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# A lower bound for the depression of trees

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**Abstract**

An edge ordering of a graph $G$ is an injection $f : E(G) \to \mathbb{N}$. A (simple) path $\lambda$ for which $f$ increases along its edge sequence is an $f$-ascent, and a maximal $f$-ascent if it is not contained in a longer $f$-ascent. The depression $\varepsilon(G)$ of $G$ is the least integer $k$ such that every edge ordering of $G$ has a maximal ascent of length at most $k$.

We determine a lower bound for the depression of trees, which is exact if the set of branch vertices is independent, but not necessarily otherwise.

## 1 Introduction

We generally follow the notation of [2]. The *neighbourhood* $N(v)$ of a vertex $v$ of a simple graph $G = (V, E)$ is defined by $N(v) = \{u \in V : uv \in E\}$. An *edge ordering* of $G$ is an injection $f : E \to \mathbb{N}$. Denote the set of all edge orderings of $G$ by $\mathcal{F}(G)$. For any $f \in \mathcal{F}(G)$ a path $a, b, c, d$ of length three such that $f(bc) = \min\{f(ab), f(bc), f(cd)\}$ or $f(bc) = \max\{f(ab), f(bc), f(cd)\}$ is called an *$f$-exchange*. A path $\lambda$ in $G$ for which $f \in \mathcal{F}(G)$ increases along its edge sequence is called an *$f$-ascent* (or simply *ascent* if the ordering is clear), and if $\lambda$ has length $k$, it will also be called a *$(k, f)$-ascent*. Thus an $f$-ascent contains no $f$-exchanges. If the path $\lambda$ with vertex sequence $v_0, v_1, ..., v_k$ or edge sequence $e_1, e_2, ..., e_k$ forms an $f$-ascent, we denote this fact by writing $\lambda$ as $v_0v_1...v_k$ or $e_1e_2...e_k$. An $f$-ascent is *maximal* if it is not contained in a longer $f$-ascent. Let $h(f)$ denote the length of a shortest maximal $f$-ascent and define the *depression* $\varepsilon(G)$ of $G$ by

$$\varepsilon(G) = \max_{f \in \mathcal{F}(G)} \{h(f)\},$$

that is, $\varepsilon(G)$ is the smallest integer $k$ such that every edge ordering of $G$ has a maximal ascent of length at most $k$. To show that $\varepsilon(G) = k$, we must therefore show that

(a) each edge ordering of $G$ has a maximal ascent of length at most $k$ – this shows that $\varepsilon(G) \leq k$,

(b) there exists an edge ordering $f$ of $G$ with no maximal ascents of length less than $k$, i.e. for which each $(l, f)$-ascent, where $l < k$, can be extended to a $(k, f)$-ascent – this shows that $\varepsilon(G) \geq k$.

The study of the lengths of increasing paths in edge-ordered graphs was initiated by Chvátal and Komlós [3] who posed the problem of determining the *altitude* $\alpha(G)$, the greatest integer $k$ such that $G$ has a $(k, f)$-ascent for each edge ordering $f \in \mathcal{F}(G)$, for $G = K_n$. They also considered the corresponding problem in the case where $f$-ascents are trails, not necessarily paths. Let $\overline{\alpha}(G)$ be the parameter corresponding to $\alpha(G)$ in this instance. It was shown by Graham and Kleitman [5] that $\overline{\alpha}(K_3) = 3$, $\overline{\alpha}(K_5) = 5$ and $\overline{\alpha}(K_n) = n - 1$ otherwise. On the other hand, $\alpha(K_n)$ is unknown for $n \geq 9$; see e.g. [1]. For this reason subsequent work focussed on $\alpha$ rather than $\overline{\alpha}$; to the best of our knowledge $\overline{\alpha}$ has not been considered for other classes of graphs.

Let $\tau(G)$ denote the *detour length* (the length of a longest path) of $G$. If $G$ is a connected graph with at least two edges, then for any $f \in \mathcal{F}(G)$, any two adjacent edges form a $(2, f)$-ascent, hence $h(f) \geq 2$. On the other hand, each ascent is a path and thus has length at most $\tau(G)$. Therefore

$$2 \leq \varepsilon(G) \leq \tau(G). \tag{1}$$

The depression of a graph was first defined in [4], where the bound in (1) was improved by defining a parameter related to $\tau(G)$. For any path $u, v, w$ in a graph $G$, let $\tau(uvw)$ be the length of a longest path in $G$ containing the subpath $u, v, w$. Define

$$\tau'(G) = \min\{\tau(uvw)\},$$

where the minimum is taken over all paths of $G$ of length two; obviously $\tau'(G) \leq \tau(G)$. As shown in [4], $\varepsilon(G) \leq \tau'(G)$ for all graphs $G$. It follows that if $G$ has a vertex adjacent to two leaves, then $\varepsilon(G) = 2$. Indeed, graphs with depression two were characterised in [4], by no means an easy task. (Of course, $\varepsilon(G) = 1$ if and only if $K_2$ is a component of $G$.)

**Theorem 1** [4] *If $G$ is connected, then $\varepsilon(G) = 2$ if and only if $G$ has a vertex adjacent to two leaves or to two adjacent vertices of degree two.*

Theorem 1 shows that there is no forbidden subgraph characterisation of graphs with depression two, because if any vertex of an arbitrary graph $G$ is joined to two new vertices, the resulting graph has depression two.

## 2 Depression in trees

Theorem 1 gives the following characterisation of trees with depression two.

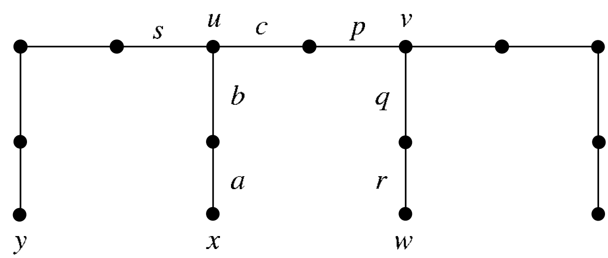


Figure 1: A tree with depression five

**Corollary 2** [4] *If $T$ is a tree, then $\varepsilon(T) = 2$ if and only if some vertex of $T$ is adjacent to at least two leaves.*

A *branch vertex* of a tree is a vertex of degree at least three and a *support vertex* is a vertex adjacent to a leaf. Let $L(T)$ and $B(T)$ respectively denote the sets of all leaves and all branch vertices of the tree $T$, and $\ell(T)$ the minimum length of a path $P$ between two leaves of $T$ such that no two consecutive vertices of $P$ are branch vertices, i.e. $B(T) \cap V(P)$ is independent. For $v \in V(T)$ and $l \in L(T)$, a $(v,l)$-*endpath*, or $v$-*endpath* if the leaf is unimportant, or *endpath* if neither $v$ nor $l$ is important, is a path $P$ from $v$ to $l$ such that each internal vertex of $P$ has degree two in $T$. A $v$-$L$ path is any path from $v$ to a leaf. A branch vertex $v$ incident with exactly one edge $e$ such that $e$ does not lie on any $v$-endpath is called a *special branch vertex*. Each tree with at least two branch vertices has at least two special branch vertices. (Root $T$ at a branch vertex $x$ and choose a branch vertex $y \neq x$ at maximum distance from $x$; now root $T$ at $y$ and choose a branch vertex $y'$ at maximum distance from $y$. The edges $e$ and $e'$ incident with $y$ and $y'$, respectively, on the $y$-$y'$ path satisfy the above requirement.) A *spider* $S(a_1, ..., a_r)$ is a tree with exactly one branch vertex $v$ and $v$-endpaths of lengths $1 \leq a_1 \leq \cdots \leq a_r$, where $r = \deg v$. The depression of spiders is given in [4].

**Proposition 3** [4] $\varepsilon(S(a_1, ..., a_r)) = \min\{a_1 + a_2, a_3 + 1\}$.

An upper bound for the depression of trees related to the above result is also given in [4]. Those spiders obtained by removing all edges of the tree $T$ that are not edges of endpaths are called *hanging spiders* of $T$. Let $\mathcal{H}(T)$ denote the set of all hanging spiders $H = S(a_1, ..., a_r)$, $r \geq 3$, of $T$ and define

$$s(T) = \min_{H \in \mathcal{H}(T)} \{a_3 + 1\}.$$

**Theorem 4** [4] For any tree $T$, $\varepsilon(T) \leq \min\{\ell(T), s(T)\}$.

This bound is not exact for all trees, not even in the case where $B(T)$ is independent. Consider the tree $T$ in Figure 1. It has no hanging spiders and $\ell(T) = 6$. Hence by Theorem 4, $\varepsilon(T) \leq 6$. Suppose $f$ is an edge ordering of $T$ with $h(f) = 6$ as (partially) indicated in Figure 1. We may assume without loss of generality that $a < b$. Suppose $c < b$. Then $p < c$, otherwise $cb$ is a maximal $(2, f)$-ascent, which is impossible. If $q < p$, then $rqpcb$ contains a maximal ascent of length at most five. Thus $p < q$. But now $pqr$ contains a maximal ascent of length at most three, a contradiction. Hence we may assume that $b < c$ and consequently $c < p$. If $q < p$, then $rqp$ contains a maximal ascent, hence $p < q$ and so $q < r$. Denote the $y$-$u$ path by $P$. If $s < b$, then $P$ followed by the edge $b$ contains a maximal ascent of length at most five, hence $b < s$. It follows that $f$ increases along $P$ from $u$ to $y$. Therefore, if $s < c$, then $scpqr$ is a maximal $(5, f)$-ascent, and if $c < s$, then $c$ followed by $P$ is a maximal $(5, f)$-ascent, a contradiction. Therefore $\varepsilon(T) \leq 5$.

In Section 3 we obtain a lower bound for the depression of all trees and show that it is not exact for trees with adjacent branch vertices, while in Section 4 we prove that this bound is exact for trees in which the set of branch vertices is independent.

## 3 Lower bound for trees

The lower bound for the depression of trees requires the following definition.

**Definition 1** For $\alpha \in B(T)$ with $\deg \alpha = r$, let $e_1(\alpha), ..., e_r(\alpha)$ be an arrangement of the edges incident with $\alpha$ and $\ell_i(\alpha)$ the length of a shortest $\alpha$-$L$ path $P_i(\alpha)$ that contains $e_i(\alpha)$. We abbreviate $e_i(\alpha)$, $\ell_i(\alpha)$ and $P_i(\alpha)$ to $e_i$, $\ell_i$ and $P_i$ if the vertex $\alpha$ is clear from the context. An arrangement $e_1, ..., e_r$ is called *suitable* if $\ell_i \leq \ell_j$ whenever $i < j$. From a suitable arrangement $e_1, ..., e_r$ of the edges incident with $\alpha$, define

$$\rho(\alpha) = \min\{\ell_1(\alpha) + \ell_2(\alpha), \ell_3(\alpha) + 1\}.$$

Generally, when we say that a path $P$ is an ascent, no direction of ascent is implied; however, when we say that the path $P \cup Q$ is an ascent, where $V(P) \cap V(Q) = \{w\}$, we mean that the ascent starts at the endvertex of $P$ other than $w$, increases towards $Q$ and ends at the endvertex of $Q$ other than $w$.

**Theorem 5** *Every tree $T$ has an edge ordering $f$ such that*

(*i*) $h(f) \geq \min_{\alpha \in B(T)}\{\rho(\alpha)\}$ *and*

(*ii*) *for each $\alpha \in B(T)$ there exist a suitable arrangement $e_1, ..., e_r$, $r = \deg \alpha$, of the edges incident with $\alpha$ and a choice of paths $P_i$ (as in Definition 1) such that $P_1 \cup P_i$, $2 \leq i \leq r$, or their reverses, are $f$-ascents.*

*Proof.* We use induction on $|B(T)|$, the result being obvious for paths. If $B(T) = \{v\}$, then $T = S(\ell_1, ..., \ell_r)$ is a spider. By Proposition 3,

$$\varepsilon(T) = \min\{\ell_1 + \ell_2, \ell_3 + 1\} = \min_{\alpha \in B(T)}\{\rho(\alpha)\}.$$

As in [4], let $f$ increase along $P_1$ from the leaf to $v$, using the integers $1, ..., \ell_1$. For $i = 2, ..., r$, and in this order, label the edges of $P_i$ to form ascents from $v$ to the leaf, using the integers $\sum_{j=1}^{i-1} \ell_j + 1, ..., \sum_{j=1}^{i} \ell_j$. It is easily seen [4] that $f$ satisfies $(i)$ and $(ii)$.

Suppose the result is true for all trees with fewer than $k \geq 2$ branch vertices. Let $T$ be a tree of size $m$, $|B(T)| = k$, $v$ with $\deg v = r$ a special branch vertex and $e$ the edge incident with $v$ not contained in any $v$-endpath. Let $Q_1, ..., Q_{r-1}$ be the $v$-endpaths, labelled such that $|E(Q_i)| \leq |E(Q_j)|$ whenever $i < j$ and $T'$ the subtree of size $m'$ obtained by deleting all vertices of $Q_i$, $i \geq 2$, except $v$, from $T$. Then $|B(T')| = k - 1$. By the induction hypothesis, $T'$ has an edge ordering $f' : E(T') \to \{1, ..., m'\}$ that satisfies $(i)$ and $(ii)$.

For any $\alpha \in B(T')$ and $1 \leq i \leq \deg \alpha$, if we define $\ell'_i(\alpha)$ in $T'$ similar to $\ell_i(\alpha)$ in $T$, then $\ell'_i(\alpha) = \ell_i(\alpha)$ by the construction of $T'$, so we only use the notation $\ell_i(\alpha)$. Consider a suitable arrangement $e_i$, $1 \leq i \leq r$, of the edges incident with $v$ and the $v$-$L$ paths $P_i$ of length $\ell_i(v)$ as in Definition 1. Then $e = e_s$ for some $1 \leq s \leq r$, and if $i \neq s$, then $Q_i = P_i$ or $Q_i = P_{i+1}$ depending on whether $i < s$ or $i > s$.

Let $u$ be the branch vertex nearest to $v$ and $Q : u, u_1, ..., u_a = v, ..., u_t$ the $u$-endpath containing $Q_1$. Then $e = vu_{a-1}$ and $u \in V(P_s)$. Let $e'$ be the edge other than $uu_1$ incident with $u$ on $P_s$. Since $Q$ is the only $u$-$L$ path in $T'$ containing $uu_1$, $Q$ is an $f'$-ascent by $(ii)$. Assume without loss of generality that $f'(uu_1) < f'(u_1u_2) < \cdots < f'(u_{t-1}u_t)$. The $u$-$L$ path $R = P_s - Q$ is obviously a shortest $u$-$L$ path containing $e'$. Hence we may assume without loss of generality that $R$ is that particular $u$-$L$ path containing $e'$ that forms an $f'$-ascent as asserted in $(ii)$ (otherwise we may redefine $P_s$ to contain the path with this property). Since $P_s$ is a shortest $v$-$L$ path containing $e$, it follows that $R$ has length $\ell_1(u)$ if $|E(R)| \leq |E(Q)|$ and length $\ell_2(u)$ if $|E(Q)| < |E(R)|$, in which case $|E(Q)| = \ell_1(u)$. In either case it follows from $(ii)$ that $R \cup Q$ is a maximal $f'$-ascent.

We obtain an edge ordering $f$ of $T$ which depends on the value of $s$. In each case we show that any maximal $f$-ascent that is not a maximal $f'$-ascent either contains a maximal $f'$-ascent, or has length at least that of some maximal $f'$-ascent for some other reason, or has length at least $\rho(v)$. This implies that $(i)$ holds. In each case it also follows from the induction hypothesis, the reconstruction of $T$ from $T'$ and the definition of $f$ that $f$ satisfies the conditions in $(ii)$ for every branch vertex of $T'$. We show that $P_1(v) \cup P_i(v)$, $i \geq 2$, or their reverses, are $f$-ascents; it follows that $f$ satisfies $(ii)$ in $T$.

**Case 1** $s = 1$. Note that for $i = 2, ..., r$, $P_i = Q_{i-1}$; also, $e_2 = vu_{a+1}$. Let $f$ agree with $f'$ on $T'$ and, using the integers $m' + 1, ..., m$, label the edges of each $P_i$, $i \geq 3$, consecutively to form $f$-ascents increasing from $v$ to the leaves, and so that $f(e_i) < f(e_j)$ if $i < j$.

Each maximal $f'$-ascent in $T' - E(Q_1)$ is a maximal $f$-ascent. Suppose $\lambda'$ is a maximal $f'$-ascent that contains $uu_1$. Since $uu_1...u_t$ is an $f'$-ascent and $t \geq 2$, $\lambda'$ does not end at $u_1$. If $\lambda'$ starts at $u_1$, then $f'(uu_1) = \min_{x \in N(u_1)}\{f'(u_1x)\}$. By definition of $f$, $f(uu_1) = \min_{x \in N(u_1)}\{f(u_1x)\}$ in $T$ also (also if $v = u_1$) and $\lambda'$ is a maximal $f$-ascent. Clearly, no other internal vertex of $Q_1$ is the start or end of a maximal $f'$-ascent. Hence each maximal $f'$-ascent in $T'$ is a maximal $f$-ascent. In particular, $P_1 \cup P_2 = P_1 \cup Q_1 = R \cup Q$ is a maximal $f$-ascent (by $(ii)$) of length $\ell_1(v) + \ell_2(v) = \ell_1(u) + \ell_2(u)$.

Therefore the $f$-ascents not contained in $T'$ are as follows; we illustrate that they either have lengths at least that of some maximal $f'$-ascent or lengths at least $\rho(v)$.

If $\lambda'$ is a maximal $f'$-ascent that contains $uu_1$ and does not start at $u_1$, then $\lambda = (\lambda' - Q_1) \cup P_i$, $i \geq 3$, is a maximal $f$-ascent of length at least that of $\lambda'$; in particular, for each $i \geq 3$, $P_1 \cup P_i$ is a maximal $f$-ascent of length at least that of the maximal $f'$-ascent $Q \cup R$. By definition of $f$, $\{e_i\} \cup P_j$, $2 \leq i < j \leq r$, are $f$-ascents and are thus contained in maximal $f$-ascents of lengths at least $\ell_j(v) + 1 \geq \ell_3(v) + 1 \geq \rho(v)$.

Hence $h(f) \geq \min\{h(f'), \ell_3(v) + 1\} \geq \min_{\alpha \in B(T)}\{\rho(\alpha)\}$. We have also shown above that $P_1(v) \cup P_i(v)$, $i \geq 2$, are $f$-ascents. Combined with the induction hypothesis this implies, as stated before Case 1, that $f$ satisfies $(ii)$.

**Case 2** $s = 2$. Then $e_2 = e = vu_{a-1}$, $P_1 = Q_1$ and for $i = 3, ..., r$, $P_i = Q_{i-1}$. For each $z \in E(T')$, let $f(z) = f'(z) + \sum_{i=3}^{r} \ell_i(v)$. Using the integers $1, ... \sum_{i=3}^{r} \ell_i(v)$, label the edges of each $P_i$, $i = 3, ..., r$, consecutively to form $f$-ascents increasing from the leaves to $v$, and so that $f(e_i) > f(e_j)$ if $i < j$. As in Case 1 each maximal $f'$-ascent in $T'$ is a maximal $f$-ascent. In particular, $P_2 \cup P_1$ is a maximal $f$-ascent of length $\ell_1(v) + \ell_2(v) = \ell_1(u) + \ell_2(u)$. The $f$-ascents not contained in $T'$ are as follows.

For each $i \geq 3$, $P_i \cup P_1$ is a maximal $f$-ascent of length at least that of the maximal $f'$-ascent $P_2 \cup P_1$. The other $f$-ascents that contain edges of $T'$ are $P_i \cup \{e_2\}$, $i \geq 3$, which have lengths at least $\ell_3(v) + 1$ and hence are contained in maximal $f$-ascents of lengths at least $\ell_3(v) + 1$. Finally, for each $3 \leq i < j \leq r$, $P_j \cup \{e_i\}$ is a maximal $f$-ascent of length at least $\ell_3(v) + 1$. Thus $h(f) \geq \min\{h(f'), \ell_3(v) + 1\} \geq \min_{\alpha \in B(T)}\{\rho(\alpha)\}$. As in Case 1, $f$ also satisfies $(ii)$.

**Case 3** $s \geq 3$. Then $P_i = Q_i$ for $i = 1, ..., s-1$ and $P_i = Q_{i-1}$ for $i > s$. For each $z \in E(T') - E(Q_1)$, let $f(z) = f'(z) + \sum_{i=s+1}^{r} \ell_i(v)$. Using the integers $1, ... \sum_{i=s+1}^{r} \ell_i(v)$, label the edges of each $P_i$, $i = s+1, ..., r$, consecutively to form $f$-ascents increasing from the leaves to $v$, and $f(e_i) > f(e_j)$ if $i < j$. Using the integers $m' + \sum_{i=s+1}^{r} \ell_i(v) + 1, ..., m' + \sum_{i=2}^{s-1} \ell_i(v) + \sum_{i=s+1}^{r} \ell_i(v)$, label the edges of $P_i$, $i = 2, ..., s-1$ to form $f$-ascents increasing from the leaves to $v$, and $f(e_i) > f(e_j)$ if $i < j$. Finally, label the edges of $P_1 = Q_1$ with the largest integers to form an $f$-ascent increasing from $v$ to the leaves.

Each maximal $f'$-ascent in $T' - E(Q_1)$ is a maximal $f$-ascent. Also, any maximal $f'$-ascent containing $uu_1$ that does not start at $u_1$ is a maximal $f$-ascent; in particular, $P_s \cup P_1$ is a maximal $f'$- and $f$-ascent. The maximal $f$-ascents that are not maximal $f'$-ascents are as follows. For each $i = 2, ..., r$, $i \neq s$, $P_i \cup P_1$ is a maximal $f$-ascent of length at least $\ell_1(v) + \ell_2(v)$. For $2 \leq i < j \leq r$, $P_j \cup \{e_i\}$ is an $f$-ascent of length at least $\ell_3(v) + 1$, hence is contained in a maximal $f$-ascent of at least this length. Therefore $h(f) \geq \min\{h(f'), \ell_1(v) + \ell_2(v), \ell_3(v) + 1\} \geq \min_{\alpha \in B(T)}\{\rho(\alpha)\}$. Since we have also shown that $P_i \cup P_1$, $2 \leq i \leq r$, are $f$-ascents, it follows as before that $f$

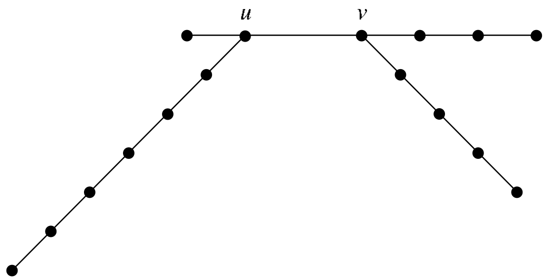


Figure 2: The double spider $S(1,6:3,4)$

satisfies $(ii)$ in $T$. ■

**Corollary 6** *For any tree $T$, $\varepsilon(T) \geq \min_{\alpha \in B(T)}\{\rho(\alpha)\}$.*

The above bound is not exact if $\langle B(T)\rangle$ contains edges, as we show next. A *double spider* is a tree $T$ such that $\langle B(T)\rangle = K_2$. If $T$ is the double spider that consists of two adjacent vertices $u$ and $v$ with $\deg u = k+1$ and $\deg v = k'+1$, together with $k \geq 2$ $u$-endpaths $P_i$ of lengths $a_i$, $1 \leq i \leq k$, and $k' \geq 2$ $v$-endpaths $P'_j$ of lengths $b_j$, $1 \leq j \leq k'$, where $a_1 \leq \cdots \leq a_k$ and $b_1 \leq \cdots \leq b_{k'}$, then we denote $T$ by $S(a_1,...,a_k : b_1,...,b_{k'})$. The double spider $S(1,6:3,4)$ is illustrated in Figure 2.

**Proposition 7** *Let $T = S(a_1,...,a_k : b_1,...,b_{k'})$. Then*

$$\varepsilon(T) \geq \min\{a_1+a_2, a_3+1, b_1+b_2, b_3+1, a_1+b_2+1, a_2+b_1+1\},$$

*where we ignore the term $a_3+1$ if $k=2$, and the term $b_3+1$ if $k'=2$.*

*Proof.* Let $u_i$ be the neighbour of $u$ on $P_i$ and $v_j$ the neighbour of $v$ on $P'_j$. Define the edge ordering $f$ as follows. From the leaf to $u$, label the edges of $P_1$ with the integers $1,...,a_1$; from the leaf to $v$, label the edges of $P'_1$ with $a_1+1,...,a_1+b_1$. Let $f(uv) = a_1+b_1+1$. For $i = 2,...,k$, and in this order, label the edges of $P_i$ with consecutive integers $a_1+b_1+2,...,\sum_{i=1}^{k} a_i + b_1 + 1$ to form ascents from $u$ to the leaf. For $j = 2,...,k'$, and in this order, label the edges of $P'_j$ with consecutive integers $\sum_{i=1}^{k} a_i + b_1 + 2, ..., \sum_{i=1}^{k} a_i + \sum_{j=1}^{k'} b_j + 1$ to form ascents from $v$ to the leaf. The maximal $f$-ascents are $P_1 \cup P_i$ for $i = 2,...,k$, $\{u_i u\} \cup P_j$ for $2 \leq i < j \leq k$, $P'_1 \cup P'_i$ for $i = 2,...,k'$, $\{v_i v\} \cup P'_j$ for $2 \leq i < j \leq k'$, $P_1 \cup \{uv\} \cup P'_j$ for $j = 2,...,k'$ and $P'_1 \cup \{vu\} \cup P_i$ for $i = 2,...,k$. Therefore

$$h(f) = \min\{a_1+a_2, a_3+1, b_1+b_2, b_3+1, a_1+b_2+1, a_2+b_1+1\}$$

(ignore the terms $a_3+1$ or $b_3+1$ if $k=2$ or $k'=2$) and the bound follows. ■

Let $T = S(1,6 : 3,4)$ – see Figure 2. Then $\ell_1(u) = 1$, $\ell_2(u) = 4$, $\ell_3(u) = 6$, $\ell_1(v) = 2$, $\ell_2(v) = 3$ and $\ell_3(v) = 4$. Therefore $\min_{x\in B(T)}\{\rho(x)\} = 5$. However, by Proposition 7, $\varepsilon(T) \geq \min\{7,7,6,10\} = 6$. This improves the bound in Corollary 6.

It is also easy to see that the difference between the bounds in Theorem 4 and Corollary 6 can be arbitrary. Consider the spider $S(1,2,2)$ with branch vertex $v$ and vertices $u$ and $w$ of degree two, and two disjoint copies of $P_n$ with vertex sequences $u_1, ..., u_n$ and $w_1, ..., w_n$. Let $T$ be the tree obtained by joining $u$ to $u_1$, and $w$ to $w_1$. Then $T$ has no hanging spiders, $\min_{x\in B(T)}\{\rho(x)\} = \rho(v) = 3$ and $\min\{\ell(T), s(T)\} = \ell(T) = n+1$.

## 4 Trees with independent branch vertices

We show that the bound in Corollary 6 is exact for trees $T$ with $B(T)$ independent. If $B(T) = \phi$, then $T$ is a path and it is easy to see that $\varepsilon(P_n) = n-1$, hence we assume that $B(T) \neq \phi$. Recall that $\ell(T)$ is the minimum length of a path $P$ between two leaves of $T$ such that no two consecutive vertices of $P$ are branch vertices.

**Theorem 8** *If $B(T)$ is independent, then $\varepsilon(T) = \min_{\alpha\in B(T)}\{\rho(\alpha)\}$.*

*Proof.* By Corollary 6 we only need to prove that $\varepsilon(T) \leq \min_{\alpha\in B(T)}\{\rho(\alpha)\}$. Suppose this is not true. Consider a tree $T$ with $B(T)$ independent and edge ordering $f$ such that $h(f) > \min_{\alpha\in B(T)}\{\rho(\alpha)\}$. By Theorem 4, $\varepsilon(T) \leq \ell(T)$. But $B(T)$ is independent, hence any path of $T$ between two leaves is without consecutive branch vertices and therefore $\ell(T) \leq \ell_1(\alpha)+\ell_2(\alpha)$ for any $\alpha \in B(T)$. Hence $h(f) > \ell_3(v)+1$ for some $v \in B(T)$ such that $\ell_3(v)+1 < \ell_1(v)+\ell_2(v)$.

Consider a suitable arrangement $e_i = vv_i$, $i = 1, ..., r = \deg v$, of the edges incident with $v$ and $v$-$L$ paths $P_i$ of length $\ell_i$ containing $e_i$. Since $B(T)$ is independent, $\deg v_i = 1$ or $2$ for each $i$. But $\ell_1 + \ell_2 > \ell_3 + 1 \geq \ell_2 + 1$, i.e. $\ell_2 \geq \ell_1 > 1$ and $\deg v_i > 1$ for each $i$. Hence $\deg v_i = 2$. We henceforth only consider $i \in \{1,2,3\}$. Let $e_i'$ be the edge on $P_i$ adjacent to $e_i$, $x_i = f(e_i)$ and $y_i = f(e_i')$. Let $P_i$ have vertex sequence $i_0 = v, i_1 = v_i, i_2, ..., i_{\ell_i}$.

If $P_i$ contains an $f$-exchange, let $j$ be the largest integer such that $i_{j-1}, i_j, i_{j+1}, i_{j+2}$ is an $f$-exchange. If $\deg i_j = 2$, then $i_j i_{j+1} ... i_{\ell_i}$ or $i_{\ell_i} i_{\ell_i -1} ... i_j$ is a maximal $f$-ascent of length less than $\ell_3$, contradicting $h(f) > \ell_3 + 1$. Therefore $i_j \in B(T)$ and $\deg i_{j-1} = \deg i_{j+1} = 2$ since $B(T)$ is independent. If it exists, let $j'$ be the largest integer less that $j$ such that $i_{j'-1}, i_{j'}, i_{j'+1}, i_{j'+2}$ is an $f$-exchange. If $\deg i_{j'} = 2$, then $i_{j'} i_{j'+1} ... i_{j+1}$ or $i_{j+1} i_j ... i_{j'}$ is a maximal $f$-ascent of length less than $\ell_3$, a contradiction. Hence $i_{j'} \in B(T)$, so that $j' \leq j-2$, and $\deg i_{j'-1} = \deg i_{j'+1} = 2$. Repeating the process if necessary we eventually obtain an integer $2 \leq t_i < \ell_i \leq \ell_3$ such that $\deg i_{t_i} = 2$ and $Q_i : i_0 i_1 ... i_{t_i}$ (or its reverse) is an $f$-ascent, while $i_0 i_1 ... i_{t_i} i_{t_i+1}$ (or its reverse) is not. If $P_i$ does not contain an $f$-exchange, then $Q_i = P_i$ is an $f$-ascent and we take $t_i = \ell_i \leq \ell_3$.

Assume without loss of generality that $x_3 < y_3$ and consider $i \in \{1,2\}$. If $x_i > x_3$, then either $e_3 e_i$ is a maximal $(2,f)$-ascent (if $y_i < x_i$), or $e_3$ followed by

$Q_i$ is a maximal $(t_i + 1, f)$-ascent, contradicting $h(f) > \ell_3 + 1$. Therefore $x_i < x_3$. Moreover, if $x_i < y_i$, then $e_i$ followed by $Q_3$ is a maximal $f$-ascent of length at most $\ell_3 + 1$, hence $y_i < x_i$. Now if $x_1 > x_2$, then $Q_2$ followed by $e_1$ is a maximal $(\ell_2 + 1, f)$-ascent, while if $x_1 < x_2$, then $Q_1$ followed by $e_2$ is a maximal $(\ell_1 + 1, f)$-ascent, where $\ell_1 \leq \ell_2 \leq \ell_3$. This contradiction shows that no edge ordering $f$ with $h(f) > \ell_3(v) + 1$ exists. Thus $\varepsilon(T) \leq \ell_3(\alpha) + 1$ for each $\alpha \in B(T)$ and the bound is established. ■

## 5 Open problems

1. The bound in Proposition 7 is not best possible in all cases. Is there a simple formula for $\varepsilon(S(a_1, ..., a_k : b_1, ..., b_{k'}))$?

2. If such a formula exists, it may be possible to determine a formula for the depression of trees when the branch vertices are not independent, or perhaps for special cases, for example where the subgraph induced by the branch vertices consists of independent edges and isolated vertices.

3. Failing the above, improve the bounds in Theorem 4 and Corollary 6.

4. For arbitrary $k \geq 0$, let $G$ be any graph with depression $k$. When any vertex of $G$ is joined to two new vertices, the resulting graph $H$ has depression two, whether the new vertices are adjacent to each other or not (Theorem 1). This can also be viewed as follows. Let $G$ be any graph and construct $H$ by identifying any vertex of $G$ with the central vertex of a new $P_3$ or with any vertex of a new $K_3$. Then $\varepsilon(H) = 2$.

   Describe similar methods to obtain larger graphs with fixed depression $\varepsilon \geq 3$ from smaller graphs with the same depression.

5. Characterise trees with $\varepsilon = 3$.

6. Characterise graphs with $\varepsilon = 3$.

**Note added in proof:** Problem 1 has been solved in [6] and Problem 5 in [7]; the rest remain open.

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