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Packing trees of bounded diameter into the complete graph

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Abstract

Let d be a positive integer. We prove that there exists a constant $c = \frac{1}{2}(\sqrt{2+(d+1)^2}-(d+1)^2)$ such that if $T_1,\ldots,T_n$ is a sequence of trees such that $|V(T_i)| = i$, $\mathrm{diam}(T_i) \le d+2$, and there exists $x_i \in V(T_i)$ such that $T_i - x_i$ has at least $(1-c)(i-1)$ isolated vertices, then $T_1,\ldots,T_n$ can be packed into K_n. This verifies a special case of the Tree Packing Conjecture. We then prove that if T is a tree of order $n+1$ and there exists $x \in V(T)$ such that $T-x$ has at least $n-\sqrt{n}/(4+2\sqrt{2})$ isolated vertices, then $2n+1$ copies of T may be packed into K_{2n+1}. Finally, we show that there exists a constant $c' = c'(d)$ such that if T is a tree of order $n+1$, $\mathrm{diam}(T) \le d+2$, and there exists $x \in V(T)$ such that $T-x$ has at least $(1-c')n$ isolated vertices, then $2n+1$ copies of T may be packed into K_{2n+1}. The last two results verify special cases of Ringel's conjecture.

1 Introduction

In 1976 Gyárfás and Lehel [7] conjectured that every sequence of trees $T_1, T_2, \ldots, T_n$ such that $|V(T_i)| = i$ for all $1 \le i \le n$, can be packed into K_n. This conjecture is usually referred to as the Tree Packing Conjecture. Since that time a variety of partial solutions to this conjecture have been obtained. The interested reader is referred to [4] and the references there for surveys of these results. In [4], the author proved that if T_i contains a vertex x_i such that $T_i - x_i$ has at least $i-1-\sqrt{6(i-1)}/4$ isolated vertices, then $T_1,\ldots,T_n$ can be packed into K_n. In [5], the author verified an "approximate" version of the Tree Packing Conjecture for similar type trees. Namely, that for $c \le .076122$, any sequence of trees $T_1,\ldots,T_n$ with $|V(T_i)| \le i - c(i-1)$ can be packed into K_n provided that for each $1 \le i \le n$ there exists a vertex $x_i \in V(T_i)$ such that $T_i - x_i$ has at least $(1-2c)(i-1)$ isolated vertices. Let d be a positive integer. We prove that there exists a constant $c = \frac{1}{2}(\sqrt{2+(d+1)^2} - (d+1)^2)$ such that if $T_1, \ldots, T_n$ is a sequence of trees such that $|V(T_i)| = i$, $\text{diam}(T_i) \leq d+2$, and there exists $x_i \in V(T_i)$ such that $T_i - x_i$ has at least $(1-c)(i-1)$ isolated vertices, then $T_1, \ldots, T_n$ can be packed into K_n. We then prove that if $T_0, \ldots, T_{2n}$ is a sequence of trees with each T_i of order $n+1$, and in each T_i there exists an $x_i \in V(T_i)$ such that $T_i - x_i$ has at least $n - \sqrt{n}/(4+2\sqrt{2})$ isolated vertices, then $T_0, \ldots, T_{2n}$ can be packed into K_{2n+1}. This proves a special case of Ringel's Conjecture, which states that $2n+1$ copies of a tree T of order $n+1$ may be packed into K_{2n+1}, and also a more general conjecture of Häggvist [8, Conjecture 2.17] stating that any list of k trees of order $\ell+1$ can be packed into any 2ℓ-regular graph of order k. Finally, we show that there exists a constant $c' = c'(d)$ such that if $T_0, \ldots, T_{2n}$ is a sequence of trees of order $n+1$ such that $\text{diam}(T_i) \leq d+2$ and there exists $x_i \in V(T_i)$ such that $T_i - x_i$ has at least $(1-c')n$ isolated vertices, then $T_0, \ldots, T_{2n}$ can be packed into K_{2n+1}. This result also verifies a special case of Ringel's Conjecture and the more general conjecture of Häggvist cited above.

Notation is standard. For terms not defined in this paper, see [2]. Let $G_1, \ldots, G_\ell$ and G be graphs. We say that $G_1, \ldots, G_\ell$ *can be packed into* G if there exists inclusions $V(G_i) \subseteq V(G)$, $1 \leq i \leq \ell$, such that if $e \in E(G_i)$, then $e \notin \cup_{j \neq i} E(G_j)$. The inclusions are said to be a *packing* P. We commonly abuse notation by saying an edge $e \in P$ if $e \in E(G_i)$ for some i and say $H \subseteq G$ if G contains a subgraph isomorphic to H. Finally, we will often have occasion to consider a directed graph $\vec{D}$ along with its underlying simple graph. Throughout this paper, if a directed graph is denoted by $\vec{D}$, then its underlying simple graph will be denoted by D.

2 Tools

We begin with a lemma that will allow us to extend a packing of trees $T_1, \ldots, T_n$ into K_t to a packing of trees $T_1, \ldots, T_{n+1}$ into K_t under appropriate circumstances. Before stating this result, we need to develop some terminology that will be used in its statement.

Let $T_1, T_2, \ldots, T_n, T_{n+1}, \ldots, T_{n+r}$ be a sequence of trees that can be packed into K_t, and fix such a packing P. Furthermore, assume that if $i \geq n+1$, then T_i is a star. Let $x_i \in V(T_i)$, $1 \leq i \leq n+r$, and $F_i \subseteq \{\vec{x_i j} : x_i j \in T_i, \deg_{T_i}(j) = 1\}$. Let $v \in V(T_{n+1})$ such that $T_{n+1} - v$ has no edges. Define a digraph $\vec{D}$ by $V(\vec{D}) = H = V(T_{n+1}) - \{v\}$ and

$$E(\vec{D}) = \{\vec{xy} : x, y \in H \text{ and } \vec{xy} \in F_i \text{ for some } i,\ 1 \leq i \leq n+r,\ \text{with } v \notin V(T_i)\}.$$

Thus $E(\vec{D})$ consists of those directed edges, each of whose endpoints is in $V(\vec{D})$, that are contained in some F_i, where $v \notin V(T_i)$. Thus no edge of F_{n+1} is contained in $E(\vec{D})$.

Lemma 1 *Let $T_1, T_2, \ldots, T_n, T_{n+1}, \ldots, T_{n+r}$ be a sequence of trees that can be packed into K_t, and fix such a packing P. Assume that if $i \geq n+1$, then T_i is a star. Let T'_{n+1} be a tree such that $|V(T'_{n+1})| = |V(T_{n+1})|$. Let $\vec{T}'_{n+1}$ be the directed tree rooted at v such that every edge of $\vec{T}'_{n+1}$ is indirected toward the root v. If $\vec{D}$ contains a subdigraph isomorphic to the digraph obtained from $\vec{T}'_{n+1} - v$ by removing all isolated vertices, then there is a packing P' of $T_1, \ldots, T_n, T'_{n+1}, T_{n+2}, \ldots, T_{n+r}$ into K_t.*

Proof. Note that if T'_{n+1} is a star, then the packing P satisfies the conclusion of this lemma. We thus assume that the subdigraph of $\vec{T}'_{n+1} - v$ obtained by removing all isolated vertices is not empty. Let $\vec{T} \subset \vec{D}$ be such that $\vec{T}$ is isomorphic to the subdigraph of $\vec{T}'_{n+1} - v$ obtained by removing all isolated vertices. For each edge $e = x_e\vec{y}_e \in E(\vec{T})$, let $1 \leq i_e \leq n+r$ such that $e \in F_{i_e}$. Then $x_e = x_{i_e}$. We now modify our packing P of $T_1, \ldots, T_{n+r}$ in the following fashion. For each $x_e\vec{y}_e \in E(\vec{T})$, remove edge x_ey_e from tree T_{i_e} and replace it with x_ev, and denote the resulting graph by T'_{i_e} (so that $T'_{i_e} = (T_{i_e} - x_ey_e) \cup \{x_ev\}$). As $x_ey_e \in E(T_{i_e})$, by the definition of $\vec{D}$, we have that $v \notin V(T_{i_e})$, so that each graph T'_{i_e} is a tree isomorphic to T_{i_e}. For $1 \leq i \leq n+r$, let $U_i = T_i$ if $i \neq n+1$ or i_e for any $e \in E(\vec{T})$, $U_{i_e} = T'_{i_e}$ for every $e \in E(\vec{T})$, and $U_{n+1} = T_{n+1} - \{vx_e : e \in E(\vec{T})\}$. We then have a packing P_1 of $U_1, \ldots, U_{n+r}$ into K_t and none of the edges x_ey_e are used in this packing. Note that U_{n+1} is a star of order $|T_{n+1}| - |E(\vec{T})|$, and, of course, if $i \neq n+1$, then $U_i \cong T_i$. Now remove the edges of each U_i, $1 \leq i \leq n+r$, $i \neq n+1$, from K_t, and then remove any isolated vertices of the resulting graph. We then have a graph G with

$$|E(\vec{T})| + |E(U_{n+1})| = |E(T_{n+1})|,$$

edges, and will be a tree isomorphic to T'_{n+1} provided that v is only adjacent in G to the vertex of a component of $\vec{T}$ that is adjacent in T'_{n+1} to v. Let $\vec{C}$ be a component of $\vec{T}$. Then there exists $x \in V(\vec{C})$ such that $xv \in E(T'_{n+1})$, and every edge of C is indirected towards x. For each edge $e = x_e\vec{y}_e \in E(C)$, $x_ev \in E(T'_{i_e}) = E(U_{i_e})$ and $i_e \neq n+1$. Furthermore, $x \neq x_e$ for any $e \in E(\vec{T})$. Thus the only vertex of $\vec{C}$ which is adjacent in G to v is x. Whence G is a tree isomorphic to T'_{n+1} and we have a packing of $T_1, \ldots, T_n, T'_{n+1}, T_{n+2}, \ldots, T_{n+r}$ into K_t. □

Remark 1 Note that for each edge of $T'_{n+1} - x_{n+1}$, there are two edges of the packing P that are changed. One such edge is used for an edge of $T'_{n+1} - x_{n+1}$, and the other, in some sense, has its "direction reversed". That is, in P, this edge was of the form vx_e in tree T_{n+1} (with $x_e = x_{i_e}$) but in the packing P', it is x_ev in tree T_{i_e}. Thus if we are viewing the edges of the trees T_i being indirected toward the root x_i, the direction of the edge vx_e is reversed. Finally, observe that if the packing P identifies x_i with v_i, then the packing P' identifies x_i with v_i as well, and $V(P(T_{n+1})) = V(P'(T'_{n+1}))$.

A *transitive tournament* is a directed graph on n vertices with out degree sequence $(0, 1, \ldots, n-1)$. At times it will be convenient to have a canonical labeling for a transitive tournament of order n. Let τ be the transitive tournament such that let $V(\tau) = \{0, 1, \ldots, n-1\} = \mathbb{Z}_n$ and $\deg^+_\tau(i) = i$. Thus $\deg^-_\tau(i) = n - i - 1$.

Lemma 2 *Let $\vec{D}$ be a subdigraph of the transitive tournament τ of order n such that D (the underlying simple graph of $\vec{D}$) has minimal degree at least $\delta(n-1)$, $0<\delta<1$. Let $c\geq 0$ and d a positive integer such that $d(1-\delta)+c\leq\delta$. Let $\vec{F}$ be a directed forest of order cn with r components $\vec{T}_1,\ldots,\vec{T}_r$, where each edge of $\vec{T}_i$ is indirected towards some root v_i. Let T_i be the underlying simple graph of $\vec{T}_i$. If $d\geq\max\{\mathrm{dist}_{T_i}(v_i,x_i):x_i\in V(T_i)\}$ for all $1\leq i\leq r$, then $\vec{F}$ is contained in $\vec{D}$.*

Proof. As $\vec{D}$ is a subdigraph of τ and $\delta(D)\geq\delta(n-1)$, D has some component of order at least $\delta(n-1)+1$. As an induced subdigraph of a transitive tournament is a transitive tournament (of possibly smaller order) we assume without loss of generality that D is connected and has order at least $\delta(n-1)+1$. For convenience, we will also assume that $\{0,\ldots,\delta(n-1)+1\}\subset V(D)$. Let F be the underlying simple graph of $\vec{F}$. Let $N_F(i)=\{x_j\in V(T_j):\mathrm{dist}_T(v_j,x_j)=i,1\leq j\leq r\}$, and $n_i=|N_F(i)|$, $0\leq i\leq d$. For $0\leq i\leq d$, let $M_F(i)=\cup_{j=0}^{i}N_F(j)$ and $m_i=|M_F(i)|$. Thus $m_i=\Sigma_{j=0}^{i}n_j$ and $m_d=cn$. We will show by induction on $0\leq i\leq d$ that $\vec{F}[M_F(i)]$ is contained in $\vec{D}[\{0,1,\ldots,\lfloor i(1-\delta)(n-1)+m_i-1\rfloor\}]$. If $i=0$, then $V(\vec{F}[M_F(0)])=\{v_j:1\leq j\leq r\}$ and we identify v_j with $j-1$, $1\leq j\leq r$. Thus $\vec{F}[M_F(0)]$ is trivially contained in $\vec{D}[\{0,\ldots,m_0-1\}]$. We thus assume that $i\geq 0$ and $\vec{F}[M_F(i)]$ is contained in $\vec{D}[\{0,1,\ldots,\lfloor i(1-\delta)(n-1)+m_i-1\rfloor\}]$. We remark that it suffices to show that each vertex of $\vec{D}$ identified with a vertex of $N_F(i)$ is inadjacent to at least n_{i+1} vertices of $\vec{D}$ contained in $\{0,1,\ldots,\lfloor(i+1)(1-\delta)(n-1)+m_{i+1}-1\rfloor\}$. Let $u_1,\ldots,u_{n_i}$ be the n_i vertices of $\vec{D}$ that have been identified with the vertices in $N_F(i)$. Then $u_j\leq\lfloor i(1-\delta)(n-1)+m_i-1\rfloor$ for every $1\leq j\leq n_i$, and so $\deg^-_\tau(u_j)\geq n-1-\lfloor i(1-\delta)(n-1)+m_i-1\rfloor$. Furthermore, as $\delta(D)\geq\delta(n-1)$, each vertex of $\vec{D}$ has at most $(n-1-\delta(n-1))=(n-1)(1-\delta)$ fewer edges incident with it than in τ. Thus

$$\deg^-_{\vec{D}}(u_j)\geq n-1-\lfloor i(1-\delta)(n-1)+m_i-1\rfloor-(n-1)(1-\delta)$$

for every $1\leq j\leq n_i$. As $\deg^-_{\vec{D}}(u_j)$ is an integer, we have that

$$\deg^-_{\vec{D}}(u_j)\geq n-1-\lfloor(i+1)(1-\delta)(n-1)+m_i-1\rfloor$$

for every $1\leq j\leq n_i$. Note that in τ, the edge $x\vec{u}_j\in E(\tau)$ if and only if $x>u_j$. Hence at most $\lfloor(i+1)(1-\delta)(n-1)+m_i-1\rfloor$ integers $u_j\neq x\in\mathbb{Z}_n$ satisfy $x\vec{u}_j\notin E(\vec{D})$. By the pigeon-hole principal, u_j is thus inadjacent to at least n_{i+1} vertices in the set

$$\{0,1,\ldots,\lfloor(i+1)(1-\delta)(n-1)+m_i-1\rfloor+n_{i+1}\}=\{0,1,\ldots,\lfloor(i+1)(1-\delta)(n-1)+m_{i+1}-1\rfloor\},$$

provided that $\lfloor(i+1)(1-\delta)(n-1)+m_{i+1}-1\rfloor\leq\delta(n-1)+1$ (as $\{0,\ldots,\delta(n-1)+1\}\subseteq V(\vec{D})$). Note that $\lfloor(i+1)(1-\delta)(n-1)+m_{i+1}-1\rfloor\leq d(1-\delta)(n-1)+cn$. As $d(1-\delta)+c\leq\delta$, we have that $d(1-\delta)(n-1)+c(n-1)\leq\delta(n-1)$ so that

$$d(1-\delta)(n-1)+cn\leq\delta(n-1)+1\leq|V(\vec{D})|,$$

Thus $\vec{F}[M_F(i+1)]$ is contained in $\vec{D}[\{0,1,\ldots,\lfloor (i+1)(1-\delta)(n-1)+m_{i+1}-1\rfloor\}]$, and the result follows by induction. □

We will have occasion to find complete subgraphs of a graph, and will use the following weak form of Turan's Theorem [3], stated here for completeness. For a graph G, let $\tilde{G}$ denote the complement of G.

Lemma 3 *If G is a graph of order $m \geq t^2$ and $|E(\tilde{G})| \leq m^2/2t$, then $K_t \subset G$.*

We will also need to find a subgraph of a graph with large minimal degree. For this we will use the following result (see [1], page xvii).

Lemma 4 *Let ℓ be a positive integer. Suppose that H is a graph of order $n \geq \ell+1$. If*

$$|E(H)| \geq (\ell-1)\Big(n-\frac{\ell}{2}\Big)+1,$$

then H contains a subgraph F such that $\delta(F) \geq \ell$.

3 The Tree Packing Conjecture

Theorem 5 *Let d be a non-negative integer, and $c = \frac{1}{2}(\sqrt{2+(d+1)^2}-(d+1))^2$. Let $T_1,\ldots,T_n$ be a sequence of trees such that $|V(T_i)| = i$, there exists $x_i \in V(T_i)$ such that $T_i - x_i$ has at least $(1-c)(i-1)$ isolated vertices, and $\mathrm{dist}_{T_i}(x_i,v) \leq d+1$ for every $v \in V(T_i)$. Then $T_1,\ldots,T_n$ can be packed into K_n.*

Proof. If $T_1,\ldots,T_n$ are all stars the result is straightforward. Indeed, assume that $T_1,\ldots,T_n$ are stars with tree T_i of order i and have been packed into K_n. Add a vertex v to K_n, and an edge from every vertex of K_n to v. The resulting graph is isomorphic to K_{n+1}, and the graph G defined by $V(G) = V(K_n) \cup \{v\}$ and $E(G) = \{vx : x \in V(K_n)\}$ is a star of order $n+1$. Note that by the same argument, if $T_1,\ldots,T_n$ is any sequence of trees with $|V(T_i)| = i$ that can be packed into K_n, then there is a packing of $T_1,\ldots,T_n,T'_{n+1}$ into K_{n+1} where T'_{n+1} is a star of order $n+1$. We may thus assume $d \geq 1$ and $c(n-1) \geq 1$.

For what follows we assume that $V(K_n) = [n] = \{1,\ldots,n\}$. We will show by induction on n that we may apply Lemma 1 in such a way that $T_1,\ldots,T_n$ can be packed into K_n so that $x_i = i$ for all $1 \leq i \leq n$. If $n = 1$, then the result is trivial. Let $n \geq 1$ and inductively assume $T_1,\ldots,T_n$ as above have been packed into K_n so that $x_i = i$. By arguments above, there then exists a packing P of $T_1,\ldots,T_n,T'_{n+1}$ into K_{n+1} (in the above argument we identify v with $n+1$), where T'_{n+1} is a star of order $n+1$. Let T_{n+1} be a tree such that there exists $x_{n+1} \in V(T_{n+1})$, $T_{n+1}-x_{n+1}$ has at least $(1-c)n$ isolated vertices and $\mathrm{dist}_{T_{n+1}}(x_{n+1},u) \leq d+1$ for every $u \in V(T_{n+1})$. Note that $H = V(T_{n+1}-x_{n+1}) = V(K_n) = [n]$, and as every tree T_i, $1 \leq i \leq n$ is packed into K_n, we have that $n+1 \notin V(T_i)$, $1 \leq i \leq n$. For $1 \leq i \leq n$, let $F_i = \{\vec{ij} : ij \in E(T_i), \deg_{T_i}(j) = 1, \text{and } j < i\}$. Define a directed graph $\vec{D}$ by $V(\vec{D}) = H$ and $E(\vec{D}) = \cup_{i=1}^{n} E(F_i)$. In order to apply Lemma 1, we must show that

$\vec{T} \subseteq \vec{D}$, where $\vec{T}$ is the directed forest obtained by first indirecting every edge of T_{n+1} towards x_{n+1}, then removing x_{n+1}, and then removing any isolated vertices.

Note that there are at least $i-1-c(i-1)$ isolated vertices of $T_i - x_i$, and if x is such an isolated vertex, then $\vec{ix} \in F_i$ unless $x > i$. By the remark following Lemma 1, the only way this can occur is if the "direction" of ix is "reversed", and there are as many of these edges in the packing P as there are edges in $\cup_{i=1}^n (T_i - x_i)$. Thus

$$\begin{aligned} |E(\vec{D})| &\geq \sum_{i=1}^n (i-1-c(i-1)) - \sum_{i=1}^n c(i-1) \\ &= \frac{(1-2c)n(n-1)}{2}. \end{aligned}$$

Let $\delta = 1-\sqrt{2c}$. We first show that D (the underlying simple graph of $\vec{D}$) contains a subgraph of minimal degree at least δn.

As $c(n-1) \geq 1$ (so that $n \geq 2$) and $1/(n-1) \geq 1/2n^2$, we have that $c \geq 1/2n^2$. It then follows that $n \geq \delta n + 1$. As $|E(\vec{D})| = |E(D)| \geq (1-2c)n(n-1)/2$, by Lemma 4 it suffices to show that the following inequality holds:

$$\frac{(1-2c)n(n-1)}{2} \geq (\delta n - 1)\Big(n - \frac{\delta n}{2}\Big) + 1.$$

As $\delta = 1-\sqrt{2c}$, the preceding inequality is equivalent to

$$2cn + \sqrt{2c}n - 2 \geq 0,$$

which clearly holds as $2cn > 2c(n-1) \geq 2$. Thus D contains a subgraph of minimal degree at least δn, so that $\vec{D}$ contains a subdigraph $\vec{D}'$ such that D' has minimal degree at least δn. Let $\vec{U_1}, \ldots, \vec{U_r}$ be the components of $\vec{T}$, and $u_i \in V(U_i)$ such that $u_i x_{n+1} \in E(T_{n+1})$. Then $\sum_{i=1}^r |V(U_i)| \leq cn$ and $\text{dist}_{U_i}(u_i, u) \leq d$ for every $u \in V(U_i)$. Let $|V(D')| = t$, $\delta' = \delta n/t$, and $c' = cn/t$ (so that $c't = cn$). If $d(1-\delta') + c' \leq \delta'$, then by Lemma 2 $\vec{D}'$ (and so $\vec{D}$) contains $\vec{T}$ provided that D' has minimal degree $\delta' t = \delta n$ and $|V(\vec{T})| = c't = cn$. Thus $\vec{D}'$ contain $\vec{T}$ if $d(1-\delta') + c' \leq \delta'$. The result will then follow by Lemma 1. It thus suffices to show that $d(1-\delta') + c' \leq \delta'$.

Substituting the values for δ', c', and $\delta = 1-\sqrt{2c}$, into $d(1-\delta') + c' \leq \delta'$, it suffices to show that

$$t \leq \frac{(1-2\sqrt{c})n + (1-\sqrt{2c})nd - cn}{d}.$$

As $t \leq n$, it thus suffices to show that

$$n \leq \frac{(1-2\sqrt{c})n + (1-\sqrt{2c})nd - cn}{d}.$$

The preceeding inequality is equivalent to

$$0 \leq 1 - \sqrt{2c} - d\sqrt{2c} - c.$$

In $\sqrt{c}$, the right-hand side of the preceeding inequality is a quadratic whose graph opens downward and is 1 at $\sqrt{c} = 0$. Thus the inequality will be true if $\sqrt{c}$ is the largest root of the right-hand side of the preceeding inequality. This inequality then holds provided that

$$\sqrt{c} = \frac{1}{\sqrt{2}}(\sqrt{3 + 2d + d^2} - 1 - d).$$

Squaring both sides of the preceeding equality, the result follows with

$$c = \frac{1}{2}(\sqrt{2 + (d+1)^2} - (d+1))^2.$$

□

Clearly we have the following result as well.

Corollary 6 *Let d be a non-negative integer, and $c = \frac{1}{2}(\sqrt{2 + (d+1)^2} - (d+1))^2$. Let $T_1, \ldots, T_n$ be a sequence of trees such that $|V(T_i)| = i$, there exists $x_i \in V(T_i)$ such that $T_i - x_i$ has at least $(1-c)(i-1)$ isolated vertices, and* $\operatorname{diam}(T_i) \leq d+2$. *Then $T_1, \ldots, T_n$ can be packed into K_n.*

4 Ringel's Conjecture

In contrast to the Tree Packing Conjecture, where there is a unique packing of the stars $T_1, \ldots, T_n$ into K_n, $|V(T_i)| = i$, there are many packings of $2n+1$ stars of order $n+1$ into K_{2n+1}. We begin by specifying a canonical packing of $2n+1$ stars of order $n+1$ into K_{2n+1} that will be used throughout this section.

Let S_0 be the star with $V(S_0) = \{0, 1, \ldots, n\}$ and $E(S_0) = \{0i : 1 \leq i \leq n\}$. With this labeling, S_0 is *graceful.* That is, if $e \in E(S_0)$, $e = 0i$, then the differences $i - 0$ are all distinct. The interested reader is referred to [6] for a more general definition of a graceful graph and a survey of known results on graceful graphs. It was shown by Rosa [10] that if G is graceful, then there exists a cyclic decomposition of K_{2n+1} into subgraphs isomorphic to G. Here a cyclic decomposition of K_{2n+1} into subgraphs isomorphic to G is just a packing of $2n+1$ copies of G into K_{2n+1} such that the function $f : \mathbb{Z}_{2n+1} \to \mathbb{Z}_{2n+1}$ by $f(i) = i + 1 \pmod{2n+1}$ leaves the packing invariant. Thus we have a packing of $2n+1$ copies of the star, say $S_0, \ldots, S_{2n}$, with n edges given by $V(S_i) = f^i(V(S_0))$ and $E(S_i) = \{f^i(0)f^i(j) : 1 \leq j \leq n\}$. Moreover, this packing has the following useful property.

Lemma 7 *Define a digraph $\vec{D}$ by $V(\vec{D}) = \mathbb{Z}_{2n+1}$ and $E(\vec{D}) = \{\vec{ij} : ij \in S_k$* and *$\deg_{S_i}(j) = 1\}$. Then $\vec{D}[V(S_i) - \{i\}]$ is a transitive tournament.*

Proof. As the packing of $S_0, \dots, S_{2n}$ into K_{2n+1} is invariant under f, it suffices to show that the result holds for $i = 0$ (as f will then cyclically permute the digraphs $\vec{D}[V(S_i) - \{i\}]$). If $i = 0$, then $V(S_0) - \{0\} = \{1, 2, \dots, n\}$. It is straightforward to verify that $\deg^{-}_{\vec{D}[V(S_0)-\{0\}]}(j) = j - 1$ for every $1 \le j \le n$. Whence $\vec{D}[(S_0) - \{0\}]$ is a transitive tournament. □

Theorem 8 *Let $T_0, \dots, T_{2n}$ be a sequence of trees of order $n+1$ such that there exists $x_i \in V(T_i)$ for which either:*

1. *$T_i - x_i$ has at least $n - \frac{\sqrt{n}}{4+2\sqrt{2}}$ isolated vertices for all $0 \le i \le 2n$, or*
2. *if d is a non-negative integer, and $c = (\sqrt{1 + (4+4d)^2} - (4+4d))^2$, then $T_i - x_i$ has at least $n - cn$ isolated vertices and $\mathrm{dist}_{T_i}(x_i, v) \le d+1$ for every $v \in V(T_i)$, for all $0 \le i \le 2n$.*

Then $T_0, \dots, T_{2n}$ can be packed into K_{2n+1}.

Proof. We will refer to the case where condition 1 holds as Case 1 and to the case where condition 2 holds as Case 2. In Case 1, let $c = 1/(\sqrt{n}(4 + 2\sqrt{2}))$. In either case, if $cn < 1$, then the only possible choice for each T_i is a star of order $n+1$. We may thus use the canonical packing of $2n+1$ stars into K_{2n+1}. We may thus assume without loss of generality that $cn \ge 1$ (so that in Case 1 we have that $n \ge 47$). We will show that if there exists $x_i \in V(T_i)$ such that $T_i - x_i$ has at least $n - cn$ isolated vertices, then $T_0, \dots, T_{2n}$ can be packed into K_{2n+1}. We begin with the canonical packing $S_0, \dots, S_{2n}$ of $2n+1$ stars of order $n+1$ into K_{2n+1} as above. We will inductively apply Lemma 1 and show by induction on i that $T_0, \dots, T_i, S'_{i+1}, \dots, S'_{2n}$ can be packed into K_{2n+1} where $S'_{i+1}, \dots, S'_{2n+1}$ are stars of order $n+1$ such that

$$|E(S_j) \cap E(S'_j)| \ge \lfloor n - 2\sqrt{c}n \rfloor, \tag{1}$$

for every $i+1 \le j \le 2n$, and if $xy = e \in E(T_j - j)$, $0 \le j \le i$, then $x, y \in V(S_j)$. As we will be applying Lemma 1, the vertex of maximal degree in T_i will always be i. If $i = 0$, then we have the packing $S_0, S_1, \dots, S_{2n}$ as above into K_{2n+1}. Define a digraph $\vec{D}$ by $V(\vec{D}) = \{0, \dots, 2n\}$ and $E(\vec{D}) = \{\vec{ij} : ij \in S_i, 0 \le i \le 2n, \text{ and } \deg_{S_i}(j) = 1\}$. By Lemma 7, $\vec{D}[V(S_0) - \{0\}]$ is a transitive tournament and thus contains every indirected tree (and thus forest) of order n (and so of order cn). Thus by Lemma 1, $T_0, S'_1, \dots, S'_{2n}$ can be packed into K_{2n+1}. Furthermore, $|E(S_j) \cap E(S'_j)| = n$ or $n-1$. To verify that Equation 1 holds, we will show that $|E(S_j) \cap E(S'_j)| \ge n - 1 \ge n - 2\sqrt{c}n$. Elementary calculations will show that Equation 1 holds provided that $1/(2\sqrt{c}) \le n$. As $cn \ge 1$, $n \ge 1/c > 1/(2\sqrt{c})$ and thus Equation 1 holds. Finally, by Remark 1, $V(T_0) = V(S_0)$. Thus if $e = xy \in E(T_0 - 0)$, then $x, y \in V(S_0)$.

We now assume that the induction hypothesis holds for $i \ge 0$, and will show that our packing of $T_0, \dots, T_i, S'_{i+1}, \dots, S'_{2n}$ into K_{2n+1} can be extended to a packing of $T_0, \dots, T_{i+1}, S''_{i+2}, \dots, S''_{2n}$ into K_{2n+1} by using Lemma 1 in such a way so that the induction hypothesis is satisfied. As Equation 1 holds, there are at least $\lfloor n-2\sqrt{c}n\rfloor$ vertices of $S_{i+1}-x_{i+1}$ that are also vertices of $S'_{i+1}-x_{i+1}$. Let $L=\{v\in (V(S_{i+1})-\{i+1\})\cap(V(S'_{i+1})-\{i+1\}) : v=x_j$ for some $i+2\leq j\leq 2n$ and $|E(S_j)\cap E(S'_j)|=\lfloor n-2\sqrt{c}n\rfloor\}$, and $\ell=|L|$. For $0\leq j\leq i$, let $F_j=\{\vec{jk} : jk\in E(S_j)\cap E(T_j)$ and $\deg_{T_j}(k)=1\}$, and for $i+2\leq j\leq 2n$, let $F_j=\{\vec{jk}:jk\in E(S_j)\cap E(S'_j)\}$. Let τ' be the transitive tournament $\tau[\mathbb{Z}_{2n+1}-\{i+1\}]$ (here τ is the canonical transitive tournament of order $2n+1$). Define a digraph $\vec{D}'$ by $V(\vec{D}')=V(\tau')$ and $E(\vec{D}')=\{\vec{xy} : \vec{xy}\in\cup_{j\in\mathbb{Z}_{2n+1}-\{i+1\}}F_j$ and $x,y\in\mathbb{Z}_{2n+1}-\{i+1\}\}$. By the remark following Lemma 1, $|E(\tau')-E(\vec{D}')|\leq 2cn\cdot i\leq 4cn^2$. Hence

$$\ell(n-\lfloor n-2\sqrt{c}n\rfloor)\leq 4cn^2.$$

Whence $\ell\leq 2\sqrt{c}n$. Let $V(D_{i+1})=(V(S_{i+1})\cap V(S'_{i+1}))-(L\cup\{i+1\})$ and $\vec{D}_{i+1}=\vec{D}'[V(D_{i+1})]$. As $|E(S_{i+1})\cap E(S'_{i+1})|\geq n-2\sqrt{c}n$ and $\ell\leq 2\sqrt{c}n$, $|V(D_{i+1})|=m\geq n-4\sqrt{c}n$. Removing every directed edge of $\vec{D}_{i+1}$ that is not contained in $E(\tau')$, we have a subdigraph $\vec{D}'_{i+1}$ of $\vec{D}_{i+1}$ such that $|V(\vec{D}'_{i+1})|=m$ whose underlying simple graph contains at least $m(m-1)/2-4cn^2$ edges. Note that $\vec{D}'_{i+1}$ is a subdigraph of a transitive tournament of order m. We now show for $x\in V(\vec{D}'_{i+1})$, that $i+1\notin V(T_x)$, $0\leq x\leq i$ and $i+1\notin V(S'_x)$, $i+2\leq x\leq 2n$.

Let $x\in V(\vec{D}'_{i+1})$. As $x\in(V(S_{i+1})\cap V(S'_{i+1}))-\{i+1\}$, and every vertex of $V(S_{i+1})\cap V(S'_{i+1})$ is a neighbor of $i+1$ in S_{i+1} or is itself $i+1$, we have that $x\in\{i+1+k \pmod{2n+1} : 1\leq k\leq n\}$. Let $e=ab\in E(S'_x)$ if $i+2\leq x\leq 2n$, $e=ab\in E(T_x)$ if $0\leq x\leq i$. It suffices to show that $a\neq i+1\neq b$. If $e\in E(S_x)$, then $x=a$ or b. Assume without loss of generality that $x=a$, so that $x\neq i+1$. As we began with the canonical packing of $S_0,\dots,S_{2n}$ into K_{2n+1}, $b\in\{x+k \pmod{2n+1} : 1\leq k\leq n\}$. Thus $b\in\{i+1+k \pmod{2n+1} : 1\leq k\leq 2n\}$. We conclude that $b\neq i+1$. If $ab=e\notin S_x$ and $e\in E(T_x-x)$ for some $0\leq j\leq i$, then by the induction hypothesis $a,b\in V(S_x)$. Thus $a,b\in\{x+k \pmod{2n+1} : 1\leq k\leq n\}$. We then have that $a,b\in\{i+1+k \pmod{2n+1} : 1\leq k\leq 2n\}$ so that $a\neq i+1\neq b$. Finally, if $e=ab\notin E(S_x)\cup E(T_x-x)$, $0\leq x\leq i$ or $e=ab\in E(S'_x)-E(S_x)$, $i+2\leq x\leq 2n$, then, following the description in the remark following Lemma 1, ab has had its "direction reversed". If, say, $b\in\{a+k \pmod{2n+1} : 1\leq k\leq n\}$ (so that $ab\in E(S_a)$), then $x=a$. Again we then have that $a,b\in\{i+1+k : 1\leq k\leq 2n\}$ and $a\neq i+1\neq b$.

Let $\vec{T}$ be the directed graph obtained by indirecting every edge of T_{i+1} to x_{i+1}, then deleting x_{i+1} and then deleting any isolated vertices. In order to apply Lemma 1, we need only show that $\vec{D}'_{i+1}$ contains a subdigraph isomorphic to $\vec{T}$. We now do this, considering Cases 1 and 2 separately.

Case 1: We wish to apply Lemma 3 to show that D'_{i+1} (the underlying simple graph of $\vec{D}'_{i+1}$) contains a complete subgraph of order cn. This will then imply that $\vec{D}'_{i+1}$ contains a transitive tournament of order cn, and hence that $\vec{D}'_{i+1}$ contains every indirected tree, and hence forest, of order cn. We must show that $m\geq(cn)^2$ and the complement of D'_{i+1} contains at most $m^2/(2cn)$ edges. It is straightforward to verify, as $n \geq 47$ and $m \geq n - 4\sqrt{c}n$, that $m \geq (cn)^2$. It was shown above that the complement of D'_{i+1} contains at most $4cn^2$ edges. An elementary calculation then shows that $m^2/(2cn) \geq 4cn^2$ will hold provided that $m \geq \sqrt{8c^2n^3}$. Using the facts that $m \geq n - 4\sqrt{c}n$ and substituting $c = 1/(\sqrt{n}(4 + 2\sqrt{2}))$, we have that $m^2/(2cn) \geq 4cn^2$ provided that $n \geq (4 + 2\sqrt{2})^2$, which is true. Hence $\vec{D}'_{i+1}$ contains a transitive tournament of order cn and $\vec{D}'_{i+1}$ contains a subdigraph isomorphic to $\vec{T}$.

Case 2: Let $\delta = 1 - 8\sqrt{c}$. In order to apply Lemma 2, we first show that D'_{i+1} contains a subgraph of minimal degree at least δn by applying Lemma 4.

As $m \geq (1-4\sqrt{c})n$ and $\delta n = (1-8\sqrt{c})n$, $m \geq \delta n + 1$ will hold provided that $n \geq 1/(4\sqrt{c})$. It is not difficult to verify that $c < 1$, and as $nc \geq 1$, $n \geq 1/c \geq 1/(4\sqrt{c})$. Thus $m \geq \delta n + 1$. In order to apply Lemma 4, it thus suffices to show that

$$\frac{m(m-1)}{2} - 4cn^2 \geq (\delta n - 1)(m - \frac{\delta n}{2}) + 1.$$

Substituting the above value of δ, we have that the above inequality is equivalent to

$$m^2 + m(1 + 16\sqrt{c}n - 2n) + n^2 - 16\sqrt{c}n^2 + 56cn^2 + 8\sqrt{c}n - n - 2 \geq 0.$$

The left-hand side of the preceeding inequality is a quadratic in m, and will hold provided that m is at least the largest root of the left-hand side of the preceeding inequality. As $m \geq (1 - 4\sqrt{c})n$, we thus need to show that

$$(1 - 4\sqrt{c})n \geq \frac{1}{2}(-1 + 2n - 16\sqrt{c}n + \sqrt{9 + 32cn^2}).$$

The preceeding inequality is equivalent to

$$8\sqrt{c}n + 1 \geq \sqrt{9 + 32cn^2},$$

which will hold provided that $4cn^2 + 2\sqrt{c}n - 1 \geq 0$. The left-hand side of this inequality is a quadratic in n, and the inequality will hold provided that $n \geq (\sqrt{5} - 1)/(4\sqrt{c})$. As $(\sqrt{5} - 1)/(4\sqrt{c}) < 1/\sqrt{c}$, it suffices to show that $n \geq 1/\sqrt{c}$, which holds as $\sqrt{c}n > cn \geq 1$. Thus D'_{i+1} contains a subdigraph of minimal degree at least δn, so that $\vec{D}'_{i+1}$ contains a subdigraph $\vec{D}''_{i+1}$ such that D''_{i+1} has minimal degree at least δn.

Let $t = |V(\vec{D}''_{i+1})|$, $\delta' = \delta n/t$, and $c' = cn/t$ (so that $c't = cn$). If $d(1 - \delta') + c' \leq \delta'$, then by Lemma 2 $\vec{D}''_{i+1}$ (and so $\vec{D}'_{i+1}$) contains every directed forest of order $c't = cn$ with every edge of each component indirected towards some root, and in the underlying simple graph of each component, the maximum distance from the root is at most d. Thus if $d(1 - \delta') + c' \leq \delta'$, then $\vec{D}'_{i+1}$ contains $\vec{T}$. Substituting the values of δ', c', and δ into this inequality, we obtain the following equivalent inequality:

$$t \leq \frac{(1-8\sqrt{c})n + (1-8\sqrt{c})dn - cn}{d}.$$

As $t \leq n$, it suffices to show that

$$n \leq \frac{(1-8\sqrt{c})n + (1-8\sqrt{c})dn - cn}{d}.$$

The preceeding inequality is equivalent to

$$0 \leq 1 - 8\sqrt{c} - 8\sqrt{c}d - c.$$

The right-hand side of this inequality is a quadratic in $\sqrt{c}$, and will hold provided $\sqrt{c}$ is less than or equal to the largest root of the right-hand side of this inequality (note that the smallest root of the right-hand side of this inequality is negative). Thus this inequality will hold provided that

$$\sqrt{c} \leq \sqrt{(4+4d)^2+1} - (4+4d).$$

Thus $\vec{D}'_{i+1}$ will contain $\vec{T}$ provided that $c \leq (\sqrt{1+(4+4d)^2} - (4+4d))^2$, which is true. Thus $\vec{D}'_{i+1}$ contains $\vec{T}$.

We now consider the two cases together. In either case, it now follows by Lemma 1 that there is a packing of $T_1, \ldots, T_{i+1}, S''_{i+2}, \ldots, S''_{2n}$ into K_{2n}, where each S''_j is isomorphic to S_j. Furthermore, as for every $j \in V(D'_{i+1})$, $i+2 \leq j \leq 2n$, $|E(S_j) - E(S'_j)| > \lfloor n - 2\sqrt{cn} \rfloor$ and $|E(S''_j) - E(S'_j)| \leq 1$, if $j \in V(D'_{i+1})$ with $i+2 \leq j \leq 2n$, then $|E(S''_j) \cap E(S_j)| \geq \lfloor n - 2\sqrt{cn} \rfloor$. Of course, if $i+2 \leq j \leq 2n$ and $j \notin V(D'_{i+1})$, then $S''_j = S'_j$. We conclude that $|E(S_j) \cap E(S''_j)| \leq \lfloor n - 2\sqrt{cn} \rfloor$ for every $i+2 \leq j \leq 2n$. Finally, as every vertex of D'_{i+1} is also a vertex of S_i, it follows by Remark 1 that if $ab \in E(T_{i+1} - \{i+1\})$, then $a, b \in V(S_{i+1}) - \{i+1\}$. The result then follows by induction. □

Clearly the preceeding theorem implies the following special cases of Ringel's Conjecture.

Corollary 9 *Let T be a tree of order $n+1$ such that there exists $x \in V(T)$ and one of the following is true:*

1. *$T - x$ has at least $n - \frac{\sqrt{n}}{4+2\sqrt{2}}$ isolated vertices, or*
2. *if d is a non-negative integer, and $c = (\sqrt{1+(4+4d)^2} - 4 - 4d)^2$, then $T - x$ has at least $n - cn$ isolated vertices and $\mathrm{dist}_{T_i}(x_i v) \leq d+1$ for every $v \in V(T_i)$. (Note that this condition is satisfied if $\mathrm{diam}(T) \leq d+2$.)*

Then $2n+1$ copies of T can be packed into K_{2n+1}.

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