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Recursive constructions for large sets of resolvable hybrid triple systems

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Abstract

An LRHTS(v) is a large set consisting of $4(v-2)$ disjoint resolvable hybrid triple systems of order v. In this paper, we present recursive constructions for LRHTSs and show some new existence results.

1 Introduction

Let X be a finite set. A *cyclic triple* on X is a set of three ordered pairs (x,y), (y,z) and (z,x) of X, which is denoted by $\langle x,y,z\rangle$ (or $\langle y,z,x\rangle$, or $\langle z,x,y\rangle$). A *transitive triple* on X is a set of three ordered pairs (x,y), (y,z) and (x,z) of X, which is denoted by (x,y,z).

An *oriented triple system* of order v is a pair $(X,\mathcal{B})$ where X is a v-set and $\mathcal{B}$ is a collection of cyclic or transitive triples on X, called *blocks*, such that every ordered pair of X belongs to exactly one block of $\mathcal{B}$. There are three types of oriented triple systems of order v, i.e., *Mendelsohn triple system* of order v (briefly, MTS(v)) in which the blocks are all cyclic triples, *directed triple system* of order v (briefly, DTS(v)) in which the blocks are all transitive triples, and *hybrid triple system* of order v (briefly, HTS(v)) in which the blocks are permitted to be either.

Most problems on hybrid triple systems can be settled by considering only cyclic or transitive triples. However, in what follows, the hybrid triple systems are always required to contain both cyclic triples and transitive triples.

An oriented triple system $(X,\mathcal{B})$ is called *resolvable* if its block set $\mathcal{B}$ can be partitioned into subsets (called *parallel class*es), each containing every element of X exactly once. A resolvable MTS(v) (or DTS(v), or HTS(v)), denoted by RMTS(v) (or RDTS(v), or RHTS(v)), is easily checked to contain $v-1$ parallel classes.

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A *large set* of MTS(v) (or DTS(v), or HTS(v)), denoted by LMTS(v) (or LDTS(v), or LHTS(v)), is a collection $\{(X, \mathcal{B}_i)\}$, where every $(X, \mathcal{B}_i)$ is an MTS(v) (or a DTS(v), or an HTS(v)) and all $\mathcal{B}_i$'s form a partition of all cyclic triples (or all transitive triples, or all cyclic triples and all transitive triples) on X. An LRMTS(v) (or LRDTS(v), or LRHTS(v)) denotes an LMTS(v) (or LDTS(v), or LHTS(v)) in which each MTS(v) (or DTS(v), or HTS(v)) is resolvable.

In [7, 10], it is conjectured that every MTS(v) supports three disjoint DTS(v)s and four disjoint HTS(v)s. But in the absence of a proof, we may consider the three types of large sets separately. What's more, the large set of HTSs is itself interesting since the number of small systems in an LHTS(v) is $4(v-2)$ (as expected, the sum of the small systems' number $v-2$ in an LMTS(v) and $3(v-2)$ in an LDTS(v)). In 1996, Kang and Lei gave the existence spectrum of LHTS(v):

Theorem 1.1 ([6]) *There exists an LHTS(v) if and only if v is a positive integer such that $v > 3$ and $v \equiv 0,\ 1 \pmod 3$.*

In this paper, we focus on the construction for LRHTSs. Obviously, we can employ LKTS(v)s to generate LRHTS(v)s. So by the latest known LKTSs listed in [3], we have the preliminary result on LRHTSs.

Theorem 1.2 *There exists an LRHTS($3^a 5^b m \Pi_{i=1}^{r}(2 \cdot 13^{n_i}+1)\Pi_{j=1}^{p}(2 \cdot 7^{m_j}+1)$) for $m \in M = \{1,\ 11,\ 17,\ 35,\ 43,\ 67,\ 91,\ 123\} \cup \{2^{2l+1}25^s + 1 : l,\ s \geq 0\}$, a, n_i, $m_j \geq 1$ ($1 \leq i \leq r$, $1 \leq j \leq p$), $b, r, p \geq 2$ when $b \geq 1$ and $m \neq 1$.*

In [11], an LRMTS(12) is constructed and the three cyclic shifts form an LRDTS(12), then by slight changes we can obtain an LRHTS(12). The specific blocks are listed in appendix.

Theorem 1.3 *There is an LRHTS(12).*

2 Preparations for construction

A *quasigroup* of order v is a pair $(X, \circ)$, where X is a v-set and $(\circ)$ is a binary operation on X such that equations $a \circ x = b$ and $y \circ a = b$ are uniquely solvable for every pair of elements a, b in X. A quasigroup $(X, \circ)$ is called *idempotent* if the identity $x \circ x = x$ holds for all x in X. An idempotent quasigroup of order v is denoted by IQ(v). Furthermore, an idempotent quasigroup $(X, \circ)$ is called *resolvable* if all $v(v-1)$ pairs of distinct elements of X can be partitioned into subsets T_i, $1 \leq i \leq 3(v-1)$, such that every $\Gamma_i = \{(x, y, x \circ y) : (x, y) \in T_i\}$ (called *parallel class*) is a partition of X. A resolvable idempotent quasigroup of order v is denoted by RIQ(v).

An IQ(v) is called *first-transitive*, if there exists a group G of order v acting transitively on X which forms an automorphism group of $(X, \circ)$. A first-transitive RIQ(v) is denoted by TRIQ(v).

Take any fixed ordered pair (i,j) $(i \neq j)$. For an IQ $(X,\circ)$ and the given ordered pair (i,j), define a set $T^X(i,j)$ of transitive triples of $X \times \{i,j\}$ as follows: for each ordered pair (x,y), $x \neq y \in X$, let $t(x,y,x\circ y)$ be the three transitive triples of $X \times \{i,j\}$ defined by

$$t(x,y,x\circ y) = \{((x,i),(y,i),(x\circ y,j)),((x,i),(x\circ y,j),(y,i)),((x\circ y,j),(x,i),(y,i))\}. \tag{2.1}$$

Set

$$T^X(i,j) = \bigcup_{x\neq y\in X} t(x,y,x\circ y). \tag{2.2}$$

The IQ $(X,\circ)$ is called *second-transitive* provided that $T^X(i,j)$ can be partitioned into three sets $T_1^X(i,j)$, $T_2^X(i,j)$ and $T_3^X(i,j)$ such that

(a) the three transitive triples in $t(x,y,x\circ y)$ belong to different $T_k^X(i,j)$s ($k=1$, 2, 3);

(b) if $a \neq b \in X$, each of the ordered pairs $((a,i),(b,j))$ and $((b,j),(a,i))$ belongs to exactly one transitive triple in each of $T_1^X(i,j)$, $T_2^X(i,j)$ and $T_3^X(i,j)$.

An IQ(v) $(X,\circ)$ with both first- and second-transitivity is called *doubly transitive.* A doubly transitive RIQ(v) is denoted by DTRIQ(v).

The existence of DTRIQ(v) is known as follows.

Theorem 2.1 ([14]) *A DTRIQ(v) exists if and only if v is a positive integer such that $3|v$ and $v \not\equiv 2 \pmod 4$.*

Another important concept is LR-design, which is introduced by Lei in [8].

Let X be a v-set. An LR-design of order v (briefly LR(v)) is a collection $\{(X,\mathcal{A}_k^j) : 1 \leq k \leq \frac{v-1}{2}, j=0,1\}$ of $v-1$ KTS(v)s with following properties:

(i) Let the resolution of $\mathcal{A}_k^j$ be $\Gamma_k^j = \{A_k^j(h) : 1 \leq h \leq \frac{v-1}{2}\}$. There is an element in each Γ_k^j, say, $A_k^j(1)$, such that

$$\bigcup_{k=1}^{\frac{v-1}{2}} A_k^0(1) = \bigcup_{k=1}^{\frac{v-1}{2}} A_k^1(1) = \mathcal{A}$$

and $(X,\mathcal{A})$ is a KTS(v).

(ii) For any triple $T=\{x,y,z\} \subseteq X$, $x \neq y \neq z \neq x$, there exist k, j such that $T \in \mathcal{A}_k^j$.

The known existence results on LR-design are as follows.

Theorem 2.2 ([3]) *There exists an LR$((3^a5^bm\Pi_{i=1}^r(2\cdot 13^{n_i}+1)\Pi_{j=1}^p(2\cdot 7^{m_j}+1))$ for n_i, $m_j \geq 1$ $(1\leq i\leq r,\ 1\leq j\leq p)$, a, b, r, $p \geq 0$ with $a+r+p\leq 1$.*

With the aid of the above mentioned structures, we obtained tripling construction and product construction for LRMTSs and LRDTSs .

Theorem 2.3 ([1, 14]) *If there exist both an LRMTS(v) (or LRDTS(v)) and a TRIQ(v) (or DTRIQ(v)), then there exists an LRMTS($3v$) (or LRDTS($3v$)).*

Theorem 2.4 [15]) *If there exist an LRMTS(u) (or LRDTS(u)) and a TRIQ(u) (or DTRIQ(u)), and there exists an LR(v), then there exists an LRMTS(uv) (or LRDTS(uv)).*

In Section 3, we present similar constructions for LRHTS, in which the concept of complete mapping is needed.

A *complete mapping* of a group $(G, \cdot)$ is a bijection mapping $x \to \theta(x)$ of G upon G such that the mapping $\eta(x) = x \cdot \theta(x)$ is again a bijection mapping of G upon G.

Lemma 2.5 ([1, Lemma 2.7]) *If there exists an IQ(v) $(X, \circ)$ with a sharply transitive automorphism group G, then G has a complete mapping.*

Remark 2.6 *Suppose that X_1 is a u-set and $(X_1, \circ)$ is a DTRIQ(u). By the definition of DTRIQ, we have:*

(A) *There is a sharply transitive automorphism group $G = \{\sigma_0, \sigma_1, \cdots, \sigma_{u-1}\}$ on $(X_1, \circ)$. By Lemma 2.5, G has a complete mapping, say, ϕ, and let $\sigma^* = [\phi(\sigma)]^{-1}$ for $\sigma \in G$. Then by the definition of complete mapping, we have*

$$\{\sigma(\sigma^*)^{-1} : \sigma \in G\} = G. \tag{2.3}$$

(B) *All $u(u-1)$ pairs of distinct elements of X_1 can be partitioned into subsets S_i $(1 \leq i \leq 3(u-1))$, such that every $\Gamma_i = \{(x, y, x \circ y) : (x, y) \in S_i\}$ is a partition of X_1.*

(C) *For any fixed ordered pair (i, j) $(i \neq j)$, $T^{X_1}(i, j) = \bigcup_{x \neq y \in X_1} t(x, y, x \circ y)$, where $t(x, y, x \circ y)$ is defined in (2.1). $T^{X_1}(i, j)$ can be partitioned into 3 sets $T_1^{X_1}(i, j)$, $T_2^{X_1}(i, j)$ and $T_3^{X_1}(i, j)$ satisfying:*

   (a) *the three transitive triples in $t(x, y, x \circ y)$ belong to different $T_l^{X_1}(i, j)$s $(l = 1, 2, 3)$;*

   (b) *if $a \neq b \in X_1$, each of the ordered pairs $((a, i), (b, j))$ and $((b, j), (a, i))$ belongs to exactly one transitive triple in each of $T_1^{X_1}(i, j)$, $T_2^{X_1}(i, j)$ and $T_3^{X_1}(i, j)$.*

Furthermore, suppose that X_2 is a v-set with a linear order "$<$" (i.e., for any $x \neq y$, $x, y \in X_2$, either $x < y$ or $y < x$). And suppose that $\{(X_2, \mathcal{A}_k^j) : 1 \leq k \leq \frac{v-1}{2}, j = 0, 1\}$ is an LR(v) satisfying condition (D):

(D) (i) *Let the resolution of $\mathcal{A}_k^j$ be $\Gamma_k^j = \{A_k^j(h) : 1 \leq h \leq \frac{v-1}{2}\}$. There is an element in each Γ_k^j, say, $A_k^j(1)$, such that*

$$\bigcup_{k=1}^{\frac{v-1}{2}} A_k^0(1) = \bigcup_{k=1}^{\frac{v-1}{2}} A_k^1(1) = \mathcal{A}$$

and $(X_2, \mathcal{A})$ is a KTS(v).

(ii) *For any triple* $T = \{x, y, z\} \subseteq X_2$, $x \neq y \neq z \neq x$, *there exist* k, j *such that* $T \in \mathcal{A}_k^j$.

The symbols and properties in Remark 2.6 will be used in the proof of Theorem 3.1. In addition, we stipulate some notations for the use in the following proof.

In (2.2), we give a symbol $T^X(i,j)$. For the convenient proof of the following theorem, we introduce an analogous symbol $C^X(i,j)$ in an IQ $(X, \circ)$. For a fixed ordered pair (i,j), define

$$C^X(i,j) = \bigcup_{x \neq y \in X} \{\langle (x,i), (y,i), (x \circ y, j) \rangle\}.$$

Moreover, if π is a permutation of X, we denote by $\pi C^X(i,j)$ (or $\pi T_l^X(i,j)$, $1 \leq l \leq 3$) the set of the cyclic (or transitive) triples in $C^X(i,j)$ (resp., $T_l^X(i,j)$, $1 \leq l \leq 3$) by replacing each occurrence of (x,j) with $(\pi(x), j)$ but keeping those occurrences with the second component "i" unchanged, say, $\pi C^X(i,j) = \bigcup_{x \neq y \in X} \{\langle (x,i), (y,i), (\pi(x \circ y), j) \rangle\}$.

3 Recursive construction for LRHTSs

Theorem 3.1 (Product Construction) *If there exist an LRHTS*(u) *and a DTRIQ*(u)*, and there exists an LR*(v)*, then there exists an LRHTS*(uv)*.*

Proof Suppose that $(X_1, \circ)$ is the DTRIQ(u) in Remark 2.6. Let $\{(X_2, \mathcal{A}_k^j) : 1 \leq k \leq \frac{v-1}{2}, j = 0, 1\}$ be the LR(v) satisfying condition (D). Let $\{(X_1, \mathcal{B}_j) : 1 \leq j \leq 4(u-2)\}$ be an LRHTS(u). We will construct an LRHTS(uv) on the set $Y = X_1 \times X_2$. The construction proceeds in 3 steps.

Step 1: For any $\{a, b, c\} \subseteq X_2$ with $a < b < c$, for $\sigma_i, \sigma_j \in G$ and $x \in X_1$, define

$$B_{ijx}^{(a,b,c)} = \{\{(x,a), (\sigma_j(x), b), (\sigma_i \sigma_j^*(x), c)\}\}.$$

We can take three fixed elements $x_1, x_2, x_3 \in X_1$ such that $x_1 \neq x_2 \neq x_3 \neq x_1$ and define

$$\begin{aligned}
P_{0ij}^{(a,b,c)} = \{&((x_1,a), (\sigma_j(x_1), b), (\sigma_i\sigma_j^*(x_1), c)), ((\sigma_i\sigma_j^*(x_1), c), (\sigma_j(x_1), b), (x_1, a)),\\
&((x_2,a), (\sigma_j(x_2), b), (\sigma_i\sigma_j^*(x_2), c)), ((\sigma_i\sigma_j^*(x_2), c), (\sigma_j(x_2), b), (x_2, a)),\\
&((x_3,a), (\sigma_j(x_3), b), (\sigma_i\sigma_j^*(x_3), c)), ((\sigma_i\sigma_j^*(x_3), c), (\sigma_j(x_3), b), (x_3, a))\}\\
&\cup (\cup_{\substack{x \in X_1 \\ x \neq x_1, x_2, x_3}} \{\langle u, v, w \rangle, \langle w, v, u \rangle : \{u, v, w\} \in B_{ijx}^{(a,b,c)}\}),\\
P_{1ij}^{(a,b,c)} = \{&\langle (x_1,a), (\sigma_j(x_1), b), (\sigma_i\sigma_j^*(x_1), c) \rangle, \langle (\sigma_i\sigma_j^*(x_1), c), (\sigma_j(x_1), b), (x_1, a) \rangle\}\\
&\cup (\cup_{\substack{x \in X_1 \\ x \neq x_1}} \{(u, v, w), (w, v, u) : \{u, v, w\} \in B_{ijx}^{(a,b,c)}\}),\\
P_{2ij}^{(a,b,c)} = \{&\langle (x_2,a), (\sigma_j(x_2), b), (\sigma_i\sigma_j^*(x_2), c) \rangle, \langle (\sigma_i\sigma_j^*(x_2), c), (\sigma_j(x_2), b), (x_2, a) \rangle\}
\end{aligned}$$

$$\cup(\cup_{\substack{x\in X_1\\ x\neq x_2}}\{(u,w,v),(v,w,u):\{u,v,w\}\in B_{ijx}^{(a,b,c)}\}),$$
$$P_{3ij}^{(a,b,c)}=\{\langle(x_3,a),(\sigma_j(x_3),b),(\sigma_i\sigma_j^*(x_3),c)\rangle,\langle(\sigma_i\sigma_j^*(x_3),c),(\sigma_j(x_3),b),(x_3,a)\rangle\}$$
$$\cup(\cup_{\substack{x\in X_1\\ x\neq x_3}}\{(w,u,v),(v,u,w):\{u,v,w\}\in B_{ijx}^{(a,b,c)}\}).$$

Then set

$$\mathcal{A}_{li}^{(a,b,c)}=\bigcup_{\sigma_j\in G}P_{lij}^{(a,b,c)},(0\le l\le 3).$$

For each $a'\in X_2$, we have $4(u-2)$ disjoint RHTS(u)s $(X_1\times\{a'\},\ \mathcal{B}_j^{(a')})$ for $j=1,2,\cdots,\ 4(u-2)$, where $\mathcal{B}_j^{(a')}=\{\langle(x,a'),(y,a'),(z,a')\rangle:\langle x,y,z\rangle\in\mathcal{B}_j\}\cup\{((x',a'),(y',a'),(z',a')):(x',y',z')\in\mathcal{B}_j\}$.

For given l and j, $0\le l\le 3$ and $1\le j\le u-2$, take $\{a,b,c\}\in\mathcal{A}$ and $a<b<c$, define

$$\mathcal{C}_{lj}=(\bigcup_{\{a,b,c\}\in\mathcal{A}}\mathcal{A}_{lj}^{(a,b,c)})\bigcup(\bigcup_{a'\in X_2}\mathcal{B}_{4j-3+l}^{(a')}).$$

Then it is not difficult to check that each $(Y,\mathcal{C}_{lj})$ is an HTS(uv) for $0\le l\le 3,\ 1\le j\le u-2$.

$(Y,\mathcal{C}_{lj})$ is resolvable because $\mathcal{C}_{lj}$ is the union of the $uv-1$ parallel classes in the following 2 parts.

Part I: For given i and k, $0\le i\le u-1$ and $1\le k\le\frac{v-1}{2}$, $\bigcup_{\{a,b,c\}\in A_k^0(1)}P_{lij}^{(a,b,c)}$ consists of 2 parallel classes. So this part gives $u(v-1)$ parallel classes.

Part II: $\bigcup_{a'\in X_2}\mathcal{B}_{4j-3+l}^{(a')}$ can be partitioned into $u-1$ parallel classes because of the resolvability of $\mathcal{B}_j$.

This step gives $4(u-2)$ disjoint RHTS(uv)s on Y.

(The remaining $\mathcal{A}_{l0}^{(a,b,c)}$ and $\mathcal{A}_{l,u-1}^{(a,b,c)}$ ($\{a,b,c\}\in\mathcal{A},\ a<b<c$) are saved for the use in the following two steps. Note that each $\mathcal{A}_{li}^{(a,b,c)}$ ($0\le l\le 3,\ 0\le i\le u-1$) contains both cyclic triples and transitive triples.)

Step 2: (making use of the block set $\mathcal{A}_{l0}^{(a,b,c)}$)

For a given $\sigma_j\in G$, $j=0,\ 1,\ \cdots,\ u-1$, define 3 permutations on X_1, namely $\alpha_j^{(s)}(s\in Z_3)$ as follows:

$$\alpha_j^{(0)}=\sigma_j,\qquad \alpha_j^{(1)}=\sigma_0\sigma_j^*\sigma_j^{-1},\qquad \alpha_j^{(2)}=(\sigma_0\sigma_j^*)^{-1}=(\alpha_j^{(1)}\alpha_j^{(0)})^{-1}.$$

For given k and j, $1\le k\le\frac{v-1}{2}$ and $0\le j\le u-1$, take $\{a,b,c\}\in A_k^0(1)$, $a<b<c$. Define

$$\mathcal{C}_{00j}^{(a,b,c)}=\alpha_j^{(0)}C^{X_1}(a,b)\cup\alpha_j^{(1)}C^{X_1}(b,c)\cup\alpha_j^{(2)}C^{X_1}(c,a),$$

$$\mathcal{C}_{l0j}^{(a,b,c)}=\alpha_j^{(0)}T_l^{X_1}(a,b)\cup\alpha_j^{(1)}T_l^{X_1}(b,c)\cup\alpha_j^{(2)}T_l^{X_1}(c,a),\ 1\le l\le 3,$$

and

$$\mathcal{D}_{lkj}^{(0)} = \Big[\bigcup_{\{a,b,c\}\in A_k^0(1)} (P_{l0j}^{(a,b,c)} \cup \mathcal{C}_{l0j}^{(a,b,c)})\Big] \bigcup \Big[\bigcup_{\{a,b,c\}\in \mathcal{A}_k^0\setminus A_k^0(1)} \mathcal{A}_{lj}^{(a,b,c)}\Big],\ 0 \le l \le 3.$$

Then it can be checked that each $(Y, \mathcal{D}_{lkj}^{(0)})$ is an RHTS(uv) for $1 \le k \le \frac{v-1}{2}$, $0 \le j \le u-1$ and $0 \le l \le 3$. Now we explain its parallel classes in 2 parts:

Part I: $\bigcup_{\{a,b,c\}\in A_k^0(1)} P_{l0j}^{(a,b,c)}$ consists of two parallel classes.

By the property (B) in Remark 2.6, for a given i, $1 \le i \le 3(u-1)$, $\Gamma_i = \{(x, y, x\circ y) : (x, y) \in S_i\}$ is a partition of X_1. Define $\pi(S_i) = \{(\pi(x), \pi(y)) : (x, y) \in S_i\}$ for some $\pi \in G$. Since $\alpha_j^{(s)} \in G$ $(s \in Z_3)$, $\alpha_j^{(2)} = (\alpha_j^{(1)}\alpha_j^{(0)})^{-1}$ and $A_k^0(1)$ is a parallel class of X_2, we can conclude that

$$\bigcup_{\{a,b,c\}\in A_k^0(1)} (\alpha_j^{(0)} C^{S_i}(a, b) \cup \alpha_j^{(1)} C^{\alpha_j^{(0)}(S_i)}(b, c) \cup \alpha_j^{(2)} C^{\alpha_j^{(1)}\alpha_j^{(0)}(S_i)}(c, a))$$

is a partition of Y, where $C^R(e, f) = \bigcup_{(x,y)\in R}\{\langle (x, e), (y, e), (x\circ y, f)\rangle\}$ for $R \in \{S_i, \alpha_j^{(0)}(S_i), \alpha_j^{(1)}\alpha_j^{(0)}(S_i)\}$ and $(e, f) \in \{(a, b), (b, c), (c, a)\}$. Note that

$$C^X(e, f) = \bigcup_{i=1}^{3(u-1)} C^{S_i}(e, f) = \bigcup_{i=1}^{3(u-1)} C^{\alpha_j^{(0)}(S_i)}(e, f) = \bigcup_{i=1}^{3(u-1)} C^{\alpha_j^{(1)}\alpha_j^{(0)}(S_i)}(e, f).$$

It is easy to see that $\cup_{\{a,b,c\}\in A_k^0(1)}\mathcal{C}_{0j}^{(a,b,c)}$ can be partitioned into $3(u-1)$ parallel classes.

Moreover, An obvious observation that $\mathcal{C}_{l0j}^{(a,b,c)}$ is just same with $\mathcal{C}_{00j}^{(a,b,c)}$ if we disregard the orientation of the triples, yields that each $\mathcal{C}_{l0j}^{(a,b,c)}$ for $1 \le l \le 3$ also consists of $3(u-1)$ parallel classes.

We have $3(u-1)+2$ parallel classes in this part.

Part II: For given m and i, $2 \le m \le \frac{v-1}{2}$ and $0 \le i \le u-1$, $\bigcup_{\{a,b,c\}\in A_k^0(m)} P_{lji}^{(a,b,c)}$ provides 2 parallel classes. So, we get $u(v-3)$ parallel classes in this part.

Obviously, $\mathcal{D}_{lkj}^{(0)}$ is the union of all the $uv-1$ parallel classes in Part I and II.

By formula (2.3), we have $\{\alpha_j^{(s)} : 0 \le j \le u-1\} = G$ $(s \in Z_3)$. With this fact, we can check that these $2u(v-1)$ RHTS(uv)s are pairwise disjoint and they are obviously disjoint with those obtained in Step 1.

Step 3. (making use of the block set $\mathcal{A}_{l,u-1}^{(a,b,c)}$)

For a given $\sigma_j \in G$, $j = 0, 1, \cdots, u-1$, define 3 permutations on X_1, namely $\beta_j^{(s)} (s \in Z_3)$ as follows:

$$\beta_j^{(0)} = \sigma_{u-1}\sigma_j^*, \qquad \beta_j^{(1)} = \sigma_j(\sigma_{u-1}\sigma_j^*)^{-1}, \qquad \beta_j^{(2)} = (\sigma_j)^{-1} = (\beta_j^{(1)}\beta_j^{(0)})^{-1}.$$

For given k and j, $1 \le k \le \frac{v-1}{2}$ and $0 \le j \le u-1$, take $\{a, b, c\} \in A_k^1(1)$, $a < b < c$. Define

$$\mathcal{C}^{(a,b,c)}_{0,u-1,j} = \beta^{(0)}_j C^{X_1}(a,c) \cup \beta^{(1)}_j C^{X_1}(c,b) \cup \beta^{(2)}_j C^{X_1}(b,a),$$
$$\mathcal{C}^{(a,b,c)}_{l,u-1,j} = \beta^{(0)}_j T^{X_1}_l(a,b) \cup \beta^{(1)}_j T^{X_1}_l(b,c) \cup \beta^{(2)}_j T^{X_1}_l(c,a),\ 1 \le l \le 3,$$

and

$$\mathcal{D}^{(u-1)}_{lkj} = \Big[\bigcup_{\{a,b,c\} \in A^0_k(1)} (P^{(a,b,c)}_{l,u-1,j} \cup \mathcal{C}^{(a,b,c)}_{l,u-1,j}) \Big] \bigcup \Big[\bigcup_{\{a,b,c\} \in \mathcal{A}^0_k \setminus A^0_k(1)} \mathcal{A}^{(a,b,c)}_{lj} \Big],\ 0 \le l \le 3.$$

The similar arguments as in Step 2 give $2u(v-1)$ RHTS(uv)s $(Y, \mathcal{D}^{(u-1)}_{lkj})$ for $0 \le l \le 3$, $1 \le k \le \frac{v-1}{2}$ and $0 \le j \le u-1$. Furthermore, these RHTS(uv)s are disjoint and also disjoint with those obtained in Step 1 and 2.

We obtain a total of $4(uv-2)$ disjoint RHTS(uv)s, a large set. This completes the proof. □

Note: There is an $LR(3)$ by Theorem 2.2. Take $LR(3)$ in Theorem 3.1, then we can obtain the tripling construction as follows.

Theorem 3.2 (Tripling Construction) *If there exist an LRHTS(v) and a DTRIQ(v), then there exists an LRHTS$(3v)$.*

4 Existence result

By Theorem 2.1 and Theorem 3.1, we get the following result.

Theorem 4.1 *Let v be a positive integer such that $v \not\equiv 2 \pmod 4$. If there exist both an LRHTS(v) and an LR(u), then there exists an LRHTS(uv).*

Applying Theorem 4.1 recursively with the LRHTS(v)s from Theorem 1.2, Theorem 1.3 and the LR(u)s from Theorem 2.2, we obtain the updated existence result on LRHTSs.

Theorem 4.2 *There exists an LRHTS$(3^a 5^b m \Pi_{i=1}^r (2 \cdot 13^{n_i} + 1) \Pi_{j=1}^p (2 \cdot 7^{m_j} + 1))$ for $m \in M = \{1, 4, 11, 17, 35, 43, 67, 91, 123\} \cup \{2^{2l+1} 25^s + 1 : l,\ s \ge 0\}$, $a, n_i, m_j \ge 1$ $(1 \le i \le r,\ 1 \le j \le p)$, $b, r, p \ge 2$ when $b \ge 1$ and $m \ne 1$.*

Appendix

An LRHTS(12) on $Z_{10} \cup \{\infty_1, \infty_2\}$ can be developed by eight base RHTS(12)s by $+2 \pmod{10}$. In each square is the block set of a base RHTS. In each RHTS, every row is a parallel class.

(∞_1 5 4)	(1 7 9)	(∞_2 0 8)	⟨6 2 3⟩	⟨∞_1 5 4⟩	(7 9 1)	(8 ∞_2 0)	(6 2 3)
⟨∞_2 2 9⟩	⟨6 0 4⟩	⟨∞_1 7 3⟩	⟨1 5 8⟩	(∞_2 2 9)	(6 0 4)	(∞_1 7 3)	(1 5 8)
⟨6 7 8⟩	⟨∞_2 5 3⟩	⟨1 2 4⟩	⟨∞_1 0 9⟩	(6 7 8)	(∞_2 5 3)	(1 2 4)	(∞_1 0 9)
⟨1 0 3⟩	⟨∞_1 2 8⟩	⟨6 5 9⟩	⟨∞_2 7 4⟩	(1 0 3)	(∞_1 2 8)	(6 5 9)	(∞_2 7 4)
(4 5 ∞_1)	⟨1 6 ∞_2⟩	⟨0 7 2⟩	⟨3 8 9⟩	⟨4 5 ∞_1⟩	(1 6 ∞_2)	(0 7 2)	(3 8 9)
⟨9 2 ∞_2⟩	⟨6 1 ∞_1⟩	⟨7 0 5⟩	⟨8 3 4⟩	(9 2 ∞_2)	(6 1 ∞_1)	(7 0 5)	(8 3 4)
⟨8 7 6⟩	⟨∞_2 ∞_1 1⟩	⟨2 5 0⟩	⟨9 4 3⟩	(8 7 6)	(∞_2 ∞_1 1)	(2 5 0)	(9 4 3)
⟨3 0 1⟩	⟨∞_1 ∞_2 6⟩	⟨5 2 7⟩	⟨4 9 8⟩	(3 0 1)	(∞_1 ∞_2 6)	(5 2 7)	(4 9 8)
(9 7 1)	⟨4 0 6⟩	⟨3 5 ∞_2⟩	⟨8 2 ∞_1⟩	(1 9 7)	(4 0 6)	(3 5 ∞_2)	(8 2 ∞_1)
(8 0 ∞_2)	⟨3 7 ∞_1⟩	⟨4 2 1⟩	⟨9 5 6⟩	(0 ∞_2 8)	(3 7 ∞_1)	(4 2 1)	(9 5 6)
⟨3 2 6⟩	⟨8 5 1⟩	⟨9 0 ∞_1⟩	⟨4 7 ∞_2⟩	(3 2 6)	(8 5 1)	(9 0 ∞_1)	(4 7 ∞_2)
(5 4 ∞_1)	⟨1 7 9⟩	(0 8 ∞_2)	(2 3 6)	(4 ∞_1 5)	(9 1 7)	⟨∞_2 0 8⟩	(3 6 2)
(2 9 ∞_2)	(0 4 6)	(7 3 ∞_1)	(5 8 1)	(9 ∞_2 2)	(4 6 0)	(3 ∞_1 7)	(8 1 5)
(7 8 6)	(5 3 ∞_2)	(2 4 1)	(0 9 ∞_1)	(8 6 7)	(3 ∞_2 5)	(4 1 2)	(9 ∞_1 0)
(0 3 1)	(2 8 ∞_1)	(5 9 6)	(7 4 ∞_2)	(3 1 0)	(8 ∞_1 2)	(9 6 5)	(4 ∞_2 7)
(∞_1 4 5)	(6 ∞_2 1)	(7 2 0)	(8 9 3)	(5 ∞_1 4)	(∞_2 1 6)	(2 0 7)	(9 3 8)
(∞_2 9 2)	(1 ∞_1 6)	(0 5 7)	(3 4 8)	(2 ∞_2 9)	(∞_1 6 1)	(5 7 0)	(4 8 3)
(6 8 7)	(∞_1 1 ∞_2)	(5 0 2)	(4 3 9)	(7 6 8)	(1 ∞_2 ∞_1)	(0 2 5)	(3 9 4)
(1 3 0)	(∞_2 6 ∞_1)	(2 7 5)	(9 8 4)	(0 1 3)	(6 ∞_1 ∞_2)	(7 5 2)	(8 4 9)
⟨9 7 1⟩	(6 4 0)	(∞_2 3 5)	(∞_1 8 2)	(7 1 9)	(0 6 4)	(5 ∞_2 3)	(2 ∞_1 8)
(∞_2 8 0)	(∞_1 3 7)	(1 4 2)	(6 9 5)	⟨8 0 ∞_2⟩	(7 ∞_1 3)	(2 1 4)	(5 6 9)
(6 3 2)	(1 8 5)	(∞_1 9 0)	(∞_2 4 7)	(2 6 3)	(5 1 8)	(0 ∞_1 9)	(7 ∞_2 4)
(∞_1 5 8)	(1 7 3)	(∞_2 0 4)	⟨6 2 9⟩	⟨∞_1 5 8⟩	(7 3 1)	(4 ∞_2 0)	(6 2 9)
⟨∞_2 2 3⟩	⟨6 0 8⟩	⟨∞_1 7 9⟩	⟨1 5 4⟩	(∞_2 2 3)	(6 0 8)	(∞_1 7 9)	(1 5 4)
⟨6 7 4⟩	⟨∞_2 5 9⟩	⟨1 2 8⟩	⟨∞_1 0 3⟩	(6 7 4)	(∞_2 5 9)	(1 2 8)	(∞_1 0 3)
⟨1 0 9⟩	⟨∞_1 2 4⟩	⟨6 5 3⟩	⟨∞_2 7 8⟩	(1 0 9)	(∞_1 2 4)	(6 5 3)	(∞_2 7 8)
(8 5 ∞_1)	⟨∞_2 6 1⟩	⟨2 7 0⟩	⟨3 4 9⟩	⟨8 5 ∞_1⟩	(∞_2 6 1)	(2 7 0)	(3 4 9)
⟨3 2 ∞_2⟩	⟨∞_1 1 6⟩	⟨5 0 7⟩	⟨8 9 4⟩	(3 2 ∞_2)	(∞_1 1 6)	(5 0 7)	(8 9 4)
⟨4 7 6⟩	⟨1 ∞_1 ∞_2⟩	⟨0 5 2⟩	⟨9 8 3⟩	(4 7 6)	(1 ∞_1 ∞_2)	(0 5 2)	(9 8 3)
⟨9 0 1⟩	(6 ∞_2 ∞_1)	(7 2 5)	(4 3 8)	(9 0 1)	(6 ∞_2 ∞_1)	(7 2 5)	(4 3 8)
(3 7 1)	⟨8 0 6⟩	⟨9 5 ∞_2⟩	⟨4 2 ∞_1⟩	(1 3 7)	(8 0 6)	(9 5 ∞_2)	(4 2 ∞_1)
(4 0 ∞_2)	⟨9 7 ∞_1⟩	⟨8 2 1⟩	⟨3 5 6⟩	(0 ∞_2 4)	(9 7 ∞_1)	(8 2 1)	(3 5 6)
⟨9 2 6⟩	⟨4 5 1⟩	⟨3 0 ∞_1⟩	⟨8 7 ∞_2⟩	(9 2 6)	(4 5 1)	(3 0 ∞_1)	(8 7 ∞_2)
(5 8 ∞_1)	⟨1 7 3⟩	(0 4 ∞_2)	(2 9 6)	(8 ∞_1 5)	(3 1 7)	⟨∞_2 0 4⟩	(9 6 2)
(2 3 ∞_2)	(0 8 6)	(7 9 ∞_1)	(5 4 1)	(3 ∞_2 2)	(8 6 0)	(9 ∞_1 7)	(4 1 5)
(7 4 6)	(5 9 ∞_2)	(2 8 1)	(0 3 ∞_1)	(4 6 7)	(9 ∞_2 5)	(8 1 2)	(3 ∞_1 0)
(0 9 1)	(2 4 ∞_1)	(5 3 6)	(7 8 ∞_2)	(9 1 0)	(4 ∞_1 2)	(3 6 5)	(8 ∞_2 7)
(∞_1 8 5)	(6 1 ∞_2)	(7 0 2)	(4 9 3)	(5 ∞_1 8)	(1 ∞_2 6)	(0 2 7)	(9 3 4)
(∞_2 3 2)	(1 6 ∞_1)	(0 7 5)	(9 4 8)	(2 ∞_2 3)	(6 ∞_1 1)	(7 5 0)	(4 8 9)
(6 4 7)	(∞_1 ∞_2 1)	(5 2 0)	(8 3 9)	(7 6 4)	(∞_2 1 ∞_1)	(2 0 5)	(3 9 8)
(1 9 0)	(∞_2 ∞_1 6)	(2 5 7)	(3 8 4)	(0 1 9)	(∞_1 6 ∞_2)	(5 7 2)	(8 4 3)
⟨3 7 1⟩	(6 8 0)	(∞_2 9 5)	(∞_1 4 2)	(7 1 3)	(0 6 8)	(5 ∞_2 9)	(2 ∞_1 4)
(∞_2 4 0)	(∞_1 9 7)	(1 8 2)	(6 3 5)	⟨4 0 ∞_2⟩	(7 ∞_1 9)	(2 1 8)	(5 6 3)
(6 9 2)	(1 4 5)	(∞_1 3 0)	(∞_2 8 7)	(2 6 9)	(5 1 4)	(0 ∞_1 3)	(7 ∞_2 8)

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