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On large cycles with lengths differing by one or two

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Abstract

Let $k \geq 3$ be an integer and G a graph with at most two vertices of degree less than k. We show that G contains two cycles of order at least k such that their lengths differ by one or two, unless G has at most two vertices. This generalizes the result of Bondy and Vince [J.A. Bondy and A. Vince, Cycles in a graph whose lengths differ by one or two, *J. Graph Theory* 27 (1998), 11–15].

1 Introduction

Bondy and Vince proved that if G is a graph with at most two vertices of degree less than 3, then G contains two cycles whose lengths differ by one or two, unless G has at most two vertices. We ask when G has two cycles of order at least k for a given integer $k \geq 3$ such that their lengths differ by one or two. By observing mK_k (the union of m vertex-disjoint copies of K_k), we need G to have vertices of degree at least k for a degree condition. In this note, we will prove the following.

Theorem *Let $k \geq 3$ be an integer and G a graph with at most two vertices of degree less than k. Then G contains two cycles of order at least k such that their lengths differ by one or two, unless G has at most two vertices.*

We discuss only finite simple graphs and use standard terminology and notation from [1] except as indicated. Let G be a graph. For a vertex $u \in V(G)$ and a subgraph H of G, $N(u, H)$ is the set of neighbors of u contained in H. We let $d(u, H) = |N(u, H)|$. Thus $d(u, G)$ is the degree of u in G. If $d(u, G) = 1$, we say that u is an endvertex of G. We use $d_G(u, v)$ to denote the distance of two vertices u and v in G. If G is a cycle or path, we use $l(G)$ to denote the length of G.

2 Proof of the Theorem

Let $k \geq 3$ be an integer and G a graph of order n with at most two vertices of degree less than k. Clearly, if G has more than two vertices, then $n \geq k + 1$. The theorem obviously holds for the graphs of order not larger than $k + 1$. The proof of the theorem is by contradiction. Assume that G is a counter example with the smallest order. Thus $n \geq k+2$. We shall derive a contradiction. To do so, we will adopt the idea from [2]. We will also use the following two lemmas from [3] and [4], respectively.

Lemma 2.1 (Theorem 1 of [3]) *Let x and y be two distinct vertices in a 2-connected graph H. Suppose that the average degree of the vertices other than x and y is r. Then H contains a path of length at least $\lceil r \rceil$ from x to y.*

Lemma 2.2 [4] *If H is a graph of order $m \geq 3$ such that $d(x)+d(y) \geq m+1$ for each pair of non-adjacent vertices x and y of H, then H is hamiltonian connected.*

By Bondy and Vince's result, $k \geq 4$. Clearly, if G was not 2-connected, then an appropriate block of G would satisfy the condition of the theorem and therefore by the minimality of G, the block would contain two required cycles, a contradiction. Hence G is 2-connected. We choose an induced cycle C of G such that

$$\textit{The order of a largest component of } G - V(C) \textit{ is maximum.} \tag{1}$$

We claim

$$G - V(C) \textit{ has exactly one component.} \tag{2}$$

Proof of (2). On the contrary, suppose that $G - V(C)$ has at least two components. Let B be a largest component of $G - V(C)$ and F be a component $G - V(C)$ with $F \neq B$. As G is 2-connected, there exist two distinct vertices x and y of C such that $d(x,B) > 0$ and $d(y,B) > 0$. There are also two distinct vertices x' and y' of C such that $d(x',F) > 0$ and $d(y',F) > 0$. If $\{x,y\} \neq \{x',y'\}$, say $x \notin \{x',y'\}$, then $G[V(C \cup F)] - x$ contains a cycle C' and $G - V(C')$ has a component containing $B + x$, contradicting (1). Hence we must have that $\{x,y\} = \{x',y'\}$. Let z be an arbitrary vertex in $V(C) - \{x,y\}$. We claim that $N(z,G) = N(z,C)$ and so $d(z,G) = 2$ for all $z \in V(C) - \{x,y\}$. If this is not true, let $u \in V(C) - \{x,y\}$ be such that $d(u, G - V(C)) \geq 1$. As G is 2-connected, there exists a component H of $G - V(C)$ and a vertex $v \in V(C)$ such that $u \neq v$, $d(u,H) \geq 1$ and $d(v,H) \geq 1$. If $H \neq B$, the above argument shows that $\{x,y\} = \{u,v\}$, a contradiction. If $H = B$, then $\{u,v\} = \{x',y'\}$, a contradiction. Therefore $N(z,G) = N(z,C)$ for all $z \in V(C) - \{x,y\}$. As G has at most two vertices of degree less than k, we see that either $B + x + y$ or $F + x + y$ satisfies the condition of the theorem. By the minimality of G, one of $B + x + y$ and $F + x + y$ contains two required cycles, a contradiction. So (2) holds.

Choose the cycle C as defined above and let B be the unique component of $G - V(C)$. Assume for the moment that $4 \leq k \leq 5$. As C contains at most two vertices of degree less than k in G, we see that there exist two distinct vertices x and y on C and two distinct vertices u and v in B such that $\{xu, yv\} \subseteq E$ and the lengths of two different paths P_1 and P_2 of C from x to y differ by one or two, unless $l(C) = 4$ and C has two non-consecutive vertices whose degrees are two in G. In the latter case, we delete the two non-consecutive vertices from G to obtain a subgraph

G' of G. Clearly, G' satisfies the condition of the theorem and therefore it contains two required cycles by the minimality of G, a contradiction. So the former case must hold. Then we choose four vertices u, v, x and y such that the length of a longest path L from u to v in B is maximum. Clearly, $L \cup C + xu + yv$ has two cycles of lengths at least 4 and their lengths differ by one or two. Therefore $k \neq 4$ and so $k = 5$ by the assumption on G. Then we see that $l(C) \leq 4$ and $l(L) = 1$ for otherwise the two cycles have lengths at least 5 and therefore G is not a counterexample as assumed, a contradiction. Thus the edge of L must be a cut-edge of B. Let B_u and B_v be the two components of $B - uv$ with $u \in V(B_u)$ and $v \in V(B_v)$. By the choice of x, y, u and v, we see that for each $w \in V(B_u) - \{u\}$, it holds that $d(w, C - y) = 0$ and for each $z \in V(B_v) - \{v\}$, it holds that $d(z, C - x) = 0$. For the same reason, we see that either $d(x, B_v - v) = 0$ or $d(y, B_u - u) = 0$. Without loss of generality, say the former holds. Then $d(z, G) \leq 4$ for all $z \in V(C) - \{y\}$. Since G has at most two vertices of degree less than 5, we see that $l(C) = 3$ and $d(v, B_v) \geq 1$. Clearly, $d(z, B_v) \geq k$ for all $z \in V(B_v) - \{v\}$. Thus B_v satisfies the condition of the theorem for $k = 5$. By the minimality of G, B_v contains two required cycles, a contradiction. Therefore $k \geq 6$.

Let I be the set of non-cut vertices of B and σ be the set of vertices of G with degree less than k. Then $B - x$ is connected for all $x \in I$ and $|\sigma| \leq 2$. We claim

$$\textit{If } l(C) \geq 4, \textit{ then } d(x, C) \leq 2 \textit{ for all } x \in I. \tag{3}$$

Proof of (3). On the contrary, suppose that $l(C) \geq 4$ and $d(x_0, C) \geq 3$ for some $x_0 \in I$. Let u and v be two distinct vertices in $N(x_0, C)$ with $d_C(u, v)$ as small as possible. First, suppose that $d_C(u, v) = 1$. Then $C' = x_0uvx_0$ is an induced cycle in G. By (1), $d(y, B - x_0) = 0$ for all $y \in V(C) - \{u, v\}$. Thus $\sigma \supseteq V(C) - \{u, v\}$. As $|\sigma| \leq 2$, we see that $l(C) = 4$ and $\sigma = V(C) - \{u, v\}$. Let $C = uvyzu$ be such that $yx_0 \in E$. Then x_0vyx_0 is a triangle in G and similarly, we must have $u \in \sigma$, a contradiction. Hence we must have that $d_C(u, v) \geq 2$. Let P be the shortest path from u to v on C. Then $C'' = P + ux_0 + x_0v$ is an induced cycle in G and $V(C) - V(P)$ has at least three distinct vertices. As $|\sigma| \leq 2$, there exists $y \in V(C) - V(P)$ with $y \notin \sigma$ such that $d(y, B - x_0) > 0$. Thus $G - V(C'')$ is connected, contradicting (1). So (3) holds.

We now choose a block B' from B as follows. If B is a block, we define $B' = B$. Otherwise, let S be the set of vertices x of B with $d(x, B) = 1$. By (3) and the fact that $k \geq 6$, we see that $S \subseteq \sigma$ and so $|S| \leq 2$. If $|S| < 2$, let B' be an endblock of B with $|B'| \geq 3$. Furthermore, we can choose B' such that if x_0 is the cut vertex of B with $x_0 \in B'$ then $|\sigma \cap V(B' - x_0)| \leq 1$. If $|S| = 2$, then B has exactly two endblocks. Moreover, they are of order 2. In this case, let x_0x_1 be an endblock of B such that x_1 is the cut vertex of B. Then we define B' to be the block of B such that $B' \neq x_0x_1$ but $x_1 \in V(B')$. We can see that B' is of order at least 3 as follows. As $k \geq 6$ and by (3), we see that σ consists of the two endvertices of B. If B' is of order 2, then $d(x_1, C) \geq 4$ and $d(w, B) \geq 4$ where $w \in V(C)$ and $wx_0 \in E$. Then we see that $C + x_1 - w$ contains a cycle C' such that $C - V(C')$ is connected with at least two vertices. Thus $G - V(C')$ is connected, contradicting (1).

For any two distinct vertices u and v of B', let P_{uv} be a longest path from u to v in B'. Let $t = |V(B')| - 2$. For each pair of distinct vertices u and v in B', we claim

$$\textit{If } l(C) \geq 4, \textit{ then } l(P_{uv}) \geq k - 3; \tag{4}$$

$$\textit{If } l(C) = 3, \textit{ then } l(P_{uv}) \geq k - 4. \tag{5}$$

Proof of (4) *and* (5). By the choice of B', we see that there are two vertices x' and x'' in B such that $V(B') - \{x', x''\} \subseteq I$ and $d(x, G) \geq k$ for each $x \in V(B') - \{x', x''\}$. Set

$$r_{uv} = \frac{1}{t} \sum_{x \in V(B') - \{u,v\}} d(x, B').$$

First, suppose that $l(C) \geq 4$. By (3), $d(x, B') \geq k - 2$ for all $x \in V(B') - \{x', x''\}$. Thus $t \geq k - 3$ and $r_{uv} \geq ((t-2)(k-2)+4)/t$. If $l(P_{uv}) < k - 3$, then $r_{uv} \leq k - 4$ by Lemma 2.1. This implies that $t \leq k - 4$, a contradiction.

Next, suppose that $l(C) = 3$. Then $d(x, B') \geq k - 3$ for all $x \in V(B') - \{x', x''\}$. Thus $t \geq k - 4$ and $r_{uv} \geq ((t-2)(k-3)+4)/t$. If $l(P_{uv}) < k - 4$, then $r_{uv} \leq k - 5$ by Lemma 2.1. This implies that $t \leq k - 5$, a contradiction. So (4) and (5) hold.

We are now in the position to complete the proof of the theorem. For each $w \in V(C)$, we let A_w be the set of vertices y of C such that the lengths of the two paths from w to y on C differ by one or two. It is easy to see that $|A_w| = 2$ for all $w \in V(C)$. We divide the proof into the following three cases.

Case 1. $B' = B$.

As G is 2-connected and $|\sigma| \leq 2$, we readily see that there exist $w \in V(C)$ and $z \in A_w$ and two distinct vertices u and v in B such that $\{wu, zv\} \subseteq E$. If $l(P_{uv}) \geq k-3$, then G contains two required cycles, a contradiction. So $l(P_{uv}) \leq k-4$. By (4) and (5), we see that $l(C) = 3$ and $l(P_{uv}) = k - 4$. Then $r_{uv} \leq k - 4$. This implies that $t \leq 2k-10$. Thus $n \leq 2k-5$. Let y_1 and y_2 be two distinct vertices of G such that $d(x, G) \geq k$ for all $x \in V(G) - \{y_1, y_2\}$. Then $k \leq n-2 = |V(G')| \leq 2k-7$ and $\delta(G') \geq k-2$ where $G' = G - y_1 - y_2$. By Lemma 2.2, G' is hamiltonian connected. Clearly, G has two required cycles, a contradiction.

Case 2. $B' \neq B$ and B' is an endblock of B.

Let x_0 be the cut vertex of B with $x_0 \in B'$. As G is 2-connected, there exists $w \in V(C)$ such that $d(w, B' - x_0) > 0$. Let $u \in V(B' - x_0)$ be such that $wu \in E$. We divide this case into the following two cases.

Case 2.1. There exists $z \in A_w$ such that $d(z, B - u) > 0$.

In this subcase, there exists a path L of G from z to a vertex v of B' with $v \neq u$ such that no internal vertex of L is in $C \cup B'$. We choose such a path L with $l(L)$ as large as possible. Clearly, if $l(P_{uv}) \geq k - 3$, or $l(P_{uv}) = k - 4$ and $l(L) \geq 2$, then the theorem holds, a contradiction. By (4) and (5), we must have that $l(C) = 3$, $l(P_{uv}) = k - 4$ and $l(L) = 1$. Say $C = w_1w_2w_3w_1$ with $w = w_1$. This argument implies that $N(w_2, B) \cup N(w_3, B) \subseteq V(B')$. Let B'' be another endblock of B. As G is 2-connected, there exists a vertex y of B'' such that $w_1y \in E$ and $y \in I$. Suppose that either $d(w_2, B' - x_0) > 0$ or $d(w_3, B' - x_0) > 0$. Say without loss of generality the former holds. Let $x \in V(B' - x_0)$ be such that $w_2x \in E$. By (5), we see that there exists a path L' of B from x to y such that $x_0 \in V(L')$ and $l(L') \geq k - 3$, a contradiction. Therefore we must have that $d(w_2, B' - x_0) = 0$ and $d(w_3, B' - x_0) = 0$. We obtain that $\sigma = \{w_2, w_3\}$. Then $B' + w_1$ has at most two vertices of degree less than k. By the minimality of G, $B' + w_1$ contains two required cycles, a contradiction.

Case 2.2. $d(z, B - u) = 0$ for each $z \in A_w$.

In this subcase, we have that $\sigma = A_w$. Let B'' be another endblock of B. As G is 2-connected, there exist a vertex y of B'' and a vertex w_1 of C such that $w_1y \in E$ and $y \in I$. As B'' can play the role of B' in the above argument, we see that $\sigma = A_{w_1}$, i.e., $w = w_1$, and $d(z, B - y) = 0$ for each $z \in A_w$. So $d(z, B) = 0$ for each $z \in A_w$. Then $G - A_w$ has at most two vertices of degree less than k. By the minimality of G, $G - A_w$ contains two required cycles, a contradiction.

Case 3. $B' \neq B$ and B' is not an endblock of B.

By the choice of B', let x_0x_1 be an endblock with $x_1 \in V(B')$. Let y_0 be the other endvertex of B. By (3), $d(x_0, G) \leq 3$ and $d(y_0, G) \leq 3$. So $\sigma = \{x_0, y_0\}$. Let $w \in V(C)$ be such that $wx_0 \in E$. Let $z \in A_w$. As $d(z, G) \geq k$, there exists $v \in V(B) - \{x_0, x_1\}$ such that $zv \in E$. By (4) and (5), we see that B has a path L'' from x_0 to v such that $l(L'') \geq k - 3$, and it follows that G has two required cycles, a contradiction. This completes the proof of the theorem.

Remark. For each integer $k \geq 3$, let $h(k)$ be the largest integer such that if G is a graph of order at least $k + 1$ with at most $h(k)$ vertices of degree less than k, then G contains two cycles of order at least k such that their lengths differ by one or two. From our theorem, we see that $h(k) \geq 2$. By observing the graph $K_{\lfloor k/2 \rfloor} + (\lceil k/2 \rceil + 1)K_1$, we see that $h(k) \leq \lceil k/2 \rceil$. So $h(3) = 2$. We conjecture that $h(k) = \lceil k/2 \rceil$.

References

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