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Strong compatibly ordered-OGDD of type h^n for $n = 6k + 1$

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Abstract

It is an open question whether a compatibly ordered-OGDD can be ordered so that it is a strong compatibly ordered-OGDD. In this article, we prove that a compatibly ordered-OGDD with base blocks can be ordered to be strong, and as its application, we obtain, for $n = 6k + 1$,

(i) there exists a strong compatibly ordered-OSTS(n);

(ii) there exists a strong compatibly ordered-OGDD of type 2^n; and

(iii) there exists a strong compatibly ordered-OGDD of type $(gv)^n$ for $g = 1$ or 2 and $v \in B(P_4)$.

1 Background and Introduction

A *Steiner triple system* of order n (briefly, STS(n)) $(X, \mathcal{A})$ is an n-set X and a set $\mathcal{A}$ of 3-subsets of X, so that each pair $\{x, y\}$ of elements of X appears once in a 3-subset of $\mathcal{A}$.

A *group divisible design* (briefly, GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$, which satisfies the three properties:

(i) $\mathcal{G}$ is a partition of X into subsets (called *groups*);

(ii) $\mathcal{A}$ is a set of subsets of X(called *blocks*), such that a group and a block contain at most one common point; and

(iii) every pair of points from distinct groups occurs in a unique block.

When all blocks of a GDD have the same size k, we use the notation k-GDD.

The *group type* of a GDD $(X, \mathcal{G}, \mathcal{A})$ is a multiset $\{|G| : G \in \mathcal{G}\}$. We use exponential notation to describe group types: a group type $g_1^{u_1} g_2^{u_2} \dots g_s^{u_s}$ denotes u_i occurrences of a group of size g_i for $1 \le i \le s$.

Definition 1.1 An *orthogonal Steiner triple system* of order n (briefly, OSTS(n)) $(X, \mathcal{A}, \mathcal{B})$ is an n-set X and two disjoint sets $\mathcal{A}$ and $\mathcal{B}$ of 3-subsets of X, so that each pair $\{x, y\}$ of elements of X appears once in a 3-subset of $\mathcal{A}$ and once in a 3-subset of $\mathcal{B}$. Moreover for two distinct intersecting triples $\{x, y, z\}$ and $\{u, v, z\}$ of $\mathcal{A}$, the triples $\{x, y, a\}$ and $\{u, v, b\}$ of $\mathcal{B}$ satisfy $a \neq b$.

OSTS(n) were introduced to construct Room squares [9] and a history of that topic can be found in [1, 8, 11].

Definition 1.2 An *orthogonal group divisible design* (briefly, OGDD) $(X, \mathcal{G}, \mathcal{A}, \mathcal{B})$ is a set X and a partition $\mathcal{G}$ of X into classes (usually called *groups*), and two disjoint sets $\mathcal{A}$ and $\mathcal{B}$ of 3-subsets of X, so that each pair $\{x, y\}$ of elements of X appears once in a 3-subset of $\mathcal{A}$ and once in a 3-subset of $\mathcal{B}$ if x and y are from different groups, and does not appear in a 3-subset of either if x and y are from the same groups. Moreover, if $\{x, y, a\} \in \mathcal{A}$ and $\{x, y, b\} \in \mathcal{B}$, then a and b are in different groups; and for two distinct intersecting triples $\{x, y, z\}$ and $\{u, v, z\}$ of $\mathcal{A}$, the triples $\{x, y, a\}$ and $\{u, v, b\}$ of $\mathcal{B}$ satisfy $a \neq b$. It is easy to see that an OSTS(n) is the same as an OGDD of type 1^n.

An OGDD is a generalization of OSTS suggested in 1991 by Stinson and Zhu [10]. OGDDs are of fundamental importance in recursive constructions for OSTSs.

Note that in this article $B(K)$ is the PBD-closure of K, $P_k = \{q : q \text{ is a prime power and } q \geq k\}$.

The existence problem for OSTSs has been given quite a lot of attention by various authors (see [2, 6, 7, 10]) since it was posed by O'Shaughnessy [11] in 1968. The problem was completely settled in 1994 (see [2]). That is,

Theorem 1.3 *An orthogonal Steiner triple system of order n exists if and only if $n \equiv 1, 3 \pmod 6, n \geq 7$ and $n \neq 9$.*

While OGDDs are useful tools in constructions for OSTSs, finding when they exist is an interesting problem in itself.

To establish recursive constructions for OGDDs, the author in 1991 introduced compatibly ordered-OGDD and strong compatibly ordered-OGDD as follows. Note that a compatibly ordered-OGDD was called a cyclic-OGDD and a strong compatibly ordered-OGDD was called an ordered-OGDD in [12].

Definition 1.4 (x, y, z) is defined to be $\{(x, y), (y, z), (z, x)\}$ and is called a *cyclically ordered* 3-subset. A *compatibly ordered orthogonal group divisible design* (briefly, compatibly ordered-*OGDD* or *COGDD*) $(X, \mathcal{G}, \mathcal{A}, \mathcal{B})$ is a set X and a partition $\mathcal{G}$ of X into classes (usually called *groups*), and two sets $\mathcal{A}$ and $\mathcal{B}$ of cyclically ordered 3-subsets of X (usually called *blocks*), so that $(X, \mathcal{G}, \mathcal{C}, \mathcal{D})$ is an orthogonal group divisible design, and if (a, b) appears in a block of $\mathcal{A}$ then (a, b) appears in a block of $\mathcal{B}$, where $\mathcal{C} = \{\{x, y, z\} : (x, y, z) \in \mathcal{A}\}$ and $\mathcal{D} = \{\{x, y, z\} : (x, y, z) \in \mathcal{B}\}$.

A *compatibly ordered orthogonal Steiner triple system* of order n (briefly, compatibly ordered-$OSTS(n)$ or $COSTS(n)$) can be viewed as a $COGDD$ of type 1^n.

Example 1.5 A compatibly ordered orthogonal Steiner triple system of order 7 is

$$\mathcal{A} = \{(0,1,3),(1,2,4),(2,3,5),(3,4,6),(4,5,0),(5,6,1),(6,0,2)\};$$

$$\mathcal{B} = \{(0,2,3),(1,3,4),(2,4,5),(3,5,6),(4,6,0),(5,0,1),(6,1,2)\}.$$

Definition 1.6 Let $\mathcal{A}$ and $\mathcal{B}$ be two collections of ordered 3-subsets of X, and $\mathcal{C} = \{(a,b,c) : [a,b,c] \in \mathcal{A}\}$ and $\mathcal{D} = \{(a,b,c) : [a,b,c] \in \mathcal{B}\}$. For $A = [a,b,c]$ we say that the three ordered pairs of points $(a,b),(b,c),(c,a)$ are in A and the seats of (a,b), (b,c) and (c,a) in A are different. We say $(X,\mathcal{G},\mathcal{A},\mathcal{B})$ is a *strong compatibly ordered-orthogonal group divisible design* (briefly, strong compatibly ordered-$OGDD$ or $SCOGDD$) if $(X,\mathcal{G},\mathcal{C},\mathcal{D})$ is a compatibly ordered-$OGDD$ and the following property is satisfied:

(∗) if (a,b) is in a block A of $\mathcal{A}$ then (a,b) is in a block B of $\mathcal{B}$ and the seat of (a,b) in A and the seat of (a,b) in B are different.

A *strong compatibly orthogonal Steiner triple system* of order n (briefly, strong compatibly ordered-$OSTS(n)$ or $SCOSTS(n)$) can be viewed as a strong compatibly ordered-$OGDD$ of type 1^n.

Example 1.7 A strong compatibly orthogonal Steiner triple system of order 7 is

$$\mathcal{A} = \{[0,1,3],[3,4,6],[6,0,2],[3,5,2],[6,1,5],[4,1,2],[0,4,5]\};$$

$$\mathcal{B} = \{[2,3,0],[5,6,3],[1,2,6],[5,2,4],[1,5,0],[1,3,4],[4,6,0]\}.$$

An example of an $OGDD$ which cannot be compatibly ordered is given in [12]. It is an open question whether every compatibly ordered-$OGDD$ can be ordered so that it is also a strong compatibly ordered-$OGDD$.

The existence problem for $OGDDs$ of type g^n has been given quite a lot of attention by various authors (see [3, 5, 10, 12, 14, 15, 16]). Colbourn and Gibbons in [3] settled it with few possible exceptions for each group size g. Furthermore, the author in [16] completely settled it for $g \le 6$.

For the existence problem of $COSTS(n)$ and $COGDD$ of type g^n, the author in [13] has obtained the following results.

Theorem 1.8 *(i) There exists a COSTS(n) for $n = 6k+1$;*
(ii) There exists a COGDD of type 2^n for $n = 6k+1$; and
(iii) There exists a COGDD of type $(gv)^n$ for $n = 6k+1, g = 1$ or 2 and $v \in B(P_4 \setminus \{5\})$.

In this article, we will prove that a compatibly ordered-$OSTS$ or a compatibly ordered-$OGDD$ with base blocks can be ordered to be strong, and as its application, we will obtain that for $n = 6k+1$,
(i) there exists a strong compatibly ordered-$OSTS(n)$;
(ii) there exists a strong compatibly ordered-$OGDD$ of type 2^n; and
(iii) there exists a strong compatibly ordered-$OGDD$ of type $(gv)^n$ for $g = 1$ or 2 and $v \in B(P_4)$.

2 Recursive Constructions

The following recursive constructions were established in [12, 13].

Theorem 2.1 *Suppose that there exists a SCOGDD of type g^u. Further suppose that $v \in B(P_4)$. Then there exists a SCOGDD of type $(gv)^u$.*

Lemma 2.2 [Filling in the Groups] *If there exists a SCOGDD of type $u^n v^m$, and there exists a SCOSTS$(u+1)$ and a SCOSTS$(v+1)$, then there exists a SCOSTS$(nu+mv+1)$.*

Lemma 2.3 *If there exists a SCOGDD of type $v^m(nh)^1$, and there exists a SCOGDD of type h^n, then there exists a SCOGDD of type $h^n v^m$.*

Lemma 2.4 *If there exists a SCOGDD of type $(u_1h)(u_2h)...(u_nh)$, and there exists a SCOGDD of type h^{u_i+1} for $i = 1, 2, ..., n$, then there exists a SCOGDD of type h^u, where $u = u_1 + u_2 + \ldots + u_n + 1$.*

3 Order $dev(0, a, a+b)$ and $dev(0, b, a+b)$

In this section, we always let $G = Z_m$, a and b be two elements of G such that any element in the set

$$\{a, b, a-b, a+b, 2a+b, 2b+a\}$$

is not zero; and let

$$A = (0, a, a+b), \quad B = (0, b, a+b)$$

be two base blocks under G. Also

$$dev\, A = \{(0, a, a+b) + g : g \in G\}, \quad dev\, B = \{(0, b, a+b) + g : g \in G\};$$

and let (X_0, X_1, X_2), (Y_0, Y_1, Y_2) be two partitions of G, and

$$\begin{aligned} \mathcal{C} &= \{x_0 + [0, a, a+b], x_1 + [a, a+b, 0], x_2 + [a+b, 0, a] : x_i \in X_i, i = 0, 1, 2\}, \\ \mathcal{D} &= \{y_0 + [0, b, a+b], y_1 + [b, a+b, 0], y_2 + [a+b, 0, b] : y_i \in Y_i, i = 0, 1, 2\}. \end{aligned}$$

It is easy to see that $\mathcal{C}$ and $\mathcal{D}$ satisfy the property $(*)$ if and only if the following condition holds:

$(**)$ $\quad x_i \neq y_i, x_i \neq y_{i+1} + b, x_i + a \neq y_{i+2}$, for any $x_i \in X_i, y_i \in Y_i, i \in Z_3$.

Hence we have

Lemma 3.1 *$dev\, A$ and $dev\, B$ can be ordered to satisfy the property $(*)$ if and only if there are two partitions of G satisfying the condition $(**)$.*

The following lemma is obvious.

Lemma 3.2 *$dev\,(0, a, a+b) = dev\,(0, b, -a) = dev\,(0, -a-b, -b)$.*

By applying Lemmas 3.1 and 3.2 we have

Lemma 3.3 *Let* $F = \{a, b, -a-b\}$ *and* c *and* d *be two different elements. Let* $c, d \in F$ *or* $c, d \in -F$*. Then* $dev\, A$ *and* $dev\, B$ *can be ordered to satisfy the property* $(*)$ *if and only if* $dev\,(0, c, c+d)$ *and* $dev\,(0, d, c+d)$ *can be ordered to satisfy the property* $(*)$*.*

Lemma 3.4 *Let* $c = a + w$ *and* $d = b - w$*. Then* $dev\, A$ *and* $dev\, B$ *can be ordered to satisfy the property* $(*)$ *if and only if* $dev\,(0, c, c+d)$ *and* $dev\,(0, d, c+d)$ *can be ordered to satisfy the property* $(*)$*.*

Proof. It is easily checked that for a and b the two partitions of G, (X_0, X_1, X_2), (Y_0, Y_1, Y_2), satisfy the condition $(**)$ if and only if for c and d the two partitions of G,

$$(X_0, X_1, X_2),\ (Y_0 + w, Y_1 + w, Y_2 + w)$$

satisfy the condition $(**)$.

Lemma 3.5 *Let* $G = Z_m,\ m = 3k+1,$ *and* $1 \le a = 3t+1,\ b = 3s+2 \le 3k$*. Then* $dev\, A$ *and* $dev\, B$ *can be ordered to satisfy the property* $(*)$*.*

Proof. Apply Lemma 3.1 with

$$X_0 = Y_1 = \{0, 3, \ldots, 3k\},\ X_1 = Y_2 = \{2, 5, \ldots, 3k-1\},\ X_2 = Y_0 = \{1, 4, \ldots, 3k-2\}.$$

Lemma 3.6 *Let* $G = Z_m, m = 3k+2,$ *and* $1 \le a = 3t+1 \le 3k+1,\ 3 \le b = 3s \le 3k$*. Then* $dev\, A$ *and* $dev\, B$ *can be ordered to satisfy the property* $(*)$*.*

Proof. Apply Lemma 3.1 with

$$X_1 = Y_0 = \{0, 3, \ldots, 3k\},\ X_2 = Y_1 = \{2, 5, \ldots, 3k-1\},\ X_0 = Y_2 = \{1, 4, \ldots, 3k+1\}.$$

The following lemma is Lemma 5.2 in [13].

Lemma 3.7 *Let* $G = Z_m, m = 3k,$ *and* $1 \le a, b \le 3k-1,\ a \not\equiv 0 \pmod 3$ *or* $b \not\equiv 0 \pmod 3$*. Then* $dev\, A$ *and* $dev\, B$ *can be ordered to satisfy the property* $(*)$*.*

Lemma 3.8 *Let* $c = 3a$ *and* $d = 3b$*. If* $dev\, A = \{(0, a, a+b) + g : g \in Z_n\}$ *and* $dev\, B = \{(0, b, a+b) + g : g \in Z_n\}$ *can be ordered to satisfy the property* $(*)$*, then* $dev\, C = \{(0, c, c+d) + g : g \in Z_{3n}\}$ *and* $dev\, D = \{(0, d, c+d) + g : g \in Z_{3n}\}$ *can be ordered to satisfy the property* $(*)$*.*

Proof. It is easily checked that

$$dev\, C = (3\, dev\, A) \cup (3\, dev\, A + 1) \cup (3\, dev\, A + 2),$$

and

$$dev\, D = (3\, dev\, B) \cup (3\, dev\, B + 1) \cup (3\, dev\, B + 2).$$

From the above results, we have

Theorem 3.9 *Let $G = Z_m$, and let a and b be two elements of G such that any element in the set*

$$\{a, b, a-b, a+b, 2a+b, 2b+a\}$$

is not zero. Also let

$$A = (0, a, a+b), B = (0, b, a+b).$$

Then $\mathrm{dev}\,A$ and $\mathrm{dev}\,B$ can be ordered to satisfy the property $(*)$.

4 Direct Constructions

Let $G = Z_{hn}$, $H = \langle n \rangle = \{0, n, 2n, \ldots, (h-1)n\}$ be a subgroup of G, and let $\mathcal{A}$ and $\mathcal{B}$ be two sets of base blocks under G, and their ordered difference sets be $S_{\mathcal{A}} = \{(a, b, -a-b) : (0, a, a+b) \in \mathcal{A}\}$ and $S_{\mathcal{B}} = \{(c, d, -c-d) : (0, c, c+d) \in \mathcal{B}\}$. For every $A = (0, a, a+b) \in \mathcal{A}$, if there are three blocks $(a, a+b, x), (b, -a, y)$ and $(-a-b, -b, z)$ of $\mathcal{B}$ then we define $D_A = (x, y, z)$.

The following lemma is obvious.

Lemma 4.1 *If $\mathcal{B} = \{(0, b, a+b) : (0, a, a+b) \in \mathcal{A}\}$, then for every $A = (0, a, a+b) \in \mathcal{A}$, $D_A = (2a+b, b-a, -2b-a)$, and c appears in $S_{\mathcal{A}}$ implies c appears in $S_{\mathcal{B}}$.*

From the definition of a COGDD of type h^n, we have:

Lemma 4.2 *Suppose that $\mathcal{A}$ forms the first 3-GDD and $\mathcal{B}$ forms the second 3-GDD. If for any base block A of $\mathcal{A}$, D_A has no element of H, and for any two different base blocks A_1 and A_2 of $\mathcal{A}$, D_{A_1} and D_{A_2} have no common element, then $\mathcal{A}$ and $\mathcal{B}$ form an OGDD of type h^n. Furthermore, if c appears in $S_{\mathcal{A}}$ implies c appears in $S_{\mathcal{B}}$, then $\mathcal{A}$ and $\mathcal{B}$ form a COGDD of type h^n.*

By combining Lemma 4.2 with Theorem 3.9, we have:

Theorem 4.3 *Let $\mathcal{B} = \{(0, b, a+b) : (0, a, a+b) \in \mathcal{A}\}$. If $\mathcal{A}$ and $\mathcal{B}$ form an OGDD of type h^n, then $\mathrm{dev}\mathcal{A}$ and $\mathrm{dev}\mathcal{B}$ can be ordered to form an SCOGDD of type h^n.*

From the definition of an SCOGDD, we have:

Theorem 4.4 *Let $\mathcal{A}$ and $\mathcal{B}$ form an OGDD of type h^n. If c appears in $S_{\mathcal{A}}$ implies c appears in $S_{\mathcal{B}}$ and the seat of c in $S_{\mathcal{A}}$ and the seat of c in $S_{\mathcal{A}}$ are different, then $\mathcal{A}$ and $\mathcal{B}$ can be ordered to form an SCOGDD of type h^n.*

From the definition of a COGDD of type h^n, we have:

Lemma 4.5 *Let hn be an even integer, $M = \langle 2 \rangle$ and $\mathcal{B} = \{(0, b, a+b) : (0, a, a+b) \in \mathcal{A}\}$. Let d, e and f be three odd integers with $2e + 2d + 2f = 0$ and $K_0 = \{\pm 3d, \pm 3e, \pm 3f\}$, and $K_1 = \{\pm(2d+e), \pm(2e+f), \pm(2f+d)\}$. Let $\mathcal{C}$ and $\mathcal{D}$ be two sets of base blocks under M with $\mathcal{C} = \{(0, d, 2d), (0, e, 2e), (0, f, 2f), (1, 1-2d, 1-2d-2e)\}$ and $\mathcal{D} = \{(x+1, y+1, z+1) : (x, y, z) \in \mathcal{C}\}$. Let $\mathcal{A}$ and $\mathcal{C}$ consist of the base blocks of a 3-GDD of type h^n, and so do $\mathcal{B}$ and $\mathcal{D}$. Suppose that*

(i) K_i has six different elements for $i = 0, 1$;

(ii) K_i *has no element of* H *for* $i = 0, 1$*;*

(iii) for any base block $A \in \mathcal{A}$*,* D_A *has no element of* H *and no element of* K_i *for* $i = 0, 1$*;*

(iv) for any two different base blocks A_1 *and* A_2 *of* $\mathcal{A}$*,* D_{A_1} *and* D_{A_2} *have no common element.*

Then the base blocks form a COGDD of type h^n*.*

Theorem 4.6 *A COGDD of type* h^n *constructed in Lemma 4.5 can be ordered so that it is an SCOGDD of type* h^n*.*

Proof. From Theorem 3.9 we only need to order $\mathcal{C}$ and $\mathcal{D}$ to satisfy the property $(*)$ as follows.

$$\begin{aligned}
\mathcal{C} &= \{[0, d, 2d], [0, e, 2e], [f, 2f, 0], [1, 1-2d, 1+2f]\}, \text{ and} \\
\mathcal{D} &= \{[x+1, y+1, z+1] : [x, y, z] \in \mathcal{C}\}.
\end{aligned}$$

In the following, we let $hn \equiv 0 \pmod 4$, $M = \langle 2 \rangle$ and $N = \langle 4 \rangle$, and c_i be odd and d_i be even for $1 \le i \le 2s$. Furthermore we let

$$\begin{aligned}
\mathcal{C} &= \{(0, c_i, c_i + d_i) : 1 \le i \le s\}, \\
\mathcal{E}_1 &= \{(\infty_{6j-5}, 0, d_{j+s}), (\infty_{6j-5}, 1, d_{j+s} + 1), (\infty_{6j-4}, 2, d_{j+s} + 2), \\
&\qquad (\infty_{6j-4}, 3, d_{j+s} + 3) : 1 \le j \le s\}, \\
\mathcal{E}_2 &= \{(\infty_{6j-3}, 0, c_{j+s}), (\infty_{6j-2}, 1, c_{j+s} + 1), (\infty_{6j-1}, 0, -c_{j+s} - d_{j+s}), \\
&\qquad (\infty_{6j}, 1, -c_{j+s} - d_{j+s} + 1) : 1 \le j \le s\}; \\
\mathcal{D} &= \{(0, c_j, c_j + d_j) : s + 1 \le j \le 2s\}, \\
\mathcal{F}_1 &= \{(\infty_{6j-5}, 0, d_j), (\infty_{6j-5}, 1, d_j + 1), (\infty_{6j-4}, 2, d_j + 2), (\infty_{6j-4}, 3, d_j + 3) : \\
&\qquad 1 \le j \le s\}, \\
\mathcal{F}_2 &= \{(\infty_{6j-3}, 0, c_j), (\infty_{6j-2}, 1, c_j + 1), (\infty_{6j-1}, 0, -c_j - d_j), \\
&\qquad (\infty_{6j}, 1, -c_j - d_j + 1) : 1 \le j \le s\}.
\end{aligned}$$

Lemma 4.7 *Let* $\mathcal{B} = \{(0, b, a+b) : (0, a, a+b) \in \mathcal{A}\}$*, and* $\mathcal{A} \cup \mathcal{C} \cup \mathcal{E}_1 \cup \mathcal{E}_2$ *and* $\mathcal{B} \cup \mathcal{D} \cup \mathcal{F}_1 \cup \mathcal{F}_2$ *be two sets of base blocks of a 3-GDD of type* $h^n 6^s$*, where* $\mathcal{E}_1$ *and* $\mathcal{F}_1$ *are base blocks under* N *and* $\mathcal{E}_2$ *and* $\mathcal{F}_2$ *are base blocks under* M*. Suppose that*

(i) $D_1 = \{\pm(d_{s+j} - d_j) : 1 \le j \le s\}$ *has* $2s$ *different elements and* $D_2 = \{\pm(c_{s+j} - c_j), \pm(c_{s+j} + d_{s+j} - c_j - d_j) : 1 \le j \le s\}$ *has* $4s$ *different elements;*

(ii) D_i *has no element of* H *for* $i = 0, 1$*;*

(iii) for any base block $A \in \mathcal{A}$*,* D_A *has no element of* H *and no element of* D_i *for* $i = 0, 1$*;*

(iv) *for any two different base blocks A_1 and A_2 of $\mathcal{A}$, D_{A_1} and D_{A_2} have no common element.*

Then $\mathcal{A}\cup\mathcal{C}\cup\mathcal{E}_1\cup\mathcal{E}_2$ and $\mathcal{B}\cup\mathcal{D}\cup\mathcal{F}_1\cup\mathcal{F}_2$ are two sets of base blocks of a COGDD of type h^n6^s.

Theorem 4.8 *A COGDD of type h^n constructed in Lemma 4.7 can be ordered so that it is an SCOGDD of type h^n6^s.*

Proof. From Theorem 3.9 we only need to order $\mathcal{C}\cup\mathcal{E}_1\cup\mathcal{E}_2$ and $\mathcal{D}\cup\mathcal{F}_1\cup\mathcal{F}_2$ to satisfy the property $(*)$ as follows.

$$\begin{aligned}
\mathcal{C} &= \{[0,c_j,c_j+d_j] : 1\le j\le s\},\\
\mathcal{E}_1 &= \{[0,d_{j+s},\infty_{6j-5}],[1,d_{j+s}+1,\infty_{6j-5}],[2,d_{j+s}+2,\infty_{6j-4}],\\
&\qquad [3,d_{j+s}+3,\infty_{6j-4}] : 1\le j\le s\},\\
\mathcal{E}_2 &= \{[c_{j+s},\infty_{6j-3},0],[c_{j+s}+1,\infty_{6j-2},1],[-c_{j+s}-d_{j+s},\infty_{6j-1},0],\\
&\qquad [-c_{j+s}-d_{j+s}+1,\infty_{6j},1] : 1\le j\le s\},\\
\mathcal{D} &= \{[c_{j+s}+d_{j+s},0,c_{j+s}] : 1\le j\le s\},\\
\mathcal{F}_1 &= \{[d_j,\infty_{6j-5},0],[d_j+1,\infty_{6j-5},1],[d_j+2,\infty_{6j-4},2],[d_j+3,\infty_{6j-4},3] :\\
&\qquad 1\le j\le s\},\\
\mathcal{F}_2 &= \{[\infty_{6j-3},0,c_j],[\infty_{6j-2},1,c_j+1],[\infty_{6j-1},0,-c_j-d_j],[\infty_{6j},1,-c_j-d_j+1] :\\
&\qquad 1\le j\le s\}.
\end{aligned}$$

5 SCOGDD of type h^n for $n = 6k+1$

By applying Theorem 4.3 with Lemma 3.5 and Lemma 3.6 in [13] (Lemma 3.5 comes from [4]), we have:

Lemma 5.1 *There exists an SCOSTS(n) for $n = 55, 115, 145$.*

By applying Theorem 4.3 with Lemma 4.2 in [13], we have:

Lemma 5.2 *There exists an SCOGDD of type 3^n for $n \in \{17, 19, 27, 29, 31, 37, 41, 53, 59, 67, 69\}$.*

By applying Theorem 4.3 with Lemma 4.3 in [13], we have:

Lemma 5.3 *There exists an SCOGDD of type 6^n for $n \in \{12, 13, 52, 53\}$.*

By applying Theorem 4.4 with Lemma 4.4 in [13], we have:

Lemma 5.4 *There exists an SCOGDD of type 3^n for $n = 11, 15$.*

Proof. For each case, we let $X = Z_{3n}$ and $H = \{0, n, 2n\}$ and present $\mathcal{A}$ as the base blocks of the first 3-GDD, and $\mathcal{B}$ as the base blocks of the second 3-GDD.

For $n = 11$,

$$\begin{aligned}
\mathcal{A} &= \{[0,1,3],[0,4,10],[0,5,18],[0,7,21],[0,-16,-24]\};\\
\mathcal{B} &= \{[1,26,0],[2,16,0],[0,-10,-13],[4,28,0],[6,21,0]\}.
\end{aligned}$$

For $n = 15$,

$\mathcal{A} = \{[0, 1, 22], [0, 2, 11], [0, -16, -19], [0, -6, -10], [-33, -38, 0], [0, 8, 25], [0, 13, 31]\}$;

$\mathcal{B} = \{[3, 0, 1], [-9, 0, -5], [0, -11, -17], [-16, -23, 0], [8, 26, 0], [0, 10, 24], [25, 0, 12]\}$.

By applying Theorem 4.8 with Lemma 4.6 and Lemma 4.7 in [13], we have:

Lemma 5.5 *There exists an SCOGDD of type $6^n s^1$ for $n = 52, s = 36, 42$ and $n = 60, s = 54$.*

By applying Theorem 4.6 with Lemma 4.8 in [13], we have:

Lemma 5.6 *There exists an SCOGDD of type 2^{55}.*

Theorem 5.7 *Let $n = 6k + 1$.*

(i) There exists an SCOSTS(n);

(ii) There exists an SCOGDD of type 2^n;

(iii) There exists an SCOGDD of type $(gv)^n$ for $g = 1$ or 2 and $v \in B(P_4)$.

Proof. The proof of (i) and (ii) is similar to that of Theorem 6.4 and Theorem 7.3 in [13]. From (i), (ii) and Theorem 2.1, (iii) holds.

6 Open Problem

It is well known that if $v \notin \{1, 2, 3, 6, 10, 12, 14, 15, 18, 26, 30\}$, then $v \in B(P_4)$ (see [1]). Hence Theorem 5.7 means that there exists an SCOGDD of type h^{6k+1} for all $h \neq 3, 6, 12, 15, 30$. Therefore if there exists an SCOGDD of type h^{6k+1} for $h = 3, 6$, then there exists an SCOGDD of type h^{6k+1} for all h. Unfortunately, up to now, we have been unable to construct an SCOGDD of type h^7 for $h = 3, 6$. Hence we post the following open problem.

Problem 6.1 Does there exist an SCOGDD of type h^7 for $h = 3, 6$?

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