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The automorphism groups of the Laguerre near-planes of order four

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Abstract

We determine the automorphism groups of the Laguerre near-planes of order 4 found in Steinke, *Australas. J. Combin.* 25 (2002), 145–166, and give characterisations of some of these planes in terms of their automorphism groups.

1 Introduction and result

A *Laguerre near-plane of order* 4 consists of a set P of $4^2 = 16$ points, a set $\mathcal{C}$ of $4^3 = 64$ circles and a set $\mathcal{G}$ of 4 generators (subsets of P) such that the following three axioms are satisfied:

(G) $\mathcal{G}$ partitions P and each generator contains 4 points.

(C) Each circle intersects each generator in precisely one point.

(J) Three points no two of which are on the same generator can be uniquely joined by a circle.

Labelling the generators from 1 to 4 and the points on each generator from 1 to 4 and identifying each circle with the 4-tuple $(c_1, \ldots, c_4)$ where c_i is the unique point of the circle on generator i, we see that a Laguerre near-plane of order 4 corresponds to an orthogonal array of strength 3 on 4 symbols (levels), 4 constraints and index 1, cf. [1], or equivalently, a transversal design $\mathrm{TD}_3(4,4)$. Since we have a more geometric point of view we rather use the term Laguerre near-plane instead of orthogonal array or transversal design.
In [2] all Laguerre near-planes of order 4 were determined and a representation of such planes in terms of a single map was developed. It was shown that such a Laguerre near-plane is isomorphic to one of five Laguerre near-planes of order 4. The results from [2] can be summarized as follows.

THEOREM 1.1 *Let $f : \mathbb{F}_4^3 \to \mathbb{F}_4$ where $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, $\omega^2 = \omega + 1$, denotes the Galois field of order 4 be a map such that for each $x_0, y_0, z_0 \in \mathbb{F}_4$ the functions $x \mapsto f(x, y_0, z_0)$, $y \mapsto f(x_0, y, z_0)$ and $z \mapsto f(x_0, y_0, z)$ are permutations of $\mathbb{F}_4$. Such a map describes a Laguerre near-plane $\mathcal{L}(f)$ of order 4 as follows. The point set is $\mathbb{F}_4 \times \mathbb{F}_4$ and generators are the verticals $\{c\} \times \mathbb{F}_4$ for $c \in \mathbb{F}_4$. Circles are of the form*

$$\{(1, x), (\omega, y), (\omega^2, z), (0, f(x, y, z))\}$$

for $x, y, z \in \mathbb{F}_4$. Conversely, every Laguerre near-plane of order 4 can be uniquely described in this way by such a map.
A Laguerre near-plane of order 4 is isomorphic to a plane described by one of the maps

$$\begin{aligned}
f_0(x,y,z) &= x+y+z,\\
f_1(x,y,z) &= (x^2+x)(y^2+y)+(y^2+y)(z^2+z)+(x^2+x)(z^2+z)+x+y+z,\\
f_2(x,y,z) &= (x^2+x)(z^2+z)+x+y+z,\\
f_3(x,y,z) &= (x^2+x)(y^2+y)(z^2+z)+x+y+z,\\
f_4(x,y,z) &= (x^2+\omega^2x)(y^2+\omega y)(z^2+\omega z)+(x^2+\omega^2x)(y^2+\omega^2y)\\
&\quad +(x^2+\omega^2x)(z^2+\omega^2z)+(y^2+\omega y)(z^2+\omega z)+x+y+z.
\end{aligned}$$

Of these Laguerre near-planes only $\mathcal{L}(f_0)$ extends to a Laguerre plane of order 4.

In the language of transversal designs the last statement in the Theorem above means that of the corresponding transversal designs only the one associated with f_0 is resolvable and extends to a transversal design $\mathrm{TD}_3(5, 4)$.
In this paper we investigate the automorphism groups of these five Laguerre near-planes of order 4 and show that they have different orders so that the planes are mutually non-isomorphic. We further determine transitivity properties of the automorphism groups on the point set, circle set and set of generators which in turn yields characterisations of some of the planes.

THEOREM 1.2 *The automorphism group $\Gamma(f_i)$ of $\mathcal{L}(f_i)$ has order $2^{10} \cdot 3^2$, $2^{10} \cdot 3$, 2^9, $2^7 \cdot 3$ and 2^7 for $i = 0, 1, 2, 3, 4$, respectively. In particular, the Laguerre near-planes $\mathcal{L}(f_i)$, $i = 0, 1, 2, 3, 4$, are mutually non-isomorphic.*
Moreover, $\Gamma(f_0)$ and $\Gamma(f_1)$ are transitive on the collection of all incident point-circle pairs; in particular, these groups act transitively on the point set, the set of circles and the set of generators. The automorphism groups of $\mathcal{L}(f_2)$ and $\mathcal{L}(f_4)$ are circle-transitive but not transitive on the sets of generators (and thus not point-transitive); $\Gamma(f_2)$ fixes no generator whereas $\Gamma(f_4)$ fixes two generators. $\Gamma(f_3)$ is neither point- nor circle-transitive but is transitive on the set of generators.

2 Isomorphisms and automorphisms of Laguerre near-planes

Let ω be a generator of the multiplicative group of $\mathbb{F}_4$. Then the non-zero elements of $\mathbb{F}_4$ can be written in the form ω^i for $i = 0, 1, 2$. We use the notation $\omega^\infty = 0$ and

$$I = \{0, 1, 2, \infty\}.$$

Then

$$\mathbb{F}_4 = \{\omega^i \mid i \in I\}.$$

We further denote by $\mathbb{F}_2$ the prime subfield of $\mathbb{F}_4$ consisting of 0 and 1.
Each circle is described by some $(c_0, c_1, c_2, c_\infty) \in \mathbb{F}_4^4$ as

$$C_{c_0,c_1,c_2,c_\infty} = \{(1, c_0), (\omega, c_1)(\omega^2, c_2), (0, c_\infty)\} = \{(\omega^i, c_i) \mid i \in I\}.$$

Every isomorphism of a Laguerre near-plane of order 4 is of the form

$$\mathbb{F}_4 \times \mathbb{F}_4 \to \mathbb{F}_4 \times \mathbb{F}_4 : (u, v) \mapsto (\alpha(u), \beta_u(v))$$

where α and β_u are permutations of $\mathbb{F}_4$ for each $u \in \mathbb{F}_4$.
Note that the group of permutations of $\mathbb{F}_4$ is the symmetric group S_4. Every even permutation can be written as $x \mapsto ax + b$ for some $a, b \in \mathbb{F}_4$, $a \neq 0$. The automorphism $x \mapsto x^2$ of $\mathbb{F}_4$ is an odd permutation of $\mathbb{F}_4$—in fact, a transposition—and every odd permutation of $\mathbb{F}_4$ is of the form $x \mapsto ax^2 + b$ for some $a, b \in \mathbb{F}_4$.
With this notation we obtained the following in [2], 3.1 and 4.1.

1. $(u, v) \mapsto (u, \beta_u(v))$ where the β_u are permutations of $\mathbb{F}_4$ for each $u \in \mathbb{F}_4$. These permutations take C_{c_0,c_1,c_2,c_∞} to $C_{\beta_1(c_0),\beta_\omega(c_1),\beta_{\omega^2}(c_2),\beta_0(c_\infty)}$. A Laguerre near-plane $\mathcal{L}(f)$ is taken to $\mathcal{L}(f')$ where

$$f'(x, y, z) = \beta_0(f(\beta_1^{-1}(x), \beta_\omega^{-1}(y), \beta_{\omega^2}^{-1}(z))$$

   for $x, y, z \in \mathbb{F}_4$.

2. $(u, v) \mapsto (u+t, v)$ for $t \in \mathbb{F}_4$. These permutations take C_{c_0,c_1,c_2,c_∞} to C_{d_0,d_1,d_2,d_∞} where

$$(d_0, d_1, d_2, d_\infty) = \begin{cases} (c_0, c_1, c_2, c_\infty), & \text{if } t = 0, \\ (c_\infty, c_2, c_1, c_0), & \text{if } t = 1, \\ (c_2, c_\infty, c_0, c_1), & \text{if } t = \omega, \\ (c_1, c_0, c_\infty, c_2), & \text{if } t = \omega^2. \end{cases}$$

   A Laguerre near-plane $\mathcal{L}(f)$ is taken to $\mathcal{L}(f')$ where $f' = f$ for $t = 0$ and f' is an inverse of a partial map of f with the other two variables interchanged given by $f'(f(x, y, z), z, y) = x$, $f'(z, f(x, y, z), x) = y$ and $f'(y, x, f(x, y, z)) = z$ for $t = 1$, ω and ω^2, respectively, that is, the maps $(x, y, z) \mapsto f_{z,y}^{-1}(x)$, $(x, y, z) \mapsto f_{z,x}^{-1}(y)$ and $(x, y, z) \mapsto f_{y,x}^{-1}(z)$, respectively; compare [2], Corollary 2.6 and Examples 2.7, for finding these inverses.

3. $(u, v) \mapsto (ru, v)$ for $r \in \mathbb{F}_4$, $r \neq 0$. These permutations take a circle C_{c_0,c_1,c_2,c_∞} to $C_{c_{3-k},c_{1-k},c_{2-k},c_\infty}$ where $r = \omega^k$, $k = 0, 1, 2$, and the indices $3 - k$, $1 - k$ and $2 - k$ are taken modulo 3. A Laguerre near-plane $\mathcal{L}(f)$ is taken to $\mathcal{L}(f')$ where

$$f'(x, y, z) = \begin{cases} f(x, y, z), & \text{if } r = 1, \\ f(y, z, x), & \text{if } r = \omega, \\ f(z, x, y), & \text{if } r = \omega^2. \end{cases}$$

4. $(u,v) \mapsto (u^2, v)$. This permutation takes C_{c_0,c_1,c_2,c_∞} to C_{c_0,c_2,c_1,c_∞}. A Laguerre near-plane $\mathcal{L}(f)$ is taken to $\mathcal{L}(f')$ where $f'(x,y,z) = f(x,z,y)$.

More explicitly, the isomorphisms of type (1) are generated by the following permutations.

(1a) $(u,v) \mapsto \begin{cases} (u,v), & \text{if } u \neq s, \\ (u, v+t), & \text{if } u = s, \end{cases}$ for $s,t \in \mathbb{F}_4$ takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where

$$f'(x,y,z) = \begin{cases} f(x,y,z) + t, & \text{if } s = 0, \\ f(x+t,y,z), & \text{if } s = 1, \\ f(x,y+t,z), & \text{if } s = \omega, \\ f(x,y,z+t), & \text{if } s = \omega^2; \end{cases}$$

(1b) $(u,v) \mapsto \begin{cases} (u,v), & \text{if } u \neq s, \\ (u, rv), & \text{if } u = s, \end{cases}$ for $r,s \in \mathbb{F}_4$, $r \neq 0$, takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where

$$f'(x,y,z) = \begin{cases} rf(x,y,z), & \text{if } s = 0, \\ f(r^2x,y,z), & \text{if } s = 1, \\ f(x,r^2y,z), & \text{if } s = \omega, \\ f(x,y,r^2z), & \text{if } s = \omega^2; \end{cases}$$

(1c) $(u,v) \mapsto \begin{cases} (u,v), & \text{if } u \neq s, \\ (u, v^2), & \text{if } u = s, \end{cases}$ for $s \in \mathbb{F}_4$ takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where

$$f'(x,y,z) = \begin{cases} f(x,y,z)^2, & \text{if } s = 0, \\ f(x^2,y,z), & \text{if } s = 1, \\ f(x,y^2,z), & \text{if } s = \omega, \\ f(x,y,z^2), & \text{if } s = \omega^2. \end{cases}$$

Note that each isomorphism

$$(u,v) \mapsto (u, v + t_u)$$

for $t_u \in \mathbb{F}_4$ can be written in the form

$$(u,v) \mapsto (u, v + s_3u^3 + s_2u^2 + s_1u + s_0)$$

for $s_3, s_2, s_1, s_0 \in \mathbb{F}_4$ where

$$\begin{aligned} s_0 &= t_0, \\ s_1 &= \omega t_{\omega^2} + \omega^2 t_\omega + t_1, \\ s_2 &= \omega^2 t_{\omega^2} + \omega t_\omega + t_1, \\ s_3 &= t_{\omega^2} + t_\omega + t_1 + t_0. \end{aligned}$$

We are using both forms whichever is more convenient at the time.
An isomorphism

$$\mathbb{F}_4 \times \mathbb{F}_4 \to \mathbb{F}_4 \times \mathbb{F}_4 : (u, v) \mapsto (\alpha(u), \beta_u(v))$$

where α and β_u are permutations of $\mathbb{F}_4$ for each $u \in \mathbb{F}_4$ defines an automorphism of a Laguerre near-plane $\mathcal{L}(f)$ if and only is $f' = f$ where f' is found as in the above lists. The collection of all permutations with $\alpha = id$ is a normal subgroup Δ. These are the permutations of type (1). We further denote by Δ^* the collection of all permutations in Δ whose accompanying field automorphisms on generators are the identity, i.e., permutations of the form $(u, v) \mapsto (u, a_u v + b_u)$ for $a_u, b_u \in \mathbb{F}_4$, $a_u \neq 0$. Clearly, Δ^* is a normal subgroup of Δ.
For automorphisms in Δ we have the following.

LEMMA 2.1 *The transformation*

$$(u, v) \mapsto (u, \beta_u(v))$$

where β_u are permutations of $\mathbb{F}_4$ for each $u \in \mathbb{F}_4$ is an automorphism of $\mathcal{L}(f)$ if and only if

$$f(\beta_1(x), \beta_\omega(y), \beta_{\omega^2}(z)) = \beta_0(f(x, y, z)).$$

3 Automorphism groups

In [2] we have established that a Laguerre near-plane of order 4 is isomorphic to one of the Laguerre near-planes $\mathcal{L}(f_i)$, $i = 0, 1, 2, 3, 4$. In order to show that in fact the latter five planes are mutually non-isomorphic we investigate the automorphism groups $\Gamma(f_i)$ of these planes, that is, the collection of all permutations of $\mathbb{F}_4 \times \mathbb{F}_4$ that preserve the Laguerre near-plane.

3.1 The automorphism group of $\mathcal{L}(f_0)$

Since $\mathcal{L}(f_0)$ extends to the Miquelian Laguerre plane, every automorphism of the Miquelian Laguerre plane that fixes a distinguished generator induces an automorphism of the Laguerre near-plane obtained by removing the distinguished generator. It is well known that the automorphism group of the Miquelian Laguerre plane of order 4 has order $2^9 \cdot 3^2 \cdot 5$ and acts transitively on the set of all incident point-circle pairs. In particular, this group is transitive on the set of generators. Hence the stabilizer of a generator has order $2^9 \cdot 3^2$. In terms of the isomorphisms from section 2 the group induced by this stabilizer is generated by all permutations of types (2) and (3) and by the following permutations:

(i) $(u, v) \mapsto (u, v + t_u)$ for $t_0, t_1, t_\omega, t_{\omega^2} \in \mathbb{F}_4$, $t_0 + t_1 + t_\omega + t_{\omega^2} = 0$ (type (1a)),

(ii) $(u, v) \mapsto (u, rv)$ for $r \neq 0$ (type (1b)) and

(iii) $(u, v) \mapsto (u^2, v^2)$ (types (4) and (1c) combined).

However, $\mathcal{L}(f_0)$ also admits the permutation of type (4) as an automorphism. In fact, together they generate the entire automorphism group of $\mathcal{L}(f_0)$. From the transitivity properties of the automorphism group of the Miquelian Laguerre plane of order 4 we see that the group G_0 generated by the above automorphisms is transitive on the set of point-circles pairs ('flags') and induces the full symmetric group of degree 4 on the set of generators. Now let γ be an automorphism of $\mathcal{L}(f_0)$. Up to elements in G_0 we can assume that γ fixes each generator, i.e., $\gamma \in \Delta$, and that γ fixes the circle $\{(u,0) \mid u \in \mathbb{F}_4\}$. Then $\gamma(u,v) = (u, \beta_u(v))$ where $\beta_u(v) = a_u v^{m_u}$ with $a_u \in \mathbb{F}_4$, $a_u \neq 0$, and $m_u = 1, 2$. Using the automorphisms $(u,v) \mapsto (u, v^2)$ (types (4) and (iii) combined) and (ii) in Δ, if necessary, we may further assume that β_0 is the identity. By Lemma 2.1 we then must have $f_0(\beta_1(x), \beta_\omega(y), \beta_{\omega^2}(z))) = f_0(x,y,z)$, that is,

$$a_1 x^{m_1} + a_\omega y^{m_\omega} + a_{\omega^2} z^{m_{\omega^2}} = x + y + z$$

for all $x, y, z \in \mathbb{F}_4$. But this implies $a_1 = a_\omega = a_{\omega^2} = 1$ and $m_1 = m_\omega = m_{\omega^2} = 1$, that is, γ is the identity.

This shows that the automorphism group of $\mathcal{L}(f_0)$ is contained in G_0. In summary we obtain the following.

PROPOSITION 3.1 *The automorphism group $\Gamma(f_0)$ of the Laguerre near-plane $\mathcal{L}(f_0)$ has order $2^{10} \cdot 3^2$. Furthermore, $\Gamma(f_0)$ acts transitively on the set of point-circles pairs of $\mathcal{L}(f_0)$ and induces the full symmetric group S_4 of degree 4 on the set of generators. In particular, $\Gamma(f_0)$ is point-transitive and circle-transitive.*
Although $\mathcal{L}(f_0)$ extends to the Miquelian Laguerre plane of order 4 not every automorphism of $\mathcal{L}(f_0)$ extends to an automorphism of the Laguerre plane.

3.2 The automorphism group of $\mathcal{L}(f_1)$

From the list in section 2 we find that the following permutations are automorphisms of $\mathcal{L}(f_1)$.

(i) $(u,v) \mapsto (u, v + t_u)$ for $t_0, t_1, t_\omega, t_{\omega^2} \in \mathbb{F}_2$, $t_0 + t_1 + t_\omega + t_{\omega^2} = 0$;

(ii) $(u,v) \mapsto (u, v^2)$;

(iii) $(u,v) \mapsto \begin{cases} (u, v+\omega), & \text{if } u \in \{0,t\} \\ (u, v^2), & \text{if } u \in \mathbb{F}_4 \setminus \{0,t\} \end{cases}$ for $t \in \mathbb{F}_4$, $t \neq 0$;

(iv) $(u,v) \mapsto \begin{cases} (u+t, v), & \text{if } u \in \{0,t\} \\ (u+t, v^2), & \text{if } u \in \mathbb{F}_4 \setminus \{0,t\} \end{cases}$ for $t \in \mathbb{F}_4$, $t \neq 0$;

(v) $(u,v) \mapsto (ru, v)$ for $r \neq 0$;

(vi) $(u,v) \mapsto (u^2, v)$.

These automorphisms generate a group G_1. By looking at the first coordinates we see that every permutation of the set of generators can be obtained by an element of G_1.

Let γ be an automorphism of $\mathcal{L}(f_1)$. Up to elements in G_1 we can assume that γ fixes each generator, i.e., $\gamma \in \Delta$. Then $\gamma(u,v) = (u, \beta_u(v))$ where $\beta_u(v) = a_u v^{m_u} + b_u$ with $a_u, b_u \in \mathbb{F}_4$, $a \neq 0$, and $m_u = 1, 2$. Using the automorphisms (ii) and (iii) in G_1, if necessary, we may further assume that $m_0 = m_{\omega^2} = 1$. By Lemma 2.1 we then must have $f_1(\beta_1(x), \beta_\omega(y), \beta_{\omega^2}(z))) = \beta_0(f_1(x,y,z))$, that is,

$$\begin{aligned}
&(a_1^2 x^{2m_1} + a_1 x^{m_1} + b_1^2 + b_1)(a_\omega^2 y^{2m_\omega} + a_\omega y^{m_\omega} + b_\omega^2 + b_\omega) \\
&\quad +(a_\omega^2 y^{2m_\omega} + a_\omega y^{m_\omega} + b_\omega^2 + b_\omega)(a_{\omega^2}^2 z^2 + a_{\omega^2} z + b_{\omega^2}^2 + b_{\omega^2}) \\
&\quad +(a_1^2 x^{2m_1} + a_1 x^{m_1} + b_1^2 + b_1)(a_{\omega^2}^2 z^2 + a_{\omega^2} z + b_{\omega^2}^2 + b_{\omega^2}) \\
&\quad + a_1 x^{m_1} + a_\omega y^{m_\omega} + a_{\omega^2} z + b_1 + b_\omega + b_{\omega^2} \\
&= a_0[(x^2+x)(y^2+y) + (y^2+y)(z^2+z) + (x^2+x)(z^2+z) + x + y + z] + b_0
\end{aligned} \tag{1}$$

for all $x, y, z \in \mathbb{F}_4$. Looking at terms x^2 and x in (1) we find

$$(b_\omega^2 + b_\omega + b_{\omega^2}^2 + b_{\omega^2})(a_1^2 x^{2m_1} + a_1 x^{m_1}) + a_1 x^{m_1} = a_0 x.$$

Since $b^2 + b \in \mathbb{F}_2$ for each $b \in \mathbb{F}_4$, we obtain two cases. Either $b_\omega^2 + b_\omega + b_{\omega+1}^2 + b_{\omega^2} = 1$ and then $m_1 = 2$, $a_1 = a_0^2$, or $b_\omega^2 + b_\omega + b_{\omega^2}^2 + b_{\omega^2} = 0$, and then $m_1 = 1$, $a_1 = a_0$. In both cases we have $a_1 = a_0^{m_1}$ and $a_1^2 x^{2m_1} + a_1 x^{m_1} = a_0^{2m_1} x^{2m_1} + a_0^{m_1} x^{m_1} = (a_0^2 x^2 + a_0 x)^{m_1} = a_0^2 x^2 + a_0 x$. One similarly finds that $a_\lambda = a_0^{m_\lambda}$ for $\lambda = \omega, \omega^2$ and $a_\omega^2 y^{2m_\omega} + a_\omega y^{m_\omega} = a_0^2 y^2 + a_0 y$, $a_{\omega^2}^2 z^2 + a_{\omega^2} z = a_0^2 z^2 + a_0 z$. Comparing terms $x^2 y^2$ in (1) yields $a_0 = 1$ and thus $a_\lambda = 1$ for all $\lambda \in \mathbb{F}_4$.

Let $d_\lambda = b_\lambda^2 + b_\lambda$ for $\lambda = 1, \omega, \omega^2$. Then (1) becomes

$$\begin{aligned}
&(x^2 + x + d_1)(y^2 + y + d_\omega) + (y^2 + y + d_\omega)(z^2 + z + d_{\omega^2}) \\
&\quad +(x^2 + x + d_1)(z^2 + z + d_{\omega^2}) + x^{m_1} + y^{m_\omega} + z + b_1 + b_\omega + b_{\omega^2} \\
&= (x^2+x)(y^2+y) + (y^2+y)(z^2+z) + (x^2+x)(z^2+z) + x + y + z + b_0.
\end{aligned}$$

Expanding the left-hand side, we see that

$$\begin{aligned}
(d_\omega + d_{\omega^2})(x^2 + x) + x^{m_1} &= x \\
(d_1 + d_{\omega^2})(y^2 + y) + y^{m_\omega} &= y \\
(d_1 + d_\omega)(z^2 + z) + z &= z
\end{aligned}$$

and

$$d_1 d_\omega + d_\omega d_{\omega^2} + d_1 d_{\omega^2} + b_0 + b_1 + b_\omega + b_{\omega^2} = 0.$$

Thus $d_\omega = d_1$ and, as before, either $d_{\omega^2} = d_1$, $m_1 = 1$ or $d_{\omega^2} = d_1 + 1$, $m_1 = 2$, and $b_0 + b_1 + b_\omega + b_{\omega^2} = d_1$. Hence, we have two cases.

Case 1: $d_{\omega^2} = d_\omega = d_1 \in \mathbb{F}_2$, $m_\lambda = 1$ for all $\lambda \in \mathbb{F}_4$.

If $d_1 = 0$, then γ is of the form (i) and thus $\gamma \in G_1$. For $d_1 = 1$ we use the composition of all three automorphisms (iii) for $t \in \mathbb{F}_4$, $t \neq 0$, in the list above. This yields the automorphism

$$\gamma' : (u,v) \mapsto \begin{cases} (u, v+\omega), & \text{if } u \in \mathbb{F}_4 \setminus \{\omega\}, \\ (u, v+\omega^2), & \text{if } u = \omega. \end{cases}$$

Then $\gamma' \circ \gamma$ is an automorphism as before and thus again $\gamma \in G_1$.
Case 2: $d_{\omega^2} = d_1 + 1$, $d_\omega = d_1 \in \mathbb{F}_2$, $m_0 = m_{\omega^2} = 1$, $m_1 = m_\omega = 2$.
In this case we use the automorphism (iii) for $t = \omega^2$ to obtain an automorphism as in the first case.
This shows that the automorphism group of $\mathcal{L}(f_1)$ is contained in G_1. Furthermore, G_1 is of order $2^{10} \cdot 3$ since, from above, $G_1/\Delta \cong S_4$, $|\Delta/\Delta^*| = 8$ and $|\Delta^*| = 16$. Taking the first and last automorphisms in the list above one readily sees that G_1 is transitive on the generator $\{0\} \times \mathbb{F}_4$. Thus G_1 acts transitively on the point set $\mathbb{F}_4 \times \mathbb{F}_4$.
Using automorphisms (i) and (iii) above we see that Δ is transitive on the generator $\{0\} \times \mathbb{F}_4$. The stabilizer $\Delta_{(0,0)}$ of $(0,0)$ has order 2^5 and is generated by the following automorphisms:

- $(u,v) \mapsto (u, v+t_u)$ for $t_u \in \mathbb{F}_2$, $t_0 = 0$, $t_1 + t_\omega + t_{\omega^2} = 0$;
- $(u,v) \mapsto (u, v^2)$;
- $(u,v) \mapsto \begin{cases} (u,v), & \text{if } u \in \mathbb{F}_2, \\ (u, v^2+u), & \text{if } u \in \mathbb{F}_4 \setminus \mathbb{F}_2; \end{cases}$
- $(u,v) \mapsto \begin{cases} (u,v), & \text{if } u \in \{0,\omega\}, \\ (u, v^2+\omega^2), & \text{if } u = 1, \\ (u, v^2+\omega), & \text{if } u = \omega^2. \end{cases}$

The last two automorphisms are obtained as a composition of two automorphisms of the form (iii) (for $t = \omega^2, \omega$ and $t = 1, \omega^2$, respectively). The first and last automorphisms then show that $\Delta_{(0,0)}$ is transitive on the generator $\{1\} \times \mathbb{F}_4$. Finally, the stabilizer $\Delta_{(0,0),(1,0)}$ of $(0,0)$ and $(1,0)$ has order 2^3 and is generated by $(u,v) \mapsto (u, y + u^2 + u)$ and the second and third automorphisms in the list above. It now readily follows that $\Delta_{(0,0),(1,0)}$ is transitive on the generator $\{\omega\} \times \mathbb{F}_4$. Since each circle through $(0,0)$ is uniquely determined by its intersection with the two generators $\{1\} \times \mathbb{F}_4$ and $\{\omega\} \times \mathbb{F}_4$, we see that $\Delta_{(0,0)}$ is transitive on the set of circles through $(0,0)$. Hence, $(G_1)_{(0,0)}$ is transitive on the set of circles through $(0,0)$, and because G_1 is point-transitive, we finally obtain that G_1 acts transitively on the set of point-circles pairs of $\mathcal{L}(f_1)$.
In summary we obtain the following.

PROPOSITION 3.2 *The automorphism group $\Gamma(f_1)$ of the Laguerre near-plane $\mathcal{L}(f_1)$ has order $2^{10} \cdot 3$. Furthermore, $\Gamma(f_1)$ acts transitively on the set of point-circles pairs of $\mathcal{L}(f_1)$ and induces the full symmetric group S_4 on the set of generators. In particular, $\Gamma(f_1)$ is point-transitive and circle-transitive.*

3.3 The automorphism group of $\mathcal{L}(f_2)$

From the list in section 2 we find that the following permutations are automorphisms of $\mathcal{L}(f_2)$.

(i) $(u, v) \mapsto (u, v + t_u)$ for $t_1, t_{\omega^2} \in \mathbb{F}_2$, $t_0, t_\omega \in \mathbb{F}_4$, $t_0 + t_1 + t_\omega + t_{\omega^2} = 0$;

(ii) $(u, v) \mapsto (u, v^2)$;

(iii) $(u, v) \mapsto \begin{cases} (u, v + \omega^2 u^2), & \text{if } u \in \mathbb{F}_4 \setminus \{\omega^2\}, \\ (u, v^2 + \omega^2 u^2), & \text{if } u = \omega^2; \end{cases}$

(iv) $(u, v) \mapsto \begin{cases} (u, v + u), & \text{if } u \in \mathbb{F}_4 \setminus \{1\}, \\ (u, v^2 + u), & \text{if } u = 1; \end{cases}$

(v) $(u, v) \mapsto (u + \omega, v)$;

(vi) $(u, v) \mapsto (\omega^2 u^2, v)$.

These automorphisms generate a group G_2. By looking at the first coordinates we see that G_2 has two orbits $\{\{0\} \times \mathbb{F}_4, \{\omega\} \times \mathbb{F}_4\}$ and $\{\{1\} \times \mathbb{F}_4, \{\omega^2\} \times \mathbb{F}_4\}$ on the set of generators.

We first show that the automorphism group $\Gamma(f_2)$ of $\mathcal{L}(f_2)$ cannot be transitive on the set of generators. Otherwise there is an automorphism γ that takes the generator $\{0\} \times \mathbb{F}_4$ to the generator $\{1\} \times \mathbb{F}_4$. Using the automorphism (vi), if necessary, we may assume that γ is of the form $(u, v) \mapsto (au + 1, \beta_u(v))$ for some $a \in \mathbb{F}_4$, $a \neq 0$, and permutations β_u of $\mathbb{F}_4$. From section 2 we see that the permutation $(u, v) \mapsto (au + 1, v)$ takes $\mathcal{L}(f_2)$ to $\mathcal{L}(f)$ where

$$f(x, y, z) = \begin{cases} (x^2 + x + z^2 + z)(y^2 + y) + x + y^2 + z, & \text{if } a = 1, \\ (x^2 + x + y^2 + y)(z^2 + z) + x + y + z^2, & \text{if } a = \omega, \\ (y^2 + y + z^2 + z)(x^2 + x) + x^2 + y + z, & \text{if } a = \omega^2. \end{cases}$$

But γ is a composition of this permutation and a permutation in Δ. Using Lemma 2.1 we now see that f_2 cannot be obtained in this way. This shows that $\Gamma(f_2)$ cannot be transitive on the set of generators. Hence, $\Gamma(f_2)$ has the same orbits as G_2 on the set of generators.

Let γ be an automorphism of $\mathcal{L}(f_2)$. Up to elements in G_2 we can assume that γ fixes each generator, i.e., $\gamma \in \Delta$. Then $\gamma(u, v) = (u, a_u v^{m_u} + b_u)$ where $a_u, b_u \in \mathbb{F}_4$, $a_u \neq 0$, and $m_u = 1, 2$. Using the automorphisms (ii), (iii) and (iv) in G_2, if necessary, we may further assume that $m_0 = m_1 = m_{\omega^2} = 1$. By Lemma 2.1 we then must have

$$\begin{aligned} &(a_1^2 x^2 + a_1 x + b_1^2 + b_1)(a_{\omega^2}^2 z^2 + a_{\omega^2} z + b_{\omega^2}^2 + b_{\omega^2}) \\ &\qquad + a_1 x + a_\omega y^{m_\omega} + a_{\omega^2} z + b_1 + b_\omega + b_{\omega^2} \\ &= a_0[(x^2 + x)(z^2 + z) + x + y + z] + b_0 \end{aligned}$$

for all $x, y, z \in \mathbb{F}_4$. As before in 3.3 we see that $m_\omega = 1$, $a_\lambda = 1$ for all $\lambda \in \mathbb{F}_4$, $b_1^2 + b_1 = b_{\omega^2}^2 + b_{\omega^2} = 0$, that is $b_1, b_{\omega^2} \in \mathbb{F}_2$ and $b_1 + b_\omega + b_{\omega^2} + b_0 = 0$. But then γ is of type (i) and thus in G_2.

This shows that the automorphism group of $\mathcal{L}(f_2)$ is contained in G_2. Furthermore, G_2 has order 2^9 since, from above, $|G_2/\Delta| = 4$, $|\Delta/\Delta^*| = 8$ and $|\Delta^*| = 16$. Using the automorphisms (i) we see that Δ is transitive on the generator $\{0\} \times \mathbb{F}_4$. The stabilizer $\Delta_{(0,0)}$ of $(0,0)$ has order 2^5 and is generated by the following automorphisms:

- $(u, v) \mapsto (u, v + t_u)$ for $t_1, t_\omega, t_{\omega^2} \in \mathbb{F}_2$, $t_0 = 0$, $t_1 + t_\omega + t_{\omega^2} = 0$;
- $(u, v) \mapsto (u, v^2)$;
- $(u, v) \mapsto \begin{cases} (u, v + \omega^2 u^2), & \text{if } u \in \mathbb{F}_4 \setminus \{\omega^2\}, \\ (u, v^2 + \omega^2 u^2), & \text{if } u = \omega^2; \end{cases}$
- $(u, v) \mapsto \begin{cases} (u, v + u), & \text{if } u \in \mathbb{F}_4 \setminus \{1\}, \\ (u, v^2 + u), & \text{if } u = 1. \end{cases}$

The first and third automorphisms then show that $\Delta_{(0,0)}$ is transitive on the generator $\{1\} \times \mathbb{F}_4$. Finally, the stabilizer $\Delta_{(0,0),(1,0)}$ of $(0,0)$ and $(1,0)$ has order 2^3 and is generated by $(u, v) \mapsto (u, v + s(u^2 + u))$ for $s \in \mathbb{F}_4$ and $(u, v) \mapsto (u, v^2)$. It now readily follows that $\Delta_{(0,0),(1,0)}$ is transitive on the generator $\{\omega\} \times \mathbb{F}_4$. Since each circle is uniquely determined by its intersection with the three generators $\{0\} \times \mathbb{F}_4$, $\{1\} \times \mathbb{F}_4$ and $\{\omega\} \times \mathbb{F}_4$, we see that Δ is transitive on the set of circles. In summary we obtain the following.

PROPOSITION 3.3 *The automorphism group $\Gamma(f_2)$ of the Laguerre near-plane $\mathcal{L}(f_2)$ has order 2^9. Furthermore, $\Gamma(f_2)$ is circle-transitive and induces a group of order 4 (the non-cyclic group $\mathbb{Z}_2 \times \mathbb{Z}_2$ where $\mathbb{Z}_2$ is the cyclic group of order 2) on the set of generators and has precisely two orbits of length two each on this set. In particular, $\Gamma(f_2)$ is not point-transitive.*

3.4 The automorphism group of $\mathcal{L}(f_3)$

From the list in section 2 we find that the following permutations are automorphisms of $\mathcal{L}(f_3)$.

(i) $(u, v) \mapsto (u, v + t_u)$ for $t_0, t_1, t_\omega, t_{\omega^2} \in \mathbb{F}_2$, $t_0 + t_1 + t_\omega + t_{\omega^2} = 0$;

(ii) $(u, v) \mapsto (u, v^2)$;

(iii) $(u, v) \mapsto (u + t, v)$ for all $t \in \mathbb{F}_4$;

(iv) $(u, v) \mapsto (ru, v)$ for $r \neq 0$;

(v) $(u, v) \mapsto (u^2, v)$.

These automorphisms generate a group G_3 of order $2^7 \cdot 3$. By looking at the first coordinates we see that every permutation of the set of generators can be obtained by an element of G_3.

Let γ be an automorphism of $\mathcal{L}(f_3)$. Up to elements in G_3 we can assume that γ fixes each generator, i.e., $\gamma \in \Delta$. Then $\gamma(u,v) = (u, a_u v^{m_u} + b_u)$ where $a_u, b_u \in \mathbb{F}_4$, $a_u \neq 0$, and $m_u = 1, 2$. Using the automorphism (ii) in G_3, if necessary, we may further assume that $m_0 = 1$. By Lemma 2.1 we then must have

$$\begin{aligned}
&(a_1^2 x^{2m_1} + a_1 x^{m_1} + b_1^2 + b_1)(a_\omega^2 y^{2m_\omega} + a_\omega y^{m_\omega} + b_\omega^2 + b_\omega) \\
&\quad \cdot (a_{\omega^2}^2 z^{2m_{\omega^2}} + a_{\omega^2} z^{m_{\omega^2}} + b_{\omega^2}^2 + b_{\omega^2}) \\
&\quad + a_1 x^{m_1} + a_\omega y^{m_\omega} + a_{\omega^2} z^{m_{\omega^2}} + b_1 + b_\omega + b_{\omega^2} \qquad (2)\\
&= a_0[(x^2+x)(y^2+y)(z^2+z) + x + y + z] + b_0
\end{aligned}$$

for all $x, y, z \in \mathbb{F}_4$. Looking at terms x^2 and x in (2) we find

$$(b_\omega^2 + b_\omega)(b_{\omega^2}^2 + b_{\omega^2})(a_1^2 x^{2m_1} + a_1 x^{m_1}) + a_1 x^{m_1} = a_0 x.$$

Since $b^2 + b \in \mathbb{F}_2$ for each $b \in \mathbb{F}_4$, we obtain two cases. Either $b_\omega^2 + b_\omega = b_{\omega+1}^2 + b_{\omega^2} = 1$ and then $m_1 = 2$, $a_1 = a_0^2$, or $(b_\omega^2 + b_\omega)(b_{\omega^2}^2 + b_{\omega^2}) = 0$, and then $m_1 = 1$, $a_1 = a_0$. In both cases we have $a_1 = a_0^{m_1}$ and $a_1^2 x^{2m_1} + a_1 x^{m_1} = a_0^2 x^2 + a_0 x$ as in 3.3. One similarly finds that $a_\lambda = a_0^{m_\lambda}$ for $\lambda = \omega, \omega^2$ and $a_\omega^2 y^{2m_\omega} + a_\omega y^{m_\omega} = a_0^2 y^2 + a_0 y$, $a_{\omega^2}^2 z^{2m_{\omega^2}} + a_{\omega^2} z^{m_{\omega^2}} = a_0^2 z^2 + a_0 z$. Comparing terms $x^2 y^2 z^2$ in (2) yields $a_0 = 1$ and thus $a_\lambda = 1$ for all $\lambda \in \mathbb{F}_4$.

Now (2) becomes

$$\begin{aligned}
&(x^2 + x + b_1^2 + b_1)(y^2 + y + b_\omega^2 + b_\omega)(z^2 + z + b_{\omega+1}^2 + b_{\omega^2}) \\
&\quad + x^{m_1} + y^{m_\omega} + z^{m_{\omega^2}} + b_0 + b_1 + b_\omega + b_{\omega^2} \\
&= (x^2+x)(y^2+y)(z^2+z) + x + y + z.
\end{aligned}$$

Expanding the left-hand side, we see that $b_1^2 + b_1 = b_\omega^2 + b_\omega = b_{\omega^2}^2 + b_{\omega^2} = 0$ and $b_0 + b_1 + b_\omega + b_{\omega^2} = 0$; in particular, $b_\lambda \in \mathbb{F}_2$ for all $\lambda \in \mathbb{F}_4$. Furthermore, $m_\lambda = 1$ for all $\lambda \in \mathbb{F}_4$. But this implies that γ is of the form (i), that is, $\gamma \in G_3$.

This shows that the automorphism group of $\mathcal{L}(f_3)$ is contained in G_3. It readily follows that each of the generators of G_3 in the list above maps $\mathbb{F}_4 \times \mathbb{F}_2$ to itself. In fact, G_3 has the two orbits $\mathbb{F}_4 \times \mathbb{F}_2$ and $\mathbb{F}_4 \times \{\omega, \omega^2\}$ in the point set. In particular, G_3 is neither point-transitive not circle transitive (a circle entirely contained in $\mathbb{F}_4 \times \mathbb{F}_2$ cannot be mapped to one having a point in the other point-orbit). In summary we obtain the following.

PROPOSITION 3.4 *The automorphism group $\Gamma(f_3)$ of the Laguerre near-plane $\mathcal{L}(f_3)$ has order $2^7 \cdot 3$. Furthermore, $\Gamma(f_3)$ is neither point- nor circle-transitive but induces the full symmetric group S_4 on the set of generators.*

3.5 The automorphism group of $\mathcal{L}(f_4)$

From the list in section 2 we find that the following permutations are automorphisms of $\mathcal{L}(f_4)$.

(i) $(u,v) \mapsto (u, v+t_u)$ for $t_0, t_1, t_\omega, t_{\omega^2} \in \mathbb{F}_4$, such that $t_0 = t_1 \in \{0, \omega^2\}$, and $t_\omega = t_{\omega^2} \in \{0, \omega\}$;

(ii) $$(u,v) \mapsto \begin{cases} (u, v+\omega^2), & \text{if } u \in \{0,\omega\}, \\ (u,v), & \text{if } u = 1, \\ (u, \omega v^2), & \text{if } u = \omega^2; \end{cases}$$

(iii) $$(u,v) \mapsto \begin{cases} (u, v+\omega^2), & \text{if } u \in \{0,\omega^2\}, \\ (u,v), & \text{if } u = 1, \\ (u, \omega v^2), & \text{if } u = \omega; \end{cases}$$

(iv) $$(u,v) \mapsto \begin{cases} (u, v+\omega), & \text{if } u \in \{0,\omega\}, \\ (u,v), & \text{if } u = \omega^2, \\ (u, \omega^2 v^2), & \text{if } u = 1; \end{cases}$$

(v) $$(u,v) \mapsto \begin{cases} (u, \omega y^2), & \text{if } u = 0, \\ (u, v+u), & \text{if } u \in \{1,\omega\}, \\ (u,v), & \text{if } u = \omega^2; \end{cases}$$

(vi) $(u,v) \mapsto (u^2, v)$.

These automorphisms generate a group G_4 of order 2^7.
Let γ be an automorphism of $\mathcal{L}(f_4)$. We first assume that γ fixes each generator, i.e., $\gamma \in \Delta$. Then $\gamma(u,v) = (u, a_u v^{m_u} + b_u)$ where $a_u, b_u \in \mathbb{F}_4$, $a_u \neq 0$, and $m_u = 1, 2$. Using the automorphisms (ii) to (v), if necessary, we may further assume that $m_0 = m_1 = m_\omega = m_{\omega^2} = 1$. By Lemma 2.1 we then obtain that

$$\begin{aligned} &(a_1^2x^2 + \omega^2 a_1 x + b_1^2 + \omega^2 b_1)(a_\omega^2 y^2 + \omega a_\omega y + b_\omega^2 + \omega b_\omega) \\ &\quad \cdot (a_{\omega^2}^2 z^2 + \omega a_{\omega^2} z + b_{\omega^2}^2 + \omega b_{\omega^2}) \\ &\quad + (a_1^2x^2 + \omega^2 a_1 x + b_1^2 + \omega^2 b_1)(a_\omega^2 y^2 + \omega^2 a_\omega y + b_\omega^2 + \omega^2 b_\omega) \\ &\quad + (a_1^2x^2 + \omega^2 a_1 x + b_1^2 + \omega^2 b_1)(a_{\omega^2}^2 z^2 + \omega^2 a_{\omega^2} z + b_{\omega^2}^2 + \omega^2 b_{\omega^2}) \\ &\quad + (a_\omega^2 y^2 + \omega a_\omega y + b_\omega^2 + \omega b_\omega)(a_{\omega^2}^2 z^2 + \omega a_{\omega^2} z + b_{\omega^2}^2 + \omega b_{\omega^2}) \\ &\quad + a_1 x + a_\omega y + a_{\omega^2} z + b_1 + b_\omega + b_{\omega^2} \\ &= a_0[(x^2+\omega^2 x)(y^2+\omega y)(z^2+\omega z) + (x^2+\omega^2 x)(y^2+\omega^2 y) \\ &\quad + (x^2+\omega^2 x)(z^2+\omega^2 z) + (y^2+\omega y)(z^2+\omega z) + x + y + z] + b_0 \end{aligned} \tag{3}$$

for all $x, y, z \in \mathbb{F}_4$. Comparing terms in which each of x, y and z occurs, i.e.,

$$\begin{aligned} &(a_1^2x^2 + \omega^2 a_1 x)(a_\omega^2 y^2 + \omega a_\omega y)(a_{\omega^2}^2 z^2 + \omega a_{\omega^2} z) \\ &= a_0(x^2+\omega^2 x)(y^2+\omega y)(z^2+\omega z), \end{aligned}$$

we obtain

$$a_0 = a_1 = a_\omega = a_{\omega^2} = 1.$$

Looking at terms involving both y and z but no x in (3) we find

$$b_1^2 + \omega^2 b_1 = 0, \text{ i.e., } b_1 \in \{0, \omega^2\}.$$

By looking at terms involving both y and x but no z and both x and z but no y, respectively, one similarly finds that

$$\begin{aligned} b_\omega^2 + \omega b_\omega = 0, &\quad \text{i.e.,} \quad b_\omega \in \{0, \omega\}, \\ b_{\omega^2}^2 + \omega b_{\omega^2} = 0, &\quad \text{i.e.,} \quad b_{\omega^2} \in \{0, \omega\}. \end{aligned}$$

Then (3) becomes

$$\begin{aligned} &(x^2 + \omega^2 x)(y^2 + \omega^2 y + b_\omega^2 + \omega^2 b_\omega) + (x^2 + \omega^2 x)(z^2 + \omega^2 z + b_{\omega^2}^2 + \omega^2 b_{\omega^2}) \\ &\qquad + b_1 + b_\omega + b_{\omega^2} \\ &= (x^2 + \omega^2 x)(y^2 + \omega^2 y) + (x^2 + \omega^2 x)(z^2 + \omega^2 z) + b_0 \end{aligned}$$

for all $x, y, z \in \mathbb{F}_4$.

Clearly, we see that $b_1 + b_\omega + b_{\omega^2} + b_0 = 0$, and by looking at the term x^2 one obtains

$$(b_\omega + b_{\omega^2})^2 + \omega^2(b_\omega + b_{\omega^2}) = 0, \text{ i.e., } b_\omega + b_{\omega^2} \in \{0, \omega^2\}.$$

Then

$$b_\omega = b_{\omega^2} \in \{0, \omega\}$$

and hence

$$b_0 = b_1 \in \{0, \omega^2\}.$$

But now γ is of the form (i) and thus belongs to G_4.

Using the automorphisms (i) and (v) we see that Δ is transitive on the generator $\{1\} \times \mathbb{F}_4$. The automorphisms (iii) and (iv) then show that $\Delta_{(1,0)}$ is transitive on the generator $\{0\} \times \mathbb{F}_4$.

The stabilizer $\Delta_{(1,0),(0,0)}$ of $(1,0)$ and $(0,0)$ contains the following automorphisms:

- $(u, v) \mapsto (u, v + \omega(u^2 + u))$;
- $(u, v) \mapsto \begin{cases} (u, v), & \text{if } u \in \mathbb{F}_2, \\ (u, \omega v^2 + \omega^2), & \text{if } u \in \mathbb{F}_4 \setminus \mathbb{F}_2. \end{cases}$

  (This is the composition of the involutory commuting automorphisms (ii) and (iii).)

It now readily follows that $\Delta_{(0,0),(1,0)}$ is transitive on the generator $\{\omega\} \times \mathbb{F}_4$. Since each circle is uniquely determined by its intersection with the three generators $\{0\} \times \mathbb{F}_4$, $\{1\} \times \mathbb{F}_4$ and $\{\omega\} \times \mathbb{F}_4$, we see that Δ is transitive on the set of circles.

We finally show that every automorphism of $\mathcal{L}(f_4)$ fixes the generators $\{0\} \times \mathbb{F}_4$ and $\{1\} \times \mathbb{F}_4$. Using the transitivity of Δ on the circle set and automorphism (vi), if necessary, we may assume that we have an automorphism γ of the form $(u, v) \mapsto (su + t, a_u v^{m_u})$ where $s, a_u \in \mathbb{F}_4 \setminus \{0\}$ and $m_u \in \{1, 2\}$ for $u \in \mathbb{F}_4$. We now write γ as the composition of the permutations

$$\begin{aligned}\gamma_1 : \quad (u, v) &\mapsto \quad (u + t, v),\\ \gamma_2 : \quad (u, v) &\mapsto \quad (su, v),\\ \gamma_3 : \quad (u, v) &\mapsto \quad (u, a_u v^{m_u}),\end{aligned}$$

as $\gamma = \gamma_1 \circ \gamma_2 \circ \gamma_3$. Then γ_3 takes $\mathcal{L}(f_4)$ to the same Laguerre near-plane as $(\gamma_1 \circ \gamma_2)^{-1} = \gamma_2^{-1} \circ \gamma_1$.
For example, if $t = 1$ and $s = \omega^2$, and using [2], Corollary 2.6, and (1), (2) and (3) from section 2 we find that

$$a_0 f_4(a_1^{2m_1} x^{m_1}, a_\omega^{2m_\omega} y^{m_\omega}, a_{\omega^2}^{2m_{\omega^2}} z^{m_{\omega^2}})^{m_0} = g(y, z, x)$$

for all $x, y, z \in \mathbb{F}_4$, where g is the inverse of the partial map f_4 with respect to x, that is,

$$\begin{aligned} g(x, y, z) \quad = \quad & (x^2 + \omega^2 x)(y^2 + \omega y)(z^2 + \omega z) + (x^2 + \omega^2 x)(y^2 + \omega^2 y) \\ & + (x^2 + \omega^2 x)(z^2 + \omega^2 z) + \omega^2 (y^2 + \omega y)(z^2 + \omega z) + x + \omega y^2 + \omega z^2 \end{aligned}$$

Explicitly one obtains

$$\begin{aligned} & a_0[(a_1^{n_1} x^{2n_1} + \omega^2 a_1^{2n_1} x^{n_1})(a_\omega^{n_\omega} y^{2n_\omega} + \omega a_\omega^{2n_\omega} y^{n_\omega})(a_{\omega^2}^{n_{\omega^2}} z^{2n_{\omega^2}} + \omega a_{\omega^2}^{2n_{\omega^2}} z^{n_{\omega^2}}) \\ & \quad + (a_1^{n_1} x^{2n_1} + \omega^2 a_1^{2n_1} x^{n_1})(a_\omega^{n_\omega} y^{2n_\omega} + \omega^2 a_\omega^{2n_\omega} y^{n_\omega}) \\ & \quad + (a_1^{n_1} x^{2n_1} + \omega^2 a_1^{2n_1} x^{n_1})(a_{\omega^2}^{n_{\omega^2}} z^{2n_{\omega^2}} + \omega^2 a_{\omega^2}^{2n_{\omega^2}} z^{n_{\omega^2}}) \\ & \quad + (a_\omega^{n_\omega} y^{2n_\omega} + \omega a_\omega^{2n_\omega} y^{n_\omega})(a_{\omega^2}^{n_{\omega^2}} z^{2n_{\omega^2}} + \omega a_{\omega^2}^{2n_{\omega^2}} z^{n_{\omega^2}}) \\ & \quad + a_1^{2n_1} x^{n_1} + a_\omega^{2n_\omega} y^{n_\omega} + a_{\omega^2}^{2n_{\omega^2}} z^{n_{\omega^2}}] \\ & = (y^2 + \omega^2 y)(z^2 + \omega z)(x^2 + \omega x) + (y^2 + \omega^2 y)(z^2 + \omega^2 z) \\ & \quad + (y^2 + \omega^2 y)(x^2 + \omega^2 x) + \omega^2 (z^2 + \omega z)(x^2 + \omega x) + y + \omega z^2 + \omega x^2 \end{aligned} \tag{4}$$

for all $x, y, z \in \mathbb{F}_4$ where $n_\lambda = m_0 m_\lambda$ for $\lambda = 1, \omega, \omega^2$. Comparing terms x^2, y and z^2 in (4) we find that

$$n_\omega = 1, \; n_1 = n_{\omega^2} = 2,$$

$$a_\omega = a_0, \; a_1 = a_{\omega^2} = \omega a_0^2.$$

But then the coefficient of $x^2 y^2 z^2$ on the left-hand side in (4) becomes ω^2 whereas on the right-hand side it is 1 - a contradiction. This shows that $t = 1$, $s = \omega^2$ is not possible. Similar arguments yield that all the other combinations for s and t except $s = 1$, $t = 0$ are not possible. From the list of automorphisms at the beginning of this section we see that only the last automorphism moves some generators. In particular, the generators $\{0\} \times \mathbb{F}_4$ and $\{1\} \times \mathbb{F}_4$ are fixed and $\{\omega\} \times \mathbb{F}_4$ and $\{\omega^2\} \times \mathbb{F}_4$ can be interchanged. In summary we obtain the following.

PROPOSITION 3.5 *The automorphism group* $\Gamma(f_4)$ *of the Laguerre near-plane* $\mathcal{L}(f_4)$ *has order* 2^7*. Furthermore,* $\Gamma(f_4)$ *is circle-transitive and induces on the set of generators a group of order* 2 *that fixes two generators.*

Looking at the orders of the automorphism groups or their transitivity properties we obtain the following.

COROLLARY 3.6 *The Laguerre near-planes* $\mathcal{L}(f_i)$*,* $i = 0, 1, 2, 3, 4$*, are mutually non-isomorphic.*

References

[1] R.C. Bose and K.A. Bush, Orthogonal arrays of strength two and three, *Ann. Math. Statistics* **23** (1952), 508–524.

[2] G.F. Steinke, A classification of Laguerre near-planes of order four, *Australas. J. Combin.* **25** (2002), 145–166.

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