	15
# shapes	1	2	1	6	6	20	26	105	185	609	1472

s	16	17	18	19	20	21
# shapes	4732	13394	43985	139538	475218	1626423

3.2 Stage 2: Colouring the edges

The second task was to produce each possible edge-colouring of each shape graph of size s. Colours were assigned to edges subject to the rules that (a) The colours of the edges incident with any vertex were distinct (b) The c-th colour could not be used before the $(c-1)$-th colour had been used (c) No colour could be used on exactly one edge and (d) No colour could be used more than $\max \deg(c_i)$ times. Condition (a) guarantees a proper colouring and conditions (b), (c) and (d) were used to trim the search. The justification for (d) is Assumption 3.

In the first version of the program it was found that the overwhelming majority of outputs from Stage 2 were bad. That is, almost all colourings did not extend to a simultaneous 2-edge colouring. For example, when $s = 15$ there were 79663 good colourings and 1293435754 bad colourings output from Stage 2. Hence, it became important to develop a test which would quickly recognise many of the bad colourings as such.

This was the condition that we used: For each triple (i, j, k) there must be another filled cell (i, j') such that symbol k occurs in column j', and there must be a third filled cell (i', j) such that symbol k occurs in row i'. This condition could be checked for all s triples in a total time of $O(s)$. The number of bad outputs from Stage 2 was thereby rapidly and dramatically cut (in the case of $s = 15$ the number of bad outputs was reduced to 470776).

However, Stage 2 remained the bottleneck for the whole algorithm because of the large number of bad colourings that were generated before being discarded. A smarter programmer might have written code that pruned the generation for any colouring destined to fail the necessary condition given above. However, early recognition that a (partial) colouring has this property does not appear to be simple to implement.

It is possible to find other necessary conditions for a PLS to be a trade that could be tested quickly at the end of Stage 2. For example, every conjugate of the PLS must have a bridgeless shape graph. However, implementing such tests is of negligible gain unless the pruning described in the previous paragraph can be achieved first.

3.3 Stage 3: Screening for trades

Even with checks of necessary conditions in place at the end of Stage 2, there were still some PLSs which reached Stage 3 but nonetheless were not trades. The smallest example was (isotopic to) this:

1	2	·
2	1	3
·	3	1

Hence, the third task was to check whether an edge colouring was part of a simultaneous 2-edge colouring; that is, whether it represented a trade. Each edge was given a second colour using a backtracking algorithm, subject to the conditions for a simultaneous 2-edge colouring. Given Assumption 1 it was advisable to treat the rows in reverse order. That is, to colour edges incident with r_n, then those incident with r_{n-1}, and so on, until finally colouring those incident with r_1. The rationale for this order is that vertices of low degree have very few choices in their second colourings. For example, a vertex of degree 2 has no choice as to how to colour to its edges in the second colouring.

3.4 Stage 4: Isomorphism testing

Now that Stage 3 had produced a comparatively small number of outputs it was feasible to perform the relatively expensive task of isomorphism testing. This task was achieved using nauty [15] in a manner analogous to its use in [16]. This allowed us to select exactly one representative from each species of trades.

3.5 Stage 5: Analysis of results

With the aid of the complete catalogue of species representatives produced by Stage 4 it was then a simple matter to

(a) Find representatives of equivalence classes using other notions of equivalence (namely isotopy and transposition classes). This was done by taking the 6 conjugates of the species representatives and using nauty to filter the results appropriately (although faster methods are possible).

(b) Screen for minimal trades. We say a trade is *minimal* if it does not contain a trade of smaller size. Checking for minimal trades was not dissimilar to the task in Stage 3. The main difference was that in the backtracking we allow the possibility for each edge that it gets removed rather than receiving a second colour. If we find a trade after at least one edge was removed, then the original trade was not minimal.

(c) Generate bi-trades and filter them to obtain exactly one representative of each equivalence class (for each different notion of equivalence).

(d) Find the genus of each bi-trade and identify all homogeneous trades (see Section 5 for the results, and an explanation of the terminology).

(e) Find the smallest latin square in which each trade could be embedded (see Section 6).

	Species		Transposition classes		Isotopy classes	
Size	Minimal	All	Minimal	All	Minimal	All
4	1	1	1	1	1	1
6	1	1	2	2	3	3
7	0	1	0	1	0	1
8	2	5	4	9	6	13
9	1	4	2	7	3	10
10	6	19	13	40	23	66
11	3	26	7	63	12	112
12	30	163	78	401	146	723
13	38	445	106	1202	205	2283
14	290	2302	839	6503	1647	12616
15	806	10795	2368	31505	4694	62163
16	5460	63624	16228	187643	32319	372147
17	23164	381626	69292	1137602	138384	2268021
18	157533	2547990	471625	7617639	942341	15209245
19	877106	17919970	2629867	53688610	5258327	107306496

Table 1: Counts of equivalence classes of trades

4 Results

In Table 1 we give counts of all latin trades of size ≤ 19, classified into the various equivalence classes discussed in Section 2. The corresponding results for bi-trades are presented in Table 2.

	Species		Transposition classes		Isotopy classes	
Size	Unordered	Ordered	Unordered	Ordered	Unordered	Ordered
4	1	1	1	1	1	1
6	1	1	2	2	3	3
7	1	1	1	1	1	1
8	5	5	9	9	13	13
9	3	4	5	7	7	10
10	17	19	36	40	59	66
11	19	26	42	63	70	112
12	131	163	313	401	553	723
13	262	445	685	1202	1277	2283
14	1448	2302	3999	6503	7661	12616
15	5743	10798	16535	31512	32395	62175
16	34514	63690	100897	187798	199205	372420
17	194239	381883	577324	1138317	1149306	2269398
18	1302209	2550827	3886229	7625934	7752187	15225551
19	9011123	17945643	26982310	53764585	53913897	107457456

Table 2: Counts of equivalence classes of bi-trades

Our numbers of transposition classes for $s \leq 10$ agree with those obtained by Donovan et al. [6], but for $s = 11$ we found 63 classes rather than 62. A representative of the class omitted from [6] is this:

1	2	3	·
4	1	·	5
3	·	5	4
·	5	2	·

Note that throughout we will adopt the convention of omitting empty rows and columns when we display a trade. Table 1 also gives data on the minimal trades of each size. Species representatives for the minimal trades of size ≤ 11 are as follows:

1	2
2	1

1	2	3
2	3	1

1	2	3	4
2	3	4	1

1	2	3
4	3	1
2	4	·

1	2	3	·
4	1	·	2
3	·	2	4

1	2	3	4	5
2	3	4	5	1

1	2	3	4
5	3	4	1
2	5	·	·

1	2	3	4
5	3	1	·
4	5	·	2

1	2	3	4
2	3	4	·
4	1	·	3

1	2	3	·
4	3	1	·
2	·	·	4
·	4	·	2

1	2	3	·
4	1	·	2
3	·	4	·
·	·	2	4

1	2	3	4	·
5	1	2	·	3
4	·	·	3	5

1	2	3	4
5	3	1	·
4	·	·	5
·	5	·	2

1	2	3	·
4	1	·	5
3	·	5	4
·	5	2	·

5 Topological properties

In this section we consider the topological properties of bi-trades. Readers unfamiliar with the area are referred to Drápal [7] (some useful discussion also appears in [11]). We now define the key terms as they are defined in [7].

We use $P[i, j]$ to denote the symbol in row i, column j of a PLS P. Suppose (T_1, T_2) is an (ordered) bi-trade of size s. Let S be the shape of T_1 (and hence also of T_2). We define three permutations of the s elements of S. First, we define the row permutation ρ which maps a cell $(i, j) \in S$ to the unique $(i, j') \in S$ such that $T_1[i, j] = T_2[i, j']$. Similarly, we define the column permutation γ which maps a cell $(i, j) \in S$ to the unique $(i', j) \in S$ such that $T_1[i, j] = T_2[i', j]$. Finally, we define the symbol permutation σ by $\sigma = \gamma^{-1}\rho$.

We say that (T_1, T_2) is *connected* if $\langle \rho, \gamma, \sigma \rangle$ (the group of permutations of S generated by ρ, γ and σ) is transitive on S.

We say that (T_1, T_2) is *separated* if $\langle \rho \rangle$ is transitive on the cells in each row of S, $\langle \gamma \rangle$ is transitive on the cells in each column of S and $\langle \sigma \rangle$ is transitive on all those cells that in T_2 contain any given symbol. It turns out that (T_1, T_2) is connected (resp. separated) if and only if (T_2, T_1) is connected (resp. separated), so we can extend these notions to unordered trades as well.

For any permutation τ let $o(\tau)$ denote the number of orbits of $\langle \tau \rangle$. We can then define the *genus* of any connected bi-trade (T_1, T_2) of size s to be

$$1 + \frac{1}{2}(s - o(\rho) - o(\gamma) - o(\sigma)).$$

This genus is the topological genus of a surface in which the latin trade can be thought of as being embedded. For the details the reader is encouraged to consult [7]. For our purposes it suffices to know that there is interest in the genus of latin trades and that for these topological questions it suffices to concentrate on separated, connected, unordered bi-trades. Note that for separated trades $o(\rho)$ is simply the number of rows which contain non-empty cells, and similar statements hold for $o(\gamma)$ and $o(\sigma)$.

In Table 3 we present counts of the species of separated unordered bi-trades. These bi-trades were classified first into disconnected or connected, and then the genus of each connected bi-trade was calculated.

The smallest bi-trades of genus 0, 1 and 3 are based on latin squares of orders 2,

Size	Disconnected	Genus 0	Genus 1	Genus 2	Genus 3	Total
4		1				1
6		1				1
7		1				1
8	1	2				3
9		2	1			3
10	1	8				9
11	1	8	1			10
12	5	32	10			47
13	4	57	21			82
14	14	185	123	2		324
15	17	466	565	5		1053
16	72	1543	3003	67	1	4686
17	121	4583	14944	689		20337
18	474	15374	75985	7985	15	99833
19	1376	50116	371549	82536	288	505865

Table 3: Separated bi-trades counted by genus

3 and 4, respectively. The two smallest bi-trades of genus 2 are:

1	2	3	4	2	3	4	1		1	2	3	4	3	4	2	1	
2	3	4	1	3	4	1	2		2	3	4	1	4	1	3	2	(1)
3	4	1	·	4	1	3	·		3	4	2	·	2	3	4	·	
4	1	·	2	1	2	·	4		4	1	·	2	1	2	·	4	

The general problem of finding the minimum size among bi-trades of a given genus has recently been solved by Lefevre et al. [12]. Their results are consistent with ours for trades of size ≤ 19.

It happens that the sequence of counts of genus 0 bi-trades in Table 3 matches sequence A007083 in Sloane's encyclopaedia [17] (see also [10]). This leads to the suspicion that the number of species of bi-trades of size n and genus 0 is equal to the number of unlabelled 3-connected cubic bipartite planar graphs with $2n$ vertices. That this is indeed the case has now been shown by Cavenagh and Lisoněk [5]. This theoretical insight confirms the value of the resource [17] and of enumerations such as those reported herein.

Topological considerations have recently generated interest [1], [2], [3], [4] in so-called *k-homogeneous* trades. These are trades in which each row and column contains either 0 or k filled cells and each symbol occurs k times or not at all.

The number of species of homogeneous latin trades of size up to 19 are shown in Table 4 (sizes for which there are no homogeneous trades have been omitted from the table). Most of the trades in Table 4 are composed of the union of disjoint (meaning they do not share any row, column or symbol) latin squares. Species representatives

	k		
Size	2	3	4
4	1		
8	1		
9		1	
12	1	2	
15		1	
16	1		2
18		3	

Table 4: Number of k-homogeneous trades

for the homogeneous trades which do *not* fit this description are given in (2). Each of these trades is 3-homogeneous.

1	2	3	·
4	1	·	2
2	·	4	3
·	3	1	4

1	2	3	·
4	3	·	1
2	·	4	3
·	4	1	2

1	2	3	·	·
2	4	·	1	·
3	·	2	·	5
·	1	·	5	4
·	·	5	4	3

1	2	3	·	·	·
2	4	·	1	·	·
3	·	2	·	5	·
·	1	·	6	·	4
·	·	5	·	3	6
·	·	·	4	6	5

1	2	3	·	·	·
4	3	·	1	·	·
·	·	5	3	6	·
·	·	1	6	·	5
2	·	·	·	4	6
·	4	·	·	5	2

(2)

6 Embedding

Finally, we looked at the question of embedding our trades in latin squares. For a PLS P we define the *embedding order* to be the order of the smallest latin square which contains P. For each species representative we found the embedding order. The results are shown in Table 5 where they are categorised according to size and order of the trade. Here we define the order of a trade T to be $\max\{n_r, n_c, n_s\}$ where n_r, n_c are respectively the number of rows and columns which contain filled cells in T and n_s is the number of different symbols used in T. The order is an obvious lower bound on the embedding order. A celebrated theorem of Evans [8] says that the embedding order is never more than twice the order. Of the trades that we generated the only ones to achieve this bound are the trades appearing in (1), both of which have an order of 4 and an embedding order of 8.

		Embedding order									
Size	Order	2	3	4	5	6	7	8	9	10	11
4	2	1									
6	3		1								
7	3			1							
8	4			2	1	2					
9	3		1								
	4				3						
10	4			3	2	6					
	5				3	2	3				
11	4				1	3	1				
	5				2	15	4				
12	4			9	2	8					
	5				12	68	20				
	6					20	11	13			
13	4			2	1		3				
	5				17	203	74	2			
	6					40	90	13			
14	4			3				2			
	5				45	532	314	6			
	6					319	810	141	1		
	7						62	45	22		
15	4			1							
	5				59	1178	1050	29			
	6					1332	5325	641	3		
	7						568	561	48		
16	4			2							
	5				123	2084	2510	78			
	6					5674	32901	5258	16		
	7						6836	6917	500	2	
	8							511	157	55	
17	5				174	2570	4749	348			
	6					17111	170911	35401	86		
	7						61503	74977	3915	10	
	8							7180	2566	125	
18	5				271	2265	5783	1279	96		
	6					47402	783566	238594	850		
	7						479481	774707	40793	45	
	8							124457	42816	1814	2
	9								3235	432	102
19	5				328	1340	2196	5014	10		
	6					113799	3093391	1367832	6793	1	
	7						3197883	7280666	428624	537	
	8							1678937	633153	16820	14
	9								82416	9869	347

Table 5: Embedding orders of trades

7 Concluding remarks

We have generated a catalogue of all latin trades of size ≤ 19 and categorised them according to various properties of interest. A major motivation for compiling this catalogue is that it should be available to any researcher who might find it useful. To that end, the author has made the catalogue freely available via his homepage, which is currently at [19].

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