AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **39** (2007), Pages 27–37

The Erdős-Sós conjecture for spiders of diameter 9

Lingli Sun*

Department of Mathematics and Informatics Sciences
Huazhong Agricultural University
Wuhan 430070
P.R. China
sunlingli@yahoo.com.cn

Abstract

The Erdős-Sós conjecture says that any graph G on n vertices with $e(G) > \frac{k-1}{2}n$ contains every tree of k edges. In *J. Graph Theory* 21 (1996), 229–234, Woźniak proved that the conjecture is true for spiders of diameter at most 4. In *Discrete Math.* (2007) (in press), Fan and Sun proved that the conjecture is true for spiders with no leg of length more than 4. In this paper, we prove that the conjecture is true for spiders of diameter at most 9, which strengthens both results mentioned above.

1 Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For $S \subseteq V(G)$, $G - S$ denotes the graph obtained from G by deleting all the vertices of S together with all the edges with at least one end in S. When $S = \{x\}$, we simplify this notation to $G - x$. If $xy \in E(G)$, we say that x is joined to y and that y is a *neighbor* of x. For a subgraph H of G, $N_H(x)$ is the set of the neighbors of x which are in H, and $d_H(x) = |N_H(x)|$ is the *degree* of x in H. When no confusion can occur, we shall write $N(x)$ and $d(x)$, instead of $N_G(x)$ and $d_G(x)$. For $A, B \subseteq V(G)$, $E(A, B)$ denotes the set, and $e(A, B)$ the number, of edges with one end in A and the other end in B. For simplicity, we write $e(A)$ for $e(A, A)$ and $e(G)$ for $e(V(G), V(G))$ $(=|E(G)|)$. When $A = \{a\}$, we simplify the notation to $e(a, B)$ $(= d_B(a))$.

A *spider* is a tree with at most one vertex of degree more than 2, called the *center* of the spider (if no vertex of degree more than two, then any vertex can be the center). A *leg* of a spider is a path from the center to a vertex of degree 1. Thus, a star with k edges is a spider of k legs, each of length 1, and a path is a spider of 1 or 2 legs. A k-edge spider is a spider with k edges.

* The project was supported financially by the fund of Huazhong Agricultural University.

A *caterpillar* is a tree in which the non-leaf vertices and the edges with two non-leaf endpoints constitute a path.

A classical result on extremal graph theory is the **Erdős-Gallai theorem:** *every graph G with $e(G) > \frac{(k-1)}{2}|V(G)|$ contains a path of k edges.* Motivated by the result, Erdős and Sós made the following conjecture (see [2]):

Erdős-Sós Conjecture: *If G is a graph on n vertices with $e(G) > \frac{k-1}{2}n$, then G contains every tree of k edges.*

The conjecture seems to be very difficult. There are only a few partial results known, mainly in two directions. One is to pose conditions on the graph G, such as graphs of girth 5 by Brandt and Dobson [1], and then this was improved to graphs with no cycle of length 4 by Saclé and Woźniak [7]. The other is to pose conditions on the tree, such as trees with a vertex joined to at least $\frac{k-1}{2}$ vertices of degree 1 by Sidorenko [8], spiders of diameter at most 4 by Woźniak [9] and spiders with no leg of length more than 4 by Fan and Sun [4]. In this paper we prove that the conjecture is true for spiders of diameter 9, which strengthens both the result of Woźniak and the result of Fan and Sun mentioned above.

2 Spiders of Diameter 9

Theorem 2.1 [6] *If G is a graph on n vertices with $e(G) > \frac{k-1}{2}n$, then G contains every k-edge caterpillar.*

Theorem 2.2 [4] *If G is a graph on n vertices with $e(G) > \frac{k-1}{2}n$, then G contains every k-edge spider of three legs.*

Theorem 2.3 [4] *If G is a graph on n vertices with $e(G) > \frac{k-1}{2}n$, then G contains every k-edge spider that has no leg of length more than 4.*

Before we prove the main theorem, we first prove an important lemma, from which the reader may see the main idea of our proof.

Lemma 2.4 *If G is a graph on n vertices with $e(G) > \frac{k-1}{2}n$, and T is a k-edge spider with the longest leg of length at most $\frac{k+2}{2}$ and other legs of lengths at most 3, then G contains a copy of T.*

Proof. Let T be a k-edge spider with the longest leg of length at most $(k+2)/2$ and other legs of lengths at most 3. Consider a minimal subgraph G' of G such that $e(G') > \frac{k-1}{2}|V(G')|$. Clearly, if G' contains T, then so does G. For simplicity, we may just assume that G is the minimal graph with $e(G) > \frac{k-1}{2}n$. For any complete subgraph $K_m \subseteq G$,

$$e(G - K_m) > \frac{(k-1)n}{2} - (\sum_{v \in K_m} d(v) - e(K_m));$$

if $\sum_{v\in K_m} d(v) \le \frac{m}{2}(k+m-2)$, then

$$e(G-K_m) > \frac{(k-1)(n-m)}{2},$$

contradicting the minimality of G. Therefore we have:

(2.1) for each $K_m \subseteq G$, $\sum_{v\in K_m} d(v) > \frac{m}{2}(k+m-2)$.

Let x be the center of T. We prove the result by induction on the degree of x in T. If $d_T(x) = k$, that is, T is a star with k edges, then clearly G has a copy of T centered at any vertex of degree at least k in G (the existence of such a vertex is guaranteed by $e(G) > \frac{k-1}{2}n$). Suppose therefore that $d_T(x) < k$ and the result holds for all k-edge spiders with the longest leg of length at most $\frac{k+2}{2}$ and other legs of lengths at most 3 and whose centers have degree more than $d_T(x)$.

Since T is not a star, T has a leg of length at least 2. Let $R = xv_1v_2\cdots v_ty$ be a longest leg of T, by theorem 2.3, we have that $t \ge 4$. Let $T' = T - y + \{xy\}$. Then $d_{T'}(x) = d_T(x)+1$, and by the induction hypothesis, G contains a copy T'' of T'. For simplicity, we use the same notations for the vertices of T'' and T', and so T'' has legs $xv_1v_2\cdots v_t$ and xy. Set

$$P_0 = v_1v_2\cdots v_t.$$

Consider a longest path L in $G - V(T''-y)$, starting at y, say

$$L = u_1u_2\cdots u_s,$$

where $u_1 = y$. We may assume that $s \le t$, for otherwise replacing xP_0 by a segment of xL with length $t+1$ yields a copy of T in G.

In what follows, we suppose, to the contrary, that G does not contain a copy of T, and shall arrive at a contradiction to the degree sum $d(u_s)+d(v_t)$.

By the maximality of L, $N(u_s) \subseteq V(T'') \cup V(L)$. Also, $N(v_t) \subseteq V(T''-y)$, for otherwise a copy of T is obtained by extending P_0 at v_t, and in particular, $e(v_t, L) = 0$.

We note that $u_sv_1 \notin E(G)$, for otherwise replacing xP_0 by $xu_1\cdots u_sv_1v_2\cdots v_{t-s+1}$ gives a copy of T in G. Furthermore, if $v_iv_t \in E(G)$, then $v_{i+1}u_s \notin E(G)$, for otherwise $xv_1\cdots v_iv_tv_{t-1}\cdots v_{i+1}u_s$ is a leg of length $t+1$ and a copy of T is obtained. In particular, $v_tu_s \notin E(G)$. Therefore, we have that

(2.2) $e(u_s, P_0) + e(v_t, P_0) \le |V(P_0)| - 1$, with equality only if $v_{i+1}u_s \in E(G)$ whenever $v_iv_t \notin E(G)$ for each i, $1 \le i \le t-1$.

Let $P_1, P_2, \ldots, P_\ell$ be the vertex-disjoint paths of $T'' - (V(P_0) \cup \{x, y\})$. We see that $|V(P_i)| \le 3$, $1 \le i \le \ell$. For any $P \in \{P_1, P_2, \ldots, P_\ell\}$, we shall prove a little stronger result which will be used in later proof. (Note: this result has been proved in [4], but we repeat the proof for the completeness of the proof of the theorem.)

(2.3) $e(u_s, P) + e(v_t, P) \le |V(P)|$ if $|V(P)| \le 4$.

Let $P = a_1a_2\cdots a_p$. If $p \le s$, then $e(v_t, P) = 0$, for otherwise, suppose that $v_ta \in E(G)$ for some $a \in V(P)$, then $xv_1v_2\cdots v_ta$ and $xu_1u_2\cdots u_p$ are legs of lengths $t+1$ and p, respectively, which yields a copy of T in G. Thus, $e(v_t, P) + e(u_s, P) = e(u_s, P) \le p$, as required by (2.3). In what follows, we assume that $p > s$. Noting that $p \le 4$, we have the following three cases.

(i) $p = s + 1$. If $e(u_s, P) > 0$, say $u_sa \in E(G)$ for some $a \in V(P)$, then $L + u_sa$ is a path of the same length as P, which means that u_s and v_t cannot be joined to two distinct vertices of P, and hence either $e(u_s, P) + e(v_t, P) \le 2 \le p$ or $e(u_s, P) + e(v_t, P) = e(u_s, P) \le p$. Otherwise, $e(u_s, P) = 0$ and so $e(u_s, P) + e(v_t, P) = e(v_t, P) \le p$.

(ii) $p = s + 2$. If $v_ta_1 \in E(G)$, then $e(u_s, P - a_1) = 0$, for otherwise, $Lu_sa_2a_3$, or $Lu_sa_{p-1}a_p$, or $Lu_sa_pa_{p-1}$ is a path of the same length as P; similarly, if $v_ta_p \in E(G)$, then $e(u_s, P - a_p) = 0$. Thus, if $e(v_t, \{a_1, a_p\}) = 2$, then $e(u_s, P) = 0$ and so $e(u_s, P) + e(v_t, P) \le p$. If $e(v_t, \{a_1, a_p\}) = 1$, then $e(u_s, P) \le 1$, and so $e(u_s, P) + e(v_t, P) \le 1 + (p - 1) = p$. Suppose therefore that $e(v_t, \{a_1, a_p\}) = 0$. Then, $e(v_t, P - a_1 - a_p) > 0$, for otherwise, $e(v_t, P) = 0$ and we have that $e(u_s, P) + e(v_t, P) \le p$. For $p = 3$, $e(v_t, P - a_1 - a_p) > 0$ implies that $v_ta_2 \in E(G)$, which means that u_s cannot be joined to both a_1 and a_3, for otherwise $a_1u_sa_3$ is a path of the same length as P, and therefore $e(u_s, P) + e(v_t, P) \le 2 + 1 = p$. For $p = 4$, as seen above, if $v_ta_2 \in E(G)$, then $e(u_s, \{a_3, a_4\}) = 0$; and if $v_ta_3 \in E(G)$, then $e(u_s, \{a_1, a_2\}) = 0$. Thus, $e(u_s, P) + e(v_t, P) \le 3 < p$.

(iii) $p = s + 3$. Then, $p = 4$ and $s = 1$. We note that if $u_sa_1 \in E(G)$ ($u_sa_3 \in E(G)$), then $u_sa_1a_2a_3$ ($u_sa_3a_2a_1$) is a path of the same length as P. Thus, if $v_ta_4 \in E(G)$, then $e(u_s, \{a_1, a_3\}) = 0$. Moreover, if $e(u_s, \{a_1, a_3\}) = 2$, then $v_ta_2 \notin E(G)$, for otherwise, $a_1u_sa_3a_4$ is a path of the same length as P. This gives that $e(u_s, \{a_1, a_3\}) + e(v_t, \{a_2, a_4\}) \le 2$. Similarly, noting that if $e(u_s, \{a_2, a_4\}) = 2$, then $a_1a_2u_sa_4$ is a path of the same length as P, which means that $v_ta_3 \notin E(G)$, and thus $e(u_s, \{a_2, a_4\}) + e(v_t, \{a_1, a_3\}) \le 2$. Consequently, $e(u_s, P) + e(v_t, P) \le 4 = p$. This completes the proof of (2.3).

By (2.2) and (2.3), we have that $d(u_s) + d(v_t) \le |V(T'' - y - x)| - 1 + e(\{u_s, v_t\}, x) + e(u_s, L)$, that is,

(2.4) $d(u_s) + d(v_t) \le k - 2 + e(\{u_s, v_t\}, x) + e(u_s, L)$, with equality only if all equalities hold in (2.2) and (2.3).

By (2.1) with $m = 1$, we have that

(2.5) $d(u_s) \ge \frac{k}{2}$ and $d(v_t) \ge \frac{k}{2}$.

Let T^* be the spider in G with legs xP_i, $0 \le i \le \ell$, and xL. From the proof above, consider spiders having $\ell + 2$ legs xQ_i, $0 \le i \le \ell$, and xL', where $|V(Q_i)| = |V(P_i)|$, $0 \le i \le \ell$, and $|V(L')| = |V(L)|$. We may suppose that T^* has been chosen such that $d(u_s) + d(v_t)$ is maximum over all such spiders in G.

For a path $P \in \{P_1, P_2, \dots, P_\ell\}$, we say that P is *usable* at u_s if the subgraph induced by $V(P) \cup \{u_s\}$ has a path of length $|V(P)|$ (a hamiltonian path of the induced subgraph), starting at u_s. Thus, if u_s is joined to an end of P, then P is usable at u_s. If each P_i, $1 \leq i \leq \ell$, is usable at u_s, then we have a copy of T centered at u_s, in which each P_i together with u_s gives a leg of length $|V(P_i)|$, $1 \leq i \leq \ell$, and $u_s u_{s-1} \cdots u_1 x v_1 v_2 \cdots v_{t-s+1}$ is a leg of length $t+1$. Therefore, there must be some $Q \in \{P_1, P_2, \dots, P_\ell\}$ such that Q is not usable at u_s. Let

$$Q = b_1 b_2 \cdots b_q.$$

Since Q is not usable at u_s, we have that

(2.6) $\quad e(u_s, \{b_1, b_q\}) = 0.$

For any $P \in \{P_1, P_2, \dots, P_\ell\}$, suppose $|V(P)| \leq s$. Let $P = a_1 a_2 \cdots a_p$ and $p \leq s$. Suppose that there exists a vertex $a \in V(P)$ with $v_t a \in E(G)$, then $x v_1 v_2 \cdots v_t a$ and $x u_1 u_2 \cdots u_p$ are legs with length $t+1$ and p, which implies that G contains a copy of T. So we have:

(2.7) for any $P \in \{P_1, P_2, \dots, P_\ell\}$, if $|V(P)| \leq s$, then $e(v_t, P) = 0$.

Now we consider the case $s \geq 3$.

Since $|V(P_i)| \leq 3$, $1 \leq i \leq \ell$, for any $P \in \{P_1, P_2, \dots, P_\ell\}$, we have that $|V(P)| \leq s$. By (2.7), we have that $e(v_t, P) = 0$. Since $t+1 \leq \frac{k+2}{2}$, we have $d(v_t) \leq t \leq \frac{k}{2}$. By (2.5), we have $d(v_t) = \frac{k}{2}$, then $N(v_t) = \{x, v_1, v_2, \dots, v_{t-1}\}$. The leg $x v_1 v_2 \cdots v_{t-2} v_t v_{t-1}$ implies that $d(v_{t-1}) \leq d(v_t)$, and applying (2.1) to $d(v_{t-1}) + d(v_t)$, we have $d(v_t) > \frac{k}{2}$, a contradiction.

The rest of the proof is divided into two cases, according to the values of s.

Case 1. $s = 1$.

Then $e(u_s, L) = 0$, by (2.4) and (2.5), we have that

$$v_t x \in E(G),\ d(u_s) = d(v_t) = \frac{k}{2}.$$

Moreover, the equalities in (2.2) and (2.3) hold.

If $v_{t-2} v_t \in E(G)$, we replace $x P_0$ by $x v_1 v_2 \cdots v_{t-2} v_t v_{t-1}$; then a copy of T^* is obtained, in which v_{t-1} plays the same role as v_t. By the choice of T^* ($d(u_s) + d(v_t)$ is maximum), we have $d(v_{t-1}) \leq d(v_t)$. By (2.1) with $m = 2$, $d(v_{t-1}) + d(v_t) > k$, then $d(v_t) > k/2$, a contradiction. So $v_{t-2} v_t \notin E(G)$. Since the equality of (2.2) holds, we have $u_s v_{t-1} \in E(G)$. It is clear that $u_s v_{t-2} \notin E(G)$, for otherwise replacing $x P_0$ by $x v_1 v_2 \cdots v_{t-2} u_s v_{t-1} v_t$ yields a copy of T. Therefore we have $v_{t-3} v_t \in E(G)$. Similarly, we have $v_{t-4} v_t \notin E(G)$, $v_{t-5} v_t \in E(G)$, $\cdots$.

If t is even, then we have $v_1 v_t \in E(G)$. With $x v_t v_{t-1} \cdots v_1$ in place of $x P_0$, we obtain a copy of T^*, in which v_1 and v_t play the same role. By the choice of T^* ($d(u_s) + d(v_t)$

is maximum), we have $d(v_1) \leq d(v_t)$. By (2.1) with $m = 2$, $d(v_1) + d(v_t) > k$, so $d(v_t) > k/2$, a contradiction.

If t is odd, then $v_4v_t \in E(G)$, $v_3v_t \notin E(G)$, $v_2v_t \in E(G)$, and $v_1v_t \notin E(G)$.

Now we consider Q. Since $|V(Q)| \leq 3$ and the equality of (2.3) holds, by (2.6), we have $e(v_t, Q) = |V(Q)|$. By (2.7), we have $q > s$, so $q = 2$ or 3.

If $q = 2$, replacing xQ by xv_1v_2 and xP_0 by $xb_1b_qv_tv_{t-1}\cdots v_3$ gives a copy of T^*. The leg $xb_1b_qv_tv_{t-1}\cdots v_3$ implies that $d(v_3) \leq d(v_t) = \frac{k}{2}$. By (2.5) we have $d(v_3) = \frac{k}{2} = d(v_t)$, which means that there is no difference between v_t and v_3. We repeat the same arguments to $xb_1b_qv_tv_{t-1}\cdots v_3$; then we have $xv_3 \in E(G)$. Replacing xQ by xv_1v_2 and xP_0 by $xv_3v_4\cdots v_tb_1b_q$ yields a copy of T^*. The leg $xv_3v_4\cdots v_tb_1b_q$ implies that $d(b_q) \leq d(v_t)$. Applying (2.1) to $d(b_q) + d(v_t)$, we have $d(v_t) > \frac{k}{2}$, a contradiction.

If $q = 3$, then replacing xQ by $xv_1v_2v_3$ and xP_0 by $xb_1b_2b_qv_tv_{t-1}\cdots v_4$ gives a copy of T^*. The leg $xb_1b_2b_qv_tv_{t-1}\cdots v_4$ implies that $d(v_4) \leq d(v_t)$. Applying (2.1) to $d(v_4) + d(v_t)$, we have $d(v_t) > \frac{k}{2}$, a contradiction.

Case 2. $s = 2$. Then $e(u_s, L) = 1$.

If the equality of (2.2) does not hold, rewriting (2.4) we have

$$d(u_s) + d(v_t) \leq k - 2 + e(\{u_s, v_t\}, x).$$

By (2.5), we have $u_sx, v_tx \in E(G)$ and $d(u_s) = d(v_t) = k/2$. Replacing xu_1u_s by xu_su_1 yields a copy of T^*; as above, $d(u_1) \leq d(u_s)$, and applying (2.1) to $d(u_1)+d(u_s)$, we have $d(u_s) > k/2$, contradiction.

So the equality of (2.2) holds. By (2.5) and (2.4), we have $e(\{u_s, v_t\}, x) \geq 1$.

If $e(\{u_s, v_t\}, x) = 1$, then $d(u_s) = d(v_t) = \frac{k}{2}$, $u_sx \in E(G)$ or $v_tx \in E(G)$.

If $u_sx \in E(G)$, as above, we have $d(u_s) > k/2$, a contradiction.

Otherwise we have $v_tx \in E(G)$. We have $u_sv_2 \notin E(G)$, for otherwise with $xu_1u_sv_2v_3\cdots v_t$ in place of xP_0, a copy of T is obtained. Since the equality of (2.2) holds, we have $v_1v_t \in E(G)$. Then the leg $xv_tv_{t-1}\cdots v_1$ implies that $d(v_t) > k/2$, a contradiction.

So $e(\{u_s, v_t\}, x) = 2$, then $u_sx, v_tx \in E(G)$. Replacing xu_1u_s by xu_su_1 gives a copy of T^*, as above, $d(u_1) \leq d(u_s)$, applying (2.1) to $d(u_1) + d(u_s)$, we have that $d(u_s) \geq \frac{k+1}{2}$. Furthermore, from above, we have that $v_1v_t \in E(G)$, then the leg $xv_tv_{t-1}\cdots v_1$ implies that $d(v_1) \leq d(v_t)$.

If $v_2v_t \in E(G)$, then the leg $xv_1v_tv_{t-1}\cdots v_2$ implies that $d(v_2) \leq d(v_t)$. Applying (2.1) to $d(v_1) + d(v_2) + d(v_t)$, we have that $d(v_t) > \frac{k+1}{2}$. Then we have that $\frac{k+1}{2} + \frac{k+1}{2} < d(u_s) + d(v_t) \leq k - 2 + 2 + 1$, contradiction.

Otherwise $v_2v_t \notin E(G)$. Since the equality of (2.2) holds, we have that $v_3u_s \in E(G)$. $xv_1v_tv_{t-1}\cdots v_3u_su_1$ in place of xP_0, we obtain a copy of T. This completes the proof of Lemma 2.4. ■

Proposition 2.5 *If G is a graph on n vertices with $e(G) > \frac{k-1}{2}n$, and T is a k-edge spider with $k \leq 11$, then G contains a copy of T.*

Proof. By the Erdős-Gallai theorem, we may assume that T is not a path, i.e., T has at least three legs.

Since $k \le 11$, if the length of the longest leg R of T is more than $\frac{k+2}{2}$, then $T - E(R)$ has at most 4 edges. Otherwise $\frac{k-3}{2} \ge e(T - E(R)) > 4$, and we have $k > 11$, a contradiction. Therefore T is a spider with three legs or a caterpillar. By Theorem 2.2 or Theorem 2.1, G contains a copy of T.

Otherwise the length of the longest leg R of T is no more than $\frac{k+2}{2}$. If T has no leg of length more than 4, by Theorem 2.3, G contains a copy of T. Otherwise $e(R) \ge 5$. If the length of each leg of $T - E(R)$ is no more than 3, then by Lemma 2.4 we have that G contains a copy of T. Otherwise there exists a leg Q in $T - E(R)$ with $e(Q) \ge 4$, and then $e(T - E(R \cup Q)) \le 2$. Then T is a spider with three legs or a caterpillar. By Theorem 2.2 or Theorem 2.1, G contains a copy of T. This completes the proof. ■

Theorem 2.6 *If G is a graph on n vertices with $e(G) > \frac{k-1}{2}n$, then G contains every k-edge spider with diameter at most* 9.

Proof. Let T be a k-edge spider with diameter at most 9. Consider a minimal subgraph G' of G such that $e(G') > \frac{k-1}{2}|V(G')|$. Clearly, if G' contains T, then so does G. For simplicity, we may just assume that G is the minimal graph with $e(G) > \frac{k-1}{2}n$. In the proof of Lemma 2.4, we have:

(2.1) for each $K_m \subseteq G$, $\sum_{v \in K_m} d(v) > \frac{m}{2}(k + m - 2)$.

Let x be the center of T. We prove the result by induction on the degree of x in T. If $d_T(x) = k$, that is, T is a star with k edges, then clearly G has a copy of T centered at any vertex of degree at least k in G (the existence of such a vertex is guaranteed by $e(G) > \frac{k-1}{2}n$). Suppose therefore that $d_T(x) < k$ and the result holds for all k-edge spiders with diameter at most 9 and whose centers have degree more than $d_T(x)$.

Since T is not a star, T has a leg of length at least 2. Let $R = xv_1v_2\cdots v_ty$ be a longest leg of T, by Theorems 2.1 and 2.3, we have $6 \ge t \ge 4$. Let $T' = T - y + \{xy\}$. Then $d_{T'}(x) = d_T(x) + 1$, and by the induction hypothesis, G contains a copy T'' of T'. For simplicity, we use the same notation for the vertices of T'' and T', and so T'' has legs $xv_1v_2\cdots v_t$ and xy. Set

$$P_0 = v_1v_2\cdots v_t.$$

Consider a longest path L in $G - V(T'' - y)$, starting at y, say

$$L = u_1u_2\cdots u_s,$$

where $u_1 = y$. We may assume that $s \le t$, for otherwise replacing xP_0 by a segment of xL with length $t + 1$ yields a copy of T in G.

In what follows, we suppose, to the contrary, that G does not contain a copy of T, and shall arrive at a contradiction to the degree sum $d(u_s) + d(v_t)$.

By the maximality of L, $N(u_s) \subseteq V(T'') \cup V(L)$. Also, $N(v_t) \subseteq V(T'' - y)$, for otherwise a copy of T is obtained by extending P_0 at v_t, and in particular, $e(v_t, L) = 0$. Also in the proof of Lemma 2.4, we have $u_s v_1 \notin E(G)$ and

(2.2) $e(u_s, P_0) + e(v_t, P_0) \leq |V(P_0)| - 1$, with equality only if $v_{i+1}u_s \in E(G)$ whenever $v_i v_t \notin E(G)$ for each i, $1 \leq i \leq t-1$.

Let $P_1, P_2, \ldots, P_\ell$ be the vertex-disjoint paths of $T'' - (V(P_0) \cup \{x, y\})$. Since the diameter of T is at most 9 and R is the longest leg of T, we see that $|V(P_i)| \leq 4$, $1 \leq i \leq \ell$. For any $P \in \{P_1, P_2, \ldots, P_\ell\}$, in the proof of Lemma 2.4, we have that

(2.3) $e(u_s, P) + e(v_t, P) \leq |V(P)|$.

By (2.2) and (2.3), we have $d(u_s) + d(v_t) \leq |V(T''-y-x)| - 1 + e(\{u_s, v_t\}, x) + e(u_s, L)$, that is,

(2.4) $d(u_s) + d(v_t) \leq k - 2 + e(\{u_s, v_t\}, x) + e(u_s, L)$, with equality only if all equalities hold in (2.2) and (2.3).

By (2.1) with $m = 1$, we have

(2.5) $d(u_s) \geq \frac{k}{2}$ and $d(v_t) \geq \frac{k}{2}$.

Let T^* be the spider in G with legs xP_i, $0 \leq i \leq \ell$, and xL. From the proof above, consider spiders having $\ell + 2$ legs xQ_i, $0 \leq i \leq \ell$, and xL', where $|V(Q_i)| = |V(P_i)|$, $0 \leq i \leq \ell$, and $|V(L')| = |V(L)|$. We may suppose that T^* has been chosen such that $d(u_s) + d(v_t)$ is maximum over all such spiders in G.

For a path $P \in \{P_1, P_2, \ldots, P_\ell\}$, we say that P is *usable* at u_s if the subgraph induced by $V(P) \cup \{u_s\}$ has a path of length $|V(P)|$ (a hamiltonian path of the induced subgraph), starting at u_s. Thus, if u_s is joined to an end of P, then P is usable at u_s. If each P_i, $1 \leq i \leq \ell$, is usable at u_s, then we have a copy of T centered at u_s, in which each P_i together with u_s gives a leg of length $|V(P_i)|$, $1 \leq i \leq \ell$, and $u_s u_{s-1} \cdots u_1 x v_1 v_2 \cdots v_{t-s+1}$ is a leg of length $t+1$. Therefore there must be some $Q \in \{P_1, P_2, \ldots, P_\ell\}$ such that Q is not usable at u_s. Let

$$Q = b_1 b_2 \cdots b_q.$$

Since Q is not usable at u_s, we have

(2.6) $e(u_s, \{b_1, b_q\}) = 0$.

For any $P \in \{P_1, P_2, \ldots, P_\ell\}$, suppose $|V(P)| \leq s$. Let $P = a_1 a_2 \cdots a_p$ and $p \leq s$. Suppose that there exists a vertex $a \in V(P)$ with $v_t a \in E(G)$; then $x v_1 v_2 \cdots v_t a$ and $x u_1 u_2 \cdots u_p$ are legs with length $t+1$ and p, which implies that G contains a copy of T. So we have:

(2.7) for any $P \in \{P_1, P_2, \ldots, P_\ell\}$, if $|V(P)| \le s$, then $e(v_t, P) = 0$.

The rest of the proof is divided into three cases, according to the values of t.

Case 1. $t = 4$.
Since $s \le t$, we have four subcases:

(i) $s = 1$. Then $e(u_s, L) = 0$, and by (2.4) and (2.5), we have that

$$v_t x \in E(G),\ d(u_s) = d(v_t) = \frac{k}{2}.$$

Moreover, all equalities in (2.2) and (2.3) hold. If $v_1 v_t \in E(G)$, replacing xP_0 by $xv_t v_3 v_2 v_1$, we have a copy of T^* in which v_1 plays the same role as v_t. By the choice of T^*(the maximality of $d(u_s)+d(v_t)$), we have that $d(v_1) \le d(v_t)$. By (2.1) with $m = 2$, $d(v_1)+d(v_t) > k$ and hence $d(v_t) > \frac{k}{2}$, a contradiction. Otherwise $v_1 v_t \notin E(G)$, since the equality of (2.2) holds, we have that $u_s v_2 \in E(G)$. It is clear that $u_s v_3 \notin E(G)$ for otherwise we have a copy of T in which $xv_1 v_2 u_s v_3 v_t$ in place of xP_0. As a result we have that $v_2 v_t \in E(G)$. The leg $xv_1 v_2 v_t v_3$ implies that $d(v_3) \le d(v_t)$, as above, from which we get $d(v_t) > k/2$, a contradiction.

(ii) $s = 2$. Then $e(u_s, L) = 1$.
If the equality of (2.2) does not hold, rewrite (2.4), we have

$$d(u_s) + d(v_t) \le k - 2 + e(\{u_s, v_t\}, x).$$

By (2.5), we have that $u_s x, v_t x \in E(G)$ and $d(u_s) = d(v_t) = \frac{k}{2}$. We have a copy of T^* in which $xu_s u_1$ in place of $xu_1 u_s$. As before, $d(u_1) \le d(u_s)$ and so $d(u_s) > \frac{k}{2}$, a contradiction again.

So the equality of (2.2) holds. But $e(u_s, P_0) = 0$, so we have $e(v_t, P_0) = |V(P_0)| - 1$. By (2.5), we have $e(\{u_s, v_t\}, x) \ge 1$.

If $e(\{u_s, v_t\}, x) = 1$, then $d(u_s) = d(v_t) = \frac{k}{2}$, $u_s x \in E(G)$ or $v_t x \in E(G)$.

If $u_s x \in E(G)$, as above, we have $d(u_s) > k/2$, a contradiction.

If $v_t x \in E(G)$, the leg $xv_t v_3 v_2 v_1$ implies that $d(v_1) \le d(v_t)$; applying (2.1) to $d(v_1) + d(v_t)$, we have $d(v_t) > k/2$, a contradiction.

If $e(\{u_s, v_t\}, x) = 2$, then $u_s x, v_t x \in E(G)$.

Since $e(v_t, P_0) = |V(P_0)| - 1$, $v_t v_2, v_t v_1 \in E(G)$. The legs $xv_1 v_2 v_t v_3$ and $xv_1 v_t v_3 v_2$ respectively imply that $d(v_3) \le d(v_t)$ and $d(v_2) \le d(v_t)$. By applying (2.1) to $d(v_2) + d(v_3) + d(v_t)$, we have $d(v_t) > \frac{k+1}{2}$. Furthermore, $u_s x \in E(G)$ implies that $d(u_s) \ge \frac{k+1}{2}$. Then $k + 1 < d(u_s) + d(v_t) \le k + 1$, a contradiction.

(iii) $s = 3$. Then $e(u_s, P_0) = 0$.
First we consider the case $e(u_s, L) = 1$.

If the equality of (2.2) does not hold, by (2.5) we have $u_sx, v_tx \in E(G)$, $d(u_s) = d(v_t) = \frac{k}{2}$ and $e(v_t, P_0) = 2$ (for otherwise $e(v_t, P_0) = 1$, then $k \le d(u_s) + d(v_t) \le k - 2 - 2 + 2 + 1 = k - 1$, a contradiction), i.e., $v_1v_t \in E(G)$ or $v_2v_t \in E(G)$.

If $v_1v_t \in E(G)$, we have a copy of T^* in which with $xv_tv_3v_2v_1$ in place of xP_0, as before, we have $d(v_t) > \frac{k}{2}$, a contradiction.

If $v_2v_t \in E(G)$, we have a copy of T^* in which with $xv_1v_2v_tv_3$ in place of xP_0, as before, we have $d(v_t) > \frac{k}{2}$, a contradiction.

So the equality of (2.2) holds. Since $e(u_s, P_0) = 0$, we have $e(v_t, P_0) = |V(P_0)| - 1$, which implies that $v_tv_2, v_tv_1 \in E(G)$. The legs $xv_1v_2v_tv_3$ and $xv_1v_tv_3v_2$ respectively imply that $d(v_3) \le d(v_t)$ and $d(v_2) \le d(v_t)$. By applying (2.1) to $d(v_2)+d(v_3)+d(v_t)$, we obtain $d(v_t) \ge \frac{k+2}{2}$, with equality only if $d(v_t) = d(v_2) = d(v_3)$. By (2.4) and (2.5), we have $e(\{u_s, v_t\}, x) = 2$ and $d(v_t) = \frac{k+2}{2}, d(u_s) = \frac{k}{2}$. Since $d(v_t) = \frac{k+2}{2}$, we have $d(v_t) = d(v_3)$, which means that there is no difference between v_3 and v_t. Repeating the same arguments to the leg $xv_1v_2v_tv_3$, we have $v_3v_1 \in E(G)$. Replacing xP_0 by $xv_tv_3v_2v_1$, we have $d(v_1) \le d(v_t)$. By applying (2.1) to $d(v_1) + d(v_2) + d(v_3) + d(v_t)$, we have $d(v_t) > \frac{k+2}{2}$, a contradiction.

Now we consider the case $e(u_s, L) = 2$. Then we have that $u_1u_s \in E(G)$.

Replacing $xu_1u_2u_s$ by $xu_1u_su_2$, as above, $d(u_2) \le d(u_s)$ and $d(u_s) \ge \frac{k+1}{2}$.

As a result, if the equality of (2.2) does not hold, since $e(u_s, P_0) = 0$, then $e(v_t, P_0) = 2$, for otherwise $e(v_t, P_0) = 1$, then $\frac{2k+1}{2} \le d(u_s) + d(v_t) \le k - 2 - 2 + 2 + 2 = k$, a contradiction. So we have $v_1v_t \in E(G)$ or $v_2v_t \in E(G)$. Moreover, $e(\{u_s, v_t\}, x) = 2$ (for otherwise $e(\{u_s, v_t\}, x) \le 1$, then $\frac{2k+1}{2} \le d(u_s) + d(v_t) \le k - 2 - 1 + 1 + 2 = k$, a contradiction).

Since $u_sx \in E(G)$, the leg $xu_su_2u_1$ implies that $d(u_1) \le d(u_s)$. By applying (2.1) to $d(u_1) + d(u_2) + d(u_s)$, we have $d(u_s) \ge \frac{k+2}{2}$. If $v_1v_t \in E(G)$, the leg $xv_tv_3v_2v_1$ implies that $d(v_t) \ge \frac{k+1}{2}$. Then $\frac{2k+3}{2} \le d(u_s) + d(v_t) \le k + 1$, a contradiction. Similarly, if $v_2v_t \in E(G)$, the leg $xv_1v_2v_tv_3$ implies that $d(v_t) \ge \frac{k+1}{2}$, which also yields a contradiction.

So the equality of (2.2) holds and $v_1v_t, v_2v_t \in E(G)$. If $xu_s \in E(G)$, the legs $xu_su_2u_1$ and $xu_1u_su_2$ respectively imply that $d(u_1) \le d(u_s)$ and $d(u_2) \le d(u_s)$. By applying (2.1) to $d(u_1) + d(u_2) + d(u_s)$, we have $d(u_s) \ge \frac{k+2}{2}$. Moreover, the legs $xv_1v_tv_3v_2$ and $xv_1v_2v_tv_3$ respectively imply that $d(v_2) \le d(v_t)$ and $d(v_3) \le d(v_t)$. By applying (2.1) to $d(v_2) + d(v_3) + d(v_t)$, we obtain $d(v_t) \ge \frac{k+2}{2}$, with equality only if $d(v_t) = d(v_2) = d(v_3)$. By (2.5) and (2.4), we have $e(\{u_s, v_t\}, x) = 2$ and $d(v_t) = \frac{k+2}{2}$. Since $d(v_t) = \frac{k+2}{2}$, we have $d(v_t) = d(v_3)$, which means that there is no difference between v_3 and v_t. Repeating the same arguments to the leg $xv_1v_2v_tv_3$, we have $v_3v_1 \in E(G)$. The leg $xv_tv_3v_2v_1$ implies that $d(v_1) \le d(v_t)$. Since $\{v_1, v_2, v_3, v_t\}$ induces a complete graph, by (2.1) with $m = 4$, we have $d(v_t) > \frac{k+2}{2}$, a contradiction. Suppose therefore that $u_sx \notin E(G)$. By applying (2.1) to $d(v_2) + d(v_3) + d(v_t)$, from above, we have $d(v_t) \ge \frac{k+2}{2}$. Since $d(u_s) \ge \frac{k+1}{2}$, we have $\frac{k+3}{2} \le d(u_s)+d(v_t) \le k+1$, a contradiction.

(iv) $s = 4$. Then for any $P \in \{P_1, P_2, \ldots, P_\ell\}$, we have $|V(P)| \le s$. By (2.7), we have $e(v_t, P) = 0$. As a result, we have $4 \geq d(v_t) \geq \frac{k}{2}$, i.e., $k \leq 8$. By Proposition 2.5, G contains a copy of T.

Case 2. $t = 5$. Then for any $P \in \{P_1, P_2, \ldots, P_\ell\}$, since the diameter of T is at most 9 and $e(R) = 6$, we have $|V(P)| \leq 3$. If $k \geq 10$, then the length of the longest leg R of T is at most $\frac{k+2}{2}$ and each length of other legs is no more than 3; by Lemma 2.4, G contains a copy of T. Otherwise $k \leq 9$, and by Proposition 2.5, G contains a copy of T.

Case 3. $t = 6$. Then for any $P \in \{P_1, P_2, \ldots, P_\ell\}$, since the diameter of T is at most 9 and $e(R) = 7$, we have $|V(P)| \leq 2$. If $k \geq 12$, then the length of the longest leg R of T is at most $\frac{k+2}{2}$ and each length of other legs is no more than 3; then by Lemma 2.4, G contains a copy of T. Otherwise $k \leq 11$, and by Proposition 2.5, G contains a copy of T. This completes the proof of the theorem. ■

References

[1] S. Brandt and E. Dobson, The Erdős-Sós conjecture for graphs of girth 5, *Discrete Math.* 150 (1996), 411–414.

[2] P. Erdős, Extremal problems in graph theory, "*Theory of Graphs and its Applications*", (M. Fiedler, eds.), Academic Press, 1965, 29–36.

[3] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar.*, 10 (1959), 337–356.

[4] G. Fan and L. Sun, The Erdős-Sós conjecture for spiders, *Discrete Math.* (2007), doi: `10.1016/j.disc.2007.03.018`.

[5] A. McLennan, The Erdős-Sós conjecture for trees of diameter four, *J. Graph Theory* 49 (2005), 291–301.

[6] W. Moser and J. Pach, Recent developments in combinatorial geometry, in: *New Trends in Discrete and Computational Geometry*, Springer, New York (1993).

[7] J.F Saclé and M. Woźniak, The Erdős-Sós conjecture for graphs without C_4, *J. Combin. Theory Ser. B* 70 (1997), 367–372.

[8] A.F. Sidorenko, Asymptotic solution for a new class of forbidden r-graphs, *Combinatorica* 9 (1989), 207–215.

[9] M. Woźniak, On the Erdős-Sós conjecture, *J. Graph Theory* 21 (1996), 229–234.

(Received 13 Mar 2006; revised 24 June 2007)