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The tree- and clique-width of bipartite graphs in special classes

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Abstract

The tree- and clique-width are two graph parameters which are of primary importance in algorithmic graph theory because of the fact that many NP-hard graph problems admit polynomial-time solutions when restricted to graphs of bounded tree- or clique-width. Both parameters are known to be unbounded in the class of all bipartite graphs. We study the tree- and clique-width of bipartite graphs in special classes. The main result is a necessary condition for the tree- and clique-width to be bounded in subclasses of bipartite graphs defined by finitely many forbidden induced bipartite subgraphs. We use this result to analyze the tree- and clique-width of bipartite graphs in classes defined by a single forbidden induced subgraph.

1 Introduction

The tree-width and clique-width are two graph parameters which are of interest due to the fact that many algorithmic problems being NP-hard in general graphs become polynomial-time solvable when restricted to graphs where one of these parameters is bounded. Both parameters are known to be unbounded in the class of general bipartite graphs. We study the tree- and clique-width of bipartite graphs in special classes. The main result is a necessary condition for the tree- and clique-width to be bounded in subclasses of bipartite graphs defined by finitely many forbidden induced bipartite subgraphs. This leads to a complete classification of monogenic classes of bipartite graphs (i.e. those defined by a single forbidden induced bipartite subgraph) with respect to bounded or unbounded tree-width. In case of clique-width such a classification remains an open problem. In this paper we reveal two new monogenic classes of bipartite graphs of bounded clique-width and distinguish all minimal and maximal monogenic classes with unknown clique-width.

We consider only simple undirected graphs without loops or multiple edges. The vertex set and the edge set of a graph G are denoted $V(G)$ and $E(G)$, respectively. A subgraph of G *induced* by a set $W \subseteq V(G)$, denoted $G[W]$, is the graph with the vertex set W and two vertices being adjacent if and only if they are adjacent in G. With some abuse of terminology, we shall say that G contains H as an induced subgraph if H is isomorphic to an induced subgraph of G. If G does not contain, we say that G is H-free. For a vertex $v \in V(G)$, we denote by $N(v)$ the neighborhood of v (i.e. the set of vertices adjacent to v) and by $\deg(v) := |N(v)|$ the degree of v. The complement of a graph G is denoted $\overline{G}$.

A bipartite graph $G = (W, B, E)$ consists of a set W of white vertices, a set B of black vertices and a set E of edges each of which connects unlike colored vertices. The fact that the vertices of a bipartite graph are colored makes it necessary to define more exactly the notion of isomorphism. We say that two bipartite graphs G and H are isomorphic if between their vertex sets there is a one-to-one correspondence that preserves the adjacency and respects the bipartition, i.e. two vertices of G are adjacent if and only if the corresponding vertices of H are adjacent, and two vertices of G have the same color if and only if the corresponding vertices of H have the same color. The bipartite complement of a bipartite graph $G = (W, B, E)$ is denoted $\widetilde{G}$ and is defined as follows: $\widetilde{G} = (W, B, (W \times B) - E)$. For instance, the bipartite complement of a cycle on 6 vertices is $3K_2$, i.e. the disjoint union of 3 copies of K_2, where K_2 is a complete graph on 2 vertices. As usual, P_n and C_n denote a chordless path and a chordless cycle on n vertices, respectively, and $K_{n,m}$ a complete bipartite graph with parts of size n and m. Also $S_{i,j,k}$ and H_n stand for the graphs represented in Figure 1.

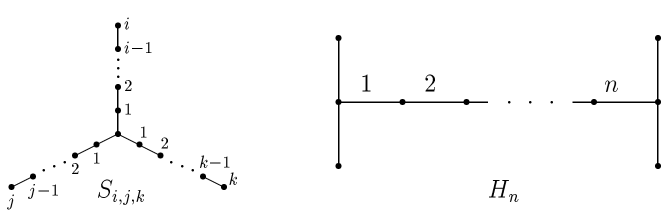

Figure 1: Graphs $S_{i,j,k}$ and H_n

We denote the class of all bipartite graphs by B and use special notations for

some other particular classes of graphs:
X_k, the class of $(C_3, C_4, \ldots, C_k)$-free graphs;
Y_l, the class of $(H_1, H_2, \ldots, H_l)$-free graphs;
Z_3, the class of graphs with maximum vertex degree at most 3;
$\mathcal{S}$, the class of graphs every connected component of which is of the form $S_{i,j,k}$.

2 Preliminaries

In this section we formally define the notions of tree- and clique-width and provide auxiliary results that will be useful in the sequel.

To introduce the notion of tree-width let us use its relationship with triangulations of graphs. A graph is said to be *chordal* or *triangulated* if it does not contain induced cycles of length at least four. Given an arbitrary graph $G = (V, E)$, a *triangulation* of G is a chordal graph $H = (V, F)$ such that $E \subseteq F$. A triangulation H of G is *minimal* if no proper subgraph of H is a triangulation of G. The tree-width of G, denoted $\mathrm{tw}(G)$, is

$$\min\{\omega(H) - 1 \mid H \, \textit{is a minimal triangulation of } G\},$$

where $\omega(H)$ is the size of a maximum clique in H.

Graphs of tree-width 0 are empty (edgeless), and graphs of tree-width at most 1 are trees, or more generally, forests. Graphs of tree-width at most k (known also as partial k-trees) appeared in the literature as a generalization of trees. Though trees are bipartite graphs, the tree-width of general bipartite graphs is unbounded, since it is unbounded even in the class of complete bipartite graphs. Indeed, in any triangulation of a complete bipartite graph, at least one of the parts is a clique (since otherwise the graph contains a C_4) and hence $\mathrm{tw}(K_{n,n}) \geq n$.

The *clique-width* of a graph G is the minimum number of labels needed to construct G using the following four operations:

(i) Create a new vertex v with label i (denoted $i(v)$).

(ii) Form the disjoint union of two labeled graphs G and H (denoted $G \oplus H$).

(iii) Join all vertices with label i to all vertices with label j ($i \neq j$, denoted $\eta_{i,j}$).

(iv) Change the label of all vertices with label i to j (denoted $\rho_{i \to j}$).

Every graph can be defined by an algebraic expression using these four operations. For instance, the cycle on five consecutive vertices a, b, c, d, e can be defined as follows:

$$\eta_{4,1}(\eta_{4,3}(4(e) \oplus \rho_{4\to 3}(\rho_{3\to 2}(\eta_{4,3}(4(d) \oplus \eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))))).$$

Such an expression is called a *k-expression* if it uses at most k different labels. Thus the clique-width of G, denoted $\mathrm{cw}(G)$, is the minimum k for which there exists a k-expression defining G. For instance, the above expression shows that $\mathrm{cw}(C_5) \leq 4$. Moreover, one can easily determine that $\mathrm{cw}(C_n) \leq 4$ for any n, and $\mathrm{cw}(T) \leq 3$ for any tree T. With some more involved analysis it has been shown that the clique-width is bounded in the class of distance-hereditary graphs [7] (generalizing trees), $S_{1,2,3}$-free bipartite graphs [8] (which includes bi-complement reducible graphs [6] and bipartite graphs totally decomposable by canonical decomposition [5]) and in the class of $(S_{2,2,2}, A)$-free bipartite graphs [1] (A is the graph obtained from a P_6 by connecting two vertices of degree 2 of distance 3 by an edge). On the other hand, the clique-width is known to be unbounded in the class of general bipartite graphs, since it is unbounded for grids [7]. Moreover, it is unbounded for chordal bipartite graphs (i.e. bipartite graphs without induced cycles of length at least 6) [2] and even for bipartite permutation graphs [3].

Many interesting relations between tree- and clique-width have been revealed by Courcelle and Olariu in [4], among which the following two results will be of particular interest in this paper.

Proposition 1 *For any class of graphs X, there is an integer function f such that* $\mathrm{tw}(X) \leq f(deg(X), \mathrm{cw}(X))$.

Proposition 2 *For any graph G,* $\mathrm{cw}(G) \leq 2^{2\mathrm{tw}(G)+2} + 1$.

Also, Courcelle and Olariu proved that for any graph G,

$$\mathrm{cw}(\overline{G}) \leq 2\mathrm{cw}(G). \tag{1}$$

A relation similar to inequality (1) has been established in [9] between a bipartite graph and its bipartite complement.

Proposition 3 *If G is a bipartite graph, then* $\mathrm{cw}(\widetilde{G}) \leq 4\mathrm{cw}(G)$.

It has been shown in [4] that the clique-width of a graph G cannot be less than the clique-width of any of its induced subgraphs. This allows us to study only *hereditary* classes of graphs, i.e. those containing with each graph G every graph isomorphic to an induced subgraph of G. If a class of graphs X is not hereditary, we can extend it to a hereditary class by adding to X all induced subgraphs of the graphs belonging to X. It is well known that a class of graphs is hereditary if and only if it can be characterized by a set of forbidden induced subgraphs. For instance, the class of bipartite graphs is precisely the class of $(C_3, C_5, C_7, \ldots)$-free graphs. If X is a proper subclass of bipartite graphs, we shall describe it only by forbidden induced *bipartite* subgraphs: these are exactly minimal bipartite graphs that are not in X.

In the study of clique-width we can obviously be restricted to connected graphs, since a disconnected graph can be obtained from its connected components with the $\oplus$-operation. Moreover, since our objective is to distinguish classes with bounded/unbounded clique-width, we may consider only those (bipartite) graphs whose (bipartite) complement is connected. A more general proposition shown in [4] states that

Proposition 4 *For any graph G,*

$$\mathrm{cw}(G) = \max\{\mathrm{cw}(H) \mid H \text{ is a prime induced subgraph of } G\}.$$

The notion of *prime* graph was introduced in the study of *modular* decomposition [10]. Restricted to bipartite graphs it can be defined as follows: a bipartite graph G is prime if G is connected and every two distinct vertices of G have different neighborhoods.

We will also need the following proposition proven in [2].

Proposition 5 *For a class of graphs Y, let $[Y]_k$ be the class of graphs G such that $G - U$ belongs to Y for a subset $U \subseteq V(G)$ of cardinality at most k. If Y is a class of graphs of bounded-clique width, then so is $[Y]_k$.*

3 Subclasses of bipartite graphs defined by finitely many forbidden induced bipartite subgraphs

We begin by establishing the following simple result.

Lemma 1 *Subdivision of an edge of a graph G does not change the tree-width of G.*

Proof. Subdivision of an edge cannot decrease the tree-width, since the tree-width of a graph is at least the tree-width of any of its minors.

Now let us show that subdivision of an edge cannot increase the tree-width. If the tree-width is 0, there is nothing to subdivide. If the tree-width is 1, then we have a forest which is a forest again after subdivision. Now let G be a graph of tree-width at least 2 and let G_T denote a minimal triangulation of G such that $\mathrm{tw}(G) = \omega(G_T) - 1$. Subdivide an edge uv of G by introducing on it a new vertex w, and denote the new graph by G'. Finally let G'_T be the graph obtained from G_T by adding a new vertex w adjacent to u and v only. Clearly G'_T is a triangulation of G' and hence $\mathrm{tw}(G') \leq \omega(G'_T) - 1$. In addition, $\omega(G'_T) = \omega(G_T)$, since $\omega(G_T) \geq 3$. Therefore, $\mathrm{tw}(G') \leq \mathrm{tw}(G)$. ■

Lemma 2 *For any integers k and l, the tree- and clique-width of graphs in the class $X_k \cap Y_l \cap Z_3 \cap B$ is unbounded.*

Proof. We will show the unboundedness of the tree-width only. The unboundedness of the clique-width will follow then from Proposition 1.

The tree-width is unbounded in the class of so-called *walls* (cf. [11]), which are planar graphs of degree at most 3 (see Figure 2). Subdividing each edge of a wall once, we increase the length of each cycle twice, and hence obtain a bipartite graph. We repeat this operation as many times as needed to get rid of induced cycles C_i with $i \leq k$ and induced graphs of the form H_i with $i \leq l$. The graph obtained in this way belongs to the class $X_k \cap Y_l \cap Z_3 \cap B$ and the tree-width of this graph coincides with the tree-width of the initial wall according to Lemma 1. Hence the lemma. ■

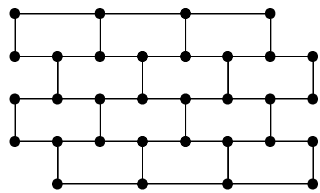

Figure 2: A Wall

Theorem 1 *Let X be a class of bipartite graphs defined by a finite set F of forbidden induced bipartite subgraphs. If F contains no graph in $\mathcal{S}$ or no graph the bipartite complement of which is in $\mathcal{S}$, then the tree- and clique-width of graphs in X is unbounded.*

Proof. We will show the unboundedness of the clique-width. Together with Proposition 2 this will imply the same conclusion for the tree-width.

Concerning the clique-width, Proposition 3 allows us to restrict ourselves to the assumption that $F \cap \mathcal{S} = \emptyset$.

Denote by k an integer greater than the number of vertices in a largest graph in F. To prove the theorem, we will show that $X_k \cap Y_k \cap Z_3 \cap B \subset X$. By contradiction, assume that a graph $G \in X_k \cap Y_k \cap Z_3 \cap B$ does not belong to X. Then G must contain a graph $A \in F$ as an induced subgraph. Since G belongs to X_k, the graph A contains no induced cycles C_j of length $j \leq k$. Moreover, A cannot contain a cycle C_j with $j > k$, because the number of vertices of A is less than k due to the choice of k. Therefore, A contains no cycles, i.e. A is a forest. Analogously, since $G \in Y_k$ and $|V(A)| < k$, the graph A contains no induced subgraphs of the form H_i, i.e. every connected component of A has at most one vertex of degree at least 3. Finally, since $G \in Z_3$, A has no vertices of degree more than 3. But then every connected component of A is of the form $S_{i,j,k}$, a contradiction. ■

The result of Theorem 1 provides a necessary condition for the tree- and clique-width to be bounded in a class of bipartite graphs defined by finitely many forbidden induced bipartite subgraphs. However, this condition is not sufficient. Indeed, the class of $(S_{2,2,2}, C_6)$-free bipartite graphs satisfies the condition of Theorem 1, since $S_{2,2,2} \in \mathcal{S}$ and $\widetilde{C}_6 \in \mathcal{S}$. However, this class contains all bipartite permutation graphs the clique-width (and hence the tree-width) of which is generally unbounded [3].

Another important observation is that the condition of Theorem 1 is necessary only for classes defined by *finitely* many forbidden induced bipartite subgraphs. Indeed, for the class of chordal bipartite graphs of vertex degree at most k, i.e. $(K_{1,k+1}, C_6, C_8, C_{10} \ldots)$-free bipartite graphs, the set of forbidden induced bipartite subgraphs is infinite and for $k > 2$ it contains no graphs in $\mathcal{S}$. However, the tree- and clique-width of graphs in this class has been shown to be bounded for any natural k [9].

4 Monogenic classes of bipartite graphs

Let us now apply Theorem 1 to *monogenic* classes of bipartite graphs, i.e. classes defined by a single forbidden induced bipartite subgraph. We denote the only forbidden subgraph by H. Together with observation that the tree-width of complete bipartite graphs is unbounded Theorem 1 provides a complete classification of H-free bipartite graphs with respect to bounded or unbounded tree-width. Indeed, in order the tree-width to be bounded in a class of H-free bipartite graphs, the graph H must be complete bipartite (otherwise the class contains all complete bipartite graphs) and must belong to $\mathcal{S}$. It is not hard to see that among complete bipartite graphs there is only one maximal graph belonging to $\mathcal{S}$, namely $K_{1,3}$. Every connected graph in the class of $K_{1,3}$-free bipartite graphs is either a cycle or a path. Therefore, the tree-width of H-free bipartite graphs is bounded if and only if H is $K_{1,3}$ or one of its subgraphs.

The analysis of the clique-width of bipartite graphs in monogenic classes is more complicated. By Theorem 1 the clique-width of H-free bipartite graphs is bounded only if both $H \in \mathcal{S}$ and $\widetilde{H} \in \mathcal{S}$. An example of a graph with this property is $S_{1,2,3}$, which is self-complementary in the bipartite sense. As mentioned before, the clique-width of $S_{1,2,3}$-free bipartite graphs is bounded [8]. Other examples of a graph H with $H \in \mathcal{S}$ and $\widetilde{H} \in \mathcal{S}$ are represented in Figure 3. Notice that none of these graphs is an induced subgraph of $S_{1,2,3}$. Therefore, the problem of determining whether the clique-width of $\mathcal{A}_j$-free graphs is bounded or not is open for any $j = 1, \ldots, 8$. In this paper we settle this problem for $j = 1$ and $j = 2$.

Theorem 2 *The clique-width of $\mathcal{A}_1$-free bipartite graphs is bounded.*

Proof. Let G be a connected prime $\mathcal{A}_1$-free bipartite graph G and a be a vertex in G. Denote by A_j the subset of vertices at distance j from a. To avoid an induced $\mathcal{A}_1$, we conclude that $A_j = \emptyset$ for any $j > 3$, every vertex in A_2 has at most two neighbors in A_3 and at most two non-neighbors in A_1. This implies that if A_3 has a vertex of degree at least 3, then $|A_3| < 5$. Indeed, assume that a vertex $b \in A_3$ has at least 3 neighbors c_1, c_2, c_3 in A_2 and suppose by contradiction that A_3 contains at least 4 vertices other than b. Then each of those vertices must have a neighbor in $\{c_1, c_2, c_3\}$ (else an induced $\mathcal{A}_1$ arises), and hence one of c_1, c_2, c_3 has at least three neighbors in A_3, contradicting the above conclusion. Complementary arguments show that if a vertex of A_1 has at least 3 non-neighbors in A_2, then $|A_1| < 5$. This discussion together with Proposition 5 permits to assume without loss of generality that every vertex of A_3 has degree at most 2, and every vertex of A_1 has at most two non-neighbors in A_2. In other words, both the subgraph $G_2 := G[A_2 \cup A_3]$ and the bipartite complement of the subgraph $G_1 := G[A_1 \cup A_2]$ are of maximum vertex degree at most 2. Besides, without loss of generality we may suppose that both $|A_1| > 1$ and $|A_3| > 1$.

Now let us show that there are finitely many vertices of A_2 that have degree 2 in G_2 or in $\widetilde{G}_1$. To prove this, consider three vertices $b_1, b_2, b_3 \in A_2$ each of which has two neighbors in A_3, and assume that $N(b_i) \cap N(b_j) \cap A_3 = \emptyset$ for any $i \neq j$. Then each

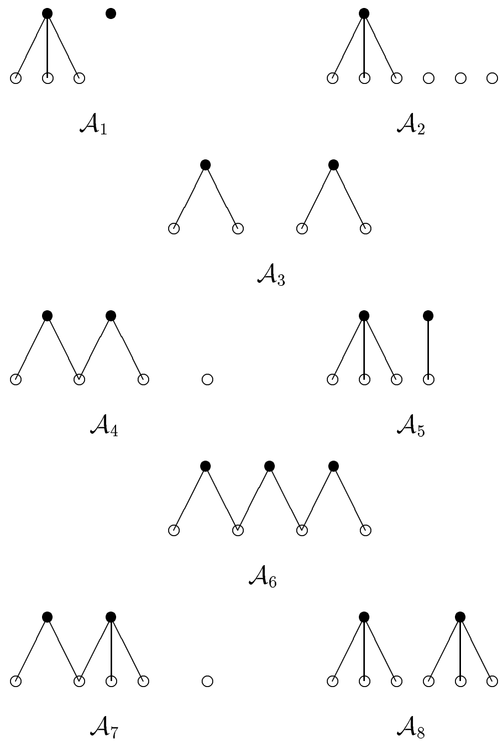

Figure 3: Graphs $\mathcal{A}_1, \ldots, \mathcal{A}_8$

of b_1, b_2, b_3 is adjacent to each vertex in A_1. Indeed, if b_1 is not adjacent to $c \in A_1$, then c must be adjacent to b_2 (or to b_3 else c has three non-neighbors in A_2), but then the vertices b_1, b_2, c and the two neighbors of b_2 in A_3 induce an $\mathcal{A}_1$. To be a prime graph, G must have a vertex $b_0 \in A_2$ that has both a neighbor and a non-neighbor in A_1. Denote a non-neighbor of b_0 in A_1 by c. Since b_0 has at most two neighbors in A_3, $N(b_0) \cap N(b_j) \cap A_3 = \emptyset$ for some j, say for $j = 2$. But now the vertices b_0, b_2, c and the two neighbors of b_2 in A_3 induce an $\mathcal{A}_1$. This contradiction shows that the graph G_2 cannot contain in A_2 three vertices of degree two with pairwise disjoint neighborhoods. Since every connected component of G_2 is either a cycle or a path, we conclude that only finitely many vertices of A_2 have degree 2 in G_2. Applying these arguments to the bipartite complement of G, we also conclude that only finitely many vertices of A_2 have degree 2 in $\widetilde{G}_1$. Therefore, we have reduced the problem in question to a subgraph G' of G induced by three independent sets $A_1' \subseteq A_1$, $A_2' \subseteq A_2$ and $A_3' \subseteq A_3$ with the property that in the subgraph $G_2' := G[A_2' \cup A_3']$ every vertex of A_3' is of degree at most 2 and every vertex of A_2' is of degree at most 1, and similarly in the subgraph $\widetilde{G}_1' := \widetilde{G}[A_1' \cup A_2']$ every vertex of A_1' is of degree at most 2 and every vertex of A_2' is of degree at most 1. In order to build an expression defining G' using finitely many different labels, we construct an auxiliary graph G" obtained from G' by complementing only the subgraph G_1' of G. The clique-width of G" is bounded, since every connected component of G" is either a cycle or a path. The subgraph of G' corresponding to a connected component C of G" can be created in a way similar to the creation of C with the difference that for the vertices in A_1', A_2' and A_3' we use disjoint sets of labels, and the operation of creation of an edge of C between a vertex in A_1' and a vertex in A_2' is replaced by the operation of creation of all other possible edges between A_1 and A_2. ■

Theorem 3 *The clique-width of $\mathcal{A}_2$-free bipartite graphs is bounded.*

Proof. Let $G = (W, B, E)$ be an $\mathcal{A}_2$-free bipartite graph. Let $W_l = \{u \in W \mid \deg(u) \leq 2\}$, $W_h = W \setminus W_l$, $B_l = \{u \in B \mid \deg(u) \leq 2\}$, $B_h = B \setminus B_l$.

Clearly, every vertex of G is adjacent either to at most two or to all but at most two of the vertices in the opposite part. Therefore, the number of edges of G between W_l and B_h is at most $2|W_l|$ and at least $|B_h|(|W_l| - 2)$. Thus $|B_h|(|W_l| - 2) \leq 2|W_l|$ which implies that either $|W_l| \leq 4$ or $|B_h| \leq 4$. Similarly, either $|B_l| \leq 4$ or $|W_h| \leq 4$. Therefore, by deleting from G at most 8 vertices, one can obtain a graph G' such that either G' or its bipartite complement is of vertex degree at most 2. In either case, the clique-width of G' is bounded, and so is the clique-width of G by Proposition 5. ■

In the rest of the section we describe the area of uncertainty for the clique-width of graphs in monogenic classes by means of three minimal and three maximal classes. To simplify our discussion, we denote by $G < G'$ the fact that G' contains G as an induced subgraph.

Proposition 6 *Let X be a class of H-free bipartite graphs such that*

(1) $H \in \mathcal{S}$ and $\widetilde{H} \in \mathcal{S}$,

(2) $H \not< S_{1,2,3}$ and $H \not< \mathcal{A}_1$ and $H \not< \mathcal{A}_2$,

then $\mathcal{A}_i < H$ for some $i \in \{3,4,5\}$ and $H < \mathcal{A}_j$ for some $j \in \{6,7,8\}$.

Proof. Clearly H is $S_{2,2,2}$-free, since the bipartite complement of $S_{2,2,2}$ contains a C_6. Assume H contains a P_7 induced by vertices $1,2,3,4,5,6,7$ with odd vertices being white. Then $\mathcal{A}_3 < P_7 < H$. Let us show that $H = P_7$. By contradiction, suppose H contains a vertex x outside the P_7. If x is black, then it must be adjacent to 1 or 3 (else $\widetilde{H}[1,3,6,x] = C_4$) and symmetrically to 5 or 7. But then H contains a cycle and hence $H \notin \mathcal{S}$. Analogously, if x is white then either H or $\widetilde{H}$ contains a cycle. From now on, H is P_7-free.

Suppose H contains a P_6 as an induced subgraph. Then H is connected, since otherwise $C_4 < \widetilde{H}$. In other words, $H = S_{i,j,k}$ for some $i \leq j \leq k$. This implies that $i \leq 1$ (else $S_{2,2,2} < H$), $j \leq 2$ (else $P_7 < H$) and hence $k \geq 4$ (else $H < S_{1,2,3}$). But then either $C_4 < \widetilde{H}$ (if $i > 0$) or $H = P_6 < S_{1,2,3}$ (if $i = 0$). This contradiction shows that H is P_6-free. Since each of the graphs $\mathcal{A}_3, \dots, \mathcal{A}_8$ is self-complementary in the bipartite sense, we may assume also that $\widetilde{H}$ is P_6-free.

Let now H contain a P_5 with the central vertex being white, and denote the component of H containing the P_5 by H_1. Since H is $(P_6, S_{2,2,2})$-free, either $H_1 = S_{i,1,3}$ or $H_1 = S_{i,2,2}$ with $i \leq 1$. In either case $H_1 < S_{1,2,3}$ and hence H must contain at least one more connected component. To avoid a C_4 in the bipartite complement to H we conclude that H contains at most one white vertex w and at most one black vertex b outside H_1. If both w and b are present in H, then $\widetilde{H}$ contains either C_4 or C_6 depending on adjacency of w to b. If H contains only b, then $H_1 = P_5$ (else $C_4 < \widetilde{H}$), but then $H < S_{1,2,3}$. Now let H contain w but not b. If H_1 has no vertex of degree 3 (i.e. if $H_1 = P_5$), then $\mathcal{A}_4 = H < \mathcal{A}_6$. If H_1 has a vertex of degree 3, then $H_1 = S_{1,1,3}$ (else $C_4 < \widetilde{H}$) and hence $\mathcal{A}_4 < H = \mathcal{A}_7$.

From now on, both H and $\widetilde{H}$ are P_5-free. Therefore, every connected component of H is an induced subgraph of $S_{1,1,2}$, which means that H has at least two connected components (else $H < S_{1,2,3}$).

If H contains two non-trivial (of size at least 2) connected components H_1 and H_2, then $H = H_1 + H_2$ (else $\widetilde{P}_5 < H$) and both components are P_4-free (else $\widetilde{P}_6 < H$). Therefore, $H < \mathcal{A}_8$. Moreover, since $H \not< S_{1,2,3}$, we have either $\mathcal{A}_3 < H$ or $\mathcal{A}_5 < H$.

Finally, let H contain only one non-trivial connected component H_1. Then $H_1 = S_{1,1,1}$ (else either $H < S_{1,2,3}$ or $C_4 < \widetilde{H}$). Denote the central vertex of H_1 by b and let b be black. Then H contains no black isolated vertex. Indeed, assuming that H contains a black isolated vertex b_1 we conclude that H contains no other black vertices (else $C_4 < \widetilde{H}$) and no white isolated vertices (else the degree of b_1 in $\widetilde{H}$ would be more than 3). But then $H = \mathcal{A}_1$. Therefore, H contains no black isolated vertices. This implies that H has no white isolated vertices, since otherwise either $H < \mathcal{A}_2$ or the degree of b in $\widetilde{H}$ is greater than 3. But now $H < S_{1,2,3}$, a contradiction. ■

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