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On the Loebl-Komlós-Sós conjecture*

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Abstract

The Loebl-Komlós-Sós conjecture says that any graph G on n vertices with at least half of its vertices of degree at least k contains every tree with k edges. We prove that the conjecture is true for trees of diameter at most 4 and for spiders of diameter at most 5.

1 Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. If $xy \in E(G)$, we say that x is joined to y and that y is a *neighbor* of x. For a subgraph H of G, $N_H(x)$ is the set of the neighbors of x which are in H, and $d_H(x) = |N_H(x)|$ is the *degree* of x in H. When no confusion can occur, we shall write $N(x)$ and $d(x)$, instead of $N_G(x)$ and $d_G(x)$. For $A, B \subseteq V(G)$, $E(A, B)$ denotes the set, and $e(A, B)$ the number, of edges with one end in A and the other end in B. For simplicity, we write $e(A)$ for $e(A, A)$ and $e(G)$ for $e(V(G), V(G))$ $(=|E(G)|)$. When $A = \{a\}$, we simplify the notation to $e(a, B)$ $(= d_B(a))$.

A *spider* is a tree with at most one vertex of degree more than 2. The vertex of degree more than 2 is called the *S-center* of the spider (if no vertex is of degree more than two, then any vertex can be the center). A *leg* of a spider is a path from the center to a vertex of degree 1. Thus, a star with k edges is a spider of k legs, each of length 1, and a path is a spider of 1 or 2 legs. A k-edge spider is a spider with k edges. An ℓ-leg of a spider is a leg with length ℓ.

A *center* of G is a vertex u such that $\max\limits_{v \in V} d(u, v)$ is as small as possible. Note that when G is a spider, the S-center and the center of G may not be the same vertex.

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If there exists a subgraph S of G which is isomorphic to T, then we say that T can be *embedded* into G.

The conjecture below was first formulated by Loebl in 1994 in the case $k = n/2$ and next generalized by Komlós and Sós (see [4]).

Loebl-Komlós-Sós Conjecture: *If at least half of the vertices of a graph G have degree at least k, then G contains every tree with k edges.*

The Loebl-Komlós-Sós conjecture has some similarity with the well-known Erdös-Sós conjecture (see [3]).

Erdös-Sós Conjecture: *If G is a graph on n vertices with $e(G) > (k-1)n/2$, then G contains every tree with k edges.*

As remarked in [5], the condition that the average degree of the graph G is greater than $k-1$ from the Erdös-Sós Conjecture is replaced in the Loebl-Komlós-Sós conjecture by the condition that the medium degree of G is at least k.

The Loebl-Komlós-Sós conjecture seems to be very difficult. There are only a few partial results known, mainly in two directions. One is to pose conditions on the graph G, such as graphs of girth 7, by Soffer [6]. The other is to pose conditions on the tree, such as stars, double stars, paths and trees with $k \geq n-3$, by Bazgan, Li and Woźniak [1]. Since stars and double stars are trees of diameter at most 3, it seems natural to consider trees of diameter at most 4. In this paper we prove that the conjecture is true for trees of diameter at most 4, and also for spiders of diameter 5.

2 Main results

For a graph satisfying the hypothesis of the Loebl-Komlós-Sós conjecture, we define $B = \{v \in V(G) \mid d_G(v) \geq k\}$ and $S = V(G) - B$.

Theorem 2.1. *If G is a graph on n vertices and it has at least $n/2$ vertices with degree at least k, then G contains every tree with k edges and diameter at most 4.*

Proof. Let G be a counterexample with as few vertices as possible. Without loss of generality, we can choose the graph G with the number of edges as small as possible. By the minimality, we have that
(1) G is connected;
(2) S is independent (otherwise the graph obtained by deleting the edges of $G[S]$ also satisfies the condition with fewer edges);
(3) each vertex of B has at most one neighbor with degree one (if there exists $b \in B$ with two 1-degree neighbors v_1, v_2, consider $G' = G - \{v_1, v_2\}$; then G' satisfies the condition with fewer vertices).

Let T be a tree with k edges and diameter at most 4, and let x be the center of T. Let $Y_1 = \{v \in N(x) \mid d(v) = 1\}$, $Y_2 = \{v \in N(x) \mid d(v) > 1\}$, $y_1 = |Y_1|$ and $y_2 = |Y_2|$. Since the diameter of T is at most 4, we have that $T = \{x\} \cup Y_1 \cup Y_2 \cup N(Y_2)$

and Y_1, Y_2, $N(Y_2)$ are pairwise disjoint. Above all, we have that

$$y_2 \leq \frac{k-y_1}{2}.$$

Note that $d_T(x) = y_1 + y_2$. If there exists a vertex $v \in S$ such that $d(v) \geq y_1 + y_2$, since $N(S) \subseteq B$ (S is an independent set) and $d(u) \geq k$ for any $u \in B$, it follows that we can embed T into G with v as the center. So we have

(1.1) $d(v) \leq y_1 + y_2 - 1$ for any vertex $v \in S$.

Similarly, if there exists $u \in B$ such that $d_B(u) \geq y_2$, since $d(u) \geq k$ and $d(x) \geq k$ for any $x \in N_B(u)$, then we can embed T into G with u as the center. So we have

(1.2) $d_B(u) \leq y_2 - 1$ for any vertex $u \in B$.

By (1.2), we have $d_S(u) \geq k - (y_2 - 1)$ for any vertex $u \in B$. Therefore, we have $e(B, S) \geq |B|[k - (y_2 - 1)] \geq \frac{n}{2}[k - (y_2 - 1)]$.

On the other hand, by (1.1), we have $e(B, S) \leq |S|(y_1 + y_2 - 1) \leq \frac{n}{2}(y_1 + y_2 - 1)$. So we have $y_2 \geq \dfrac{k-y_1}{2} + 1$, contradiction. This completes the proof. ■

Proposition 2.2. ([1]) *Let n and k be two integers, $k \leq n-1$ and let G be a graph on n vertices with at least $n/2$ vertices of degree at least k. For any three integers p, q, r such that $p + q + r = k$, denote by $T(p, q, r)$ the tree obtained from the path $P = x_0, \cdots, x_p, x_{p+1}, \cdots, x_{p+q}$ of length $p+q$ by adding r new vertices $y_1, \cdots, y_r$ and r new edges $x_p y_i, i = 1, \cdots, r$. Then G contains $T(p, q, r)$.*

Theorem 2.3. *If G is a graph on n vertices and it has at least $n/2$ vertices with degree at least k, then G contains every spider with k edges and diameter at most 5.*

Proof. Let G be a counterexample with the fewest vertices. Without loss of generality, we can choose the graph G with the fewest edges as well. By the minimality, as seen in the proof of Theorem 2.1, we have

(1) G is connected;
(2) S is independent;
(3) each vertex of B has at most one neighbor with degree one.

Let T be a spider with k edges and diameter at most 5, and let x be the S-center. Let P be a longest leg of T and $\ell = e(P)$. We have that $1 \leq \ell \leq 4$.

By Theorem 2.1, we only need to consider $3 \leq \ell \leq 4$. We divide the proof into two cases, according to the values of ℓ.

Case 1. $\ell = 3$.

Let $P = xv_1v_2v_3$. Since the diameter of T is at most 5, we have that P is the only leg of length 3. Let $T' = T - \{v_2v_3\} + \{xv_3\}$. Since the diameter of T' is at most 4, by Theorem 2.1, G contains a copy T^* of T'. For simplicity, we use the same notation for the vertices of T^* and T'. We denote all the 2-legs of T^* to be $xu_{i1}u_{i2}, 1 \leq i \leq t$ and all the 1-legs to be $xw_j, 1 \leq j \leq s$, $2t + s = k$. We first prove claims (3.1)–(3.3).

(3.1) $e(\bigcup_{j=1}^{s}\{w_j\}, \bigcup_{i=1}^{t}\{u_{i1}, u_{i2}\}) = 0$ and $d_{G[T^*]}(u_{i1}) \leq k-1, d_{G[T^*]}(w_j) \leq k-2$ $(1 \leq i \leq t, 1 \leq j \leq s)$.

Otherwise if there exists $w_j u_{i1} \in E(G)$, then $T^* - \{xu_{i1}\} + \{w_j u_{i1}\}$ is a copy of T. If there exists $w_j u_{i2} \in E(G)$, then $T^* - \{xw_j\} + \{w_j u_{i2}\}$ is a copy of T.

Since $j \geq 1$ ($v_3 = w_j$ for some j), we have $u_{i1}w_1 \notin E(G)$, so $d_{G[T^*]}(u_{i1}) \leq k-1, 1 \leq i \leq t$.

Since $i \geq 1$ ($v_1 v_2 = u_{i1}u_{i2}$ for some i), we have $w_j u_{i1}, w_j u_{i2} \notin E(G)$, so $d_{G[T^*]}(w_j) \leq k-2, 1 \leq j \leq s$.

(3.2) $u_{i2} \in S, 1 \leq i \leq t$, and so $u_{i1} \in B, 1 \leq i \leq t$.

Otherwise we may assume that there exists a 2-leg $xu_{i1}u_{i2}$ with $u_{i2} \in B$. Since $u_{i2} \in B$, by (3.1), we have that $u_{i2}w_1 \notin E(G)$, so $d_{G[T^*]}(u_{i2}) \leq k-1$, i.e., there exists a vertex y such that $y \in N(u_{i2}) \setminus V(T^*)$. Then $T^* + \{yu_{i2}\} - \{xw_1\}$ is a copy of T.

(3.3) $N(u_{i2}) \subseteq V(T^*)$ and we may assume $d(u_{i2}) \geq 2, 1 \leq i \leq t$.

We have $N(u_{i2}) \subseteq V(T^*), 1 \leq i \leq t$. Otherwise, if there exists a vertex z and some u_{i2} such that $z \in N(u_{i2}) \setminus V(T^*)$, then $T^* + \{zu_{i2}\} - \{xw_1\}$ is a copy of T. Furthermore, we may assume $d(u_{i2}) \geq 2, 1 \leq i \leq t$. Since $u_{i2} \in S$, we have that $u_{i1} \in B$. By (3.1), we have that $d_{G[T^*]}(u_{i1}) \leq k-1$, so there exists $u'_{i2} \in N(u_{i1}) \setminus V(T^*)$. Since every vertex of B has at most one neighbor with degree one, we can choose u_{i2} such that $d(u_{i2}) \geq 2$.

Now we have two subcases by x.

If $x \in S$, then $w_1 \in B$. By (3.1), we have that $d_{G[T^*]}(w_1) \leq k-2$, so there exists a vertex w'_1 such that $w'_1 \in N(w_1) \setminus V(T^*)$. By (3.1) and (3.3), $e(u_{12}, \{u_{21}, u_{31}, \ldots, u_{t1}, u_{22}, u_{32}, \ldots, u_{t2}\}) > 0$. By (3.2), $\{u_{i2}, 1 \leq i \leq t\} \subseteq S$ is an independent set. So we have that $N(u_{12}) \cap \{u_{21}, u_{31}, \ldots, u_{t1}\} \neq \emptyset$. Without loss of generality, we may assume that $u_{12}u_{i1} \in E(G), 2 \leq i \leq t$, then $T^* - \{xu_{i1}, u_{i1}u_{i2}\} + \{u_{12}u_{i1}, w_1 w'_1\}$ is a copy of T.

Otherwise $x \in B$.

If $u_{12}x \in E(G)$, since $u_{11} \in B$, by (3.1) we have that $d_{G[T^*]}(u_{11}) \leq k-1$, so there exists $u'_{12} \in N(u_{11}) \setminus V(T^*)$, then $T^* - \{xu_{11}, xw_1\} + \{xu_{12}, u_{11}u'_{12}\}$ is a copy of T.

Otherwise $u_{12}x \notin E(G)$, then $d_{G[T^*]}(x) \leq k-1$, so there exists $w_{s+1} \in N(x) \setminus V(T^*)$. We may adjust the sequence $\{w_1, w_2, \ldots, w_s, w_{s+1}\}$ such that $d(w_i) \geq d(w_j)$ when $i \leq j$. Since every vertex of B has at most one neighbor with degree one, by (3.1), we have that $w_1 w \in E(G)$, where $w = w_r$ $(2 \leq r \leq s+1)$ or $w \in N(w_1) \setminus V(T^* \cup \{w_{s+1}\})$. Similar to the proof of (3.1), we have that $e(\bigcup_{j=1}^{s+1}\{w_j\}, \bigcup_{i=1}^{t}\{u_{i1}, u_{i2}\}) = 0$. By (3.2) and (3.3), we have that $N(u_{12}) \cap \{u_{21}, u_{31}, \ldots, u_{t1}\} \neq \emptyset$. Without loss of generality, we may assume that $u_{12}u_{i1} \in E(G), 2 \leq i \leq t$. If $w = w_r$ $(2 \leq r \leq s+1)$, then $T^* - \{xu_{i1}, u_{i1}u_{i2}, xw\} + \{u_{12}u_{i1}, w_1 w, xw_{s+1}\}$ is a copy of T. If $w \in N(w_1) \setminus V(T^*)$, then $T^* - \{xu_{i1}, u_{i1}u_{i2}\} + \{u_{12}u_{i1}, w_1 w\}$ is a copy of T.

Case 2. $\ell = 4$.

In this case, we have that T has only one 4-leg and all other legs are 1-legs, which is a special case of Proposition 2.2 with $p = 4$ and $q = 1$. This completes the proof. ■

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