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# On incomparable and uncomplemented families of sets

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**Abstract**

In 1977, A. J. W. Hilton proposed the following conjecture (see D.J. Kleitman, Math. Review 53#146, 1977): if $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_k$ are collections of distinct subsets from an $n$-element set such that these collections are incomparable and uncomplemented, then $\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1}$. In this paper we try to verify this conjecture for some cases. In particular, we provide a new bound:

$$\sum_{i=1}^{k} |\mathcal{A}_i| < (4 - 2\sqrt{2})2^{n-1},$$

which improves several results in [7]. Also, under some fairly general conditions, we show that

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq (1 + \frac{1}{k})2^{n-1}.$$

## 1 Introduction and Main Results

Let $\mathcal{A}_1, \cdots, \mathcal{A}_k$ be $k$ collections of distinct subsets of set $[n] = \{1, 2, \ldots, n\}$. These $k$ collections of distinct subsets are called *incomparable* if, when $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$, $(i \neq j)$, then $A_i \not\subset A_j$. A collection of subsets $\mathcal{C}$ is called *uncomplemented* if, when $A \in \mathcal{C}$, then $\bar{A} \notin \mathcal{C}$, where $\bar{A} = [n] \backslash A$.

It is well known that if $\mathcal{C}$ is a collection of distinct subsets of set $[n]$ which are uncomplemented, then $|\mathcal{C}| \leq 2^{n-1}$. Hilton extended this result to two incomparable, uncomplemented collections.

**Theorem 1** *[3] If $\mathcal{A}_1$ and $\mathcal{A}_2$ are collections of distinct subsets of set $[n]$ such that these collections are incomparable and uncomplemented, then*

$$|\mathcal{A}_1| + |\mathcal{A}_2| \leq 2^{n-1}.$$

D. J. Kleitman [6] also proved the above result using a correlation inequality from [5]. In [6], he also pointed the following conjecture proposed by Hilton.

**Conjecture 1** *[6] If $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ are collections of distinct subsets of set $[n]$ such that these collections are incomparable and uncomplemented, then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1}.$$

In this paper, we will prove the following results in Section 2.

**Theorem 2** *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ be incomparable and uncomplemented collections of distinct subsets of set $[n]$. Then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| < (4 - 2\sqrt{2})2^{n-1}.$$

*Remark.* We note that $4 - 2\sqrt{2} \approx 1.17$. Also note that this bound improves several results in [7].

**Theorem 3** *Conjecture 1 holds when $n \leq 6$.*

For incomparable collections of subsets of set [n], using probabilistic approach, we prove the following bound in Section 3.

**Theorem 4** *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ be incomparable collections of distinct subsets of set $[n]$. For every $c \in (0, \frac{3}{16 \ln 2}]$, if $max_{1 \leq i \leq k}\{|\mathcal{A}_i|\} \leq c2^n$, then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq min\{(1 + c^2 k)2^{n-1}, (\frac{2}{1 + \sqrt{1 - (\frac{1}{2} + \frac{8 \ln 2}{3} c)^2}})2^{n-1}\}.$$

*Remark.* If we take $c = \frac{1}{k}$ in Theorem 4, then

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq (1 + \frac{1}{k})2^{n-1}.$$

If we take $c = \frac{1}{16 \ln 2}$ in Theorem 4, then

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq (\frac{9 - 3\sqrt{5}}{2})2^{n-1},$$

We note that $(\frac{9-3\sqrt{5}}{2})2^{n-1} \approx 1.14 \times 2^{n-1}$.

## 2 Proof of Theorem 2 and Theorem 3

Theorem 2 follows directly from the following lemmas.

**Lemma 5** *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ be incomparable collections of distinct subsets of set $[n]$ with $|\mathcal{A}_1| \geq |\mathcal{A}_2| \geq \cdots \geq |\mathcal{A}_k|$. If $|\mathcal{A}_1| < \frac{2^{n-1}+1+2^{\frac{n+1}{2}}}{2}$ and $|\mathcal{A}_3| < 2^{\frac{n+1}{2}}$, then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1}.$$

**Lemma 6** *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ be incomparable collections of distinct subsets of set $[n]$ with $|\mathcal{A}_1| \geq |\mathcal{A}_2| \geq \cdots \geq |\mathcal{A}_k|$.*

*(i) If $|\mathcal{A}_1| < \frac{2^{n-1}+1+2^{\frac{n+1}{2}}}{2}$ and $|\mathcal{A}_3| \geq 2^{\frac{n+1}{2}}$, then $\sum_{i=1}^{k} |\mathcal{A}_i| < (4-2\sqrt{2})2^{n-1}$.*

*(ii) If $\frac{2^{n-1}+1+2^{\frac{n+1}{2}}}{2} \leq |\mathcal{A}_1| \leq 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil} + 2$ (notice that this situation happens only when $n \geq 7$), then $\sum_{i=1}^{k} |\mathcal{A}_i| < (4-2\sqrt{2})2^{n-1}$.*

**Lemma 7** *[7] Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ be incomparable and uncomplemented collections of distinct subsets of set $[n]$ with $|\mathcal{A}_1| \geq |\mathcal{A}_2| \geq \cdots \geq |\mathcal{A}_k|$. If $|\mathcal{A}_1| > 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil} + 2$, then $\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1}$.*

Proof of Lemmas 5 and 6 will be given in Sections 2.1 and 2.2 respectively. Lemma 7 was proved in [7]. In Section 2.3, we prove Theorem 3.

The following results related to incomparable collections of subsets will be used in our proof. The first result is given by Seymour in [8].

**Lemma 8** *[8] If $\mathcal{A}$, $\mathcal{B}$ are incomparable collections of distinct subsets of $[n]$, then $|\mathcal{A}|^{1/2} + |\mathcal{B}|^{1/2} \leq 2^{n/2}$.*

In [7], the above lemma was generalized as follows.

**Proposition 1** *[7] Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ be incomparable collections of distinct subsets of set $[n]$. Let $I$ and $J$ be any partition of set $[k]$ where $[k] = \{1, \ldots, k\}$. Then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| + 2[(\sum_{j\in J} |\mathcal{A}_j|)(\sum_{i \in I} |\mathcal{A}_i|)]^{\frac{1}{2}} \leq 2^n.$$

**Proof.** It follows from Lemma 8 and the fact that $\cup_{i\in I}\mathcal{A}_i$ and $\cup_{j\in J}\mathcal{A}_j$ are incomparable. ∎

The following lemma is an implication of Proposition 1 and will be applied in proving Lemmas 5, 6 and Theorem 3.

**Lemma 9** *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ be incomparable collections of distinct subsets of set $[n]$. If there exists $I \subseteq [k]$ such that*

$$\frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2} < \sum_{i\in I} |\mathcal{A}_i| < \frac{2^{n-1}+1+2^{\frac{n+1}{2}}}{2},$$

*then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1}.$$

**Proof.** Suppose that there exists $I \subset [k]$ such that

$$\frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2} < \sum_{i\in I} |\mathcal{A}_i| < \frac{2^{n-1}+1+2^{\frac{n+1}{2}}}{2}.$$

If $\sum_{i=1}^{k} |\mathcal{A}_i| \geq 2^{n-1}+1$, then

$$\begin{aligned} &\textstyle\sum_{i=1}^{k} |\mathcal{A}_i| + 2[(\sum_{i\in I}|\mathcal{A}_i|)(\sum_{j\in[k]\setminus I} |\mathcal{A}_j|)]^{\frac{1}{2}} \\ &\geq 2^{n-1}+1+2[(\textstyle\sum_{i\in I}|\mathcal{A}_i|)(2^{n-1}+1-\sum_{i\in I}|\mathcal{A}_i|)]^{\frac{1}{2}}. \end{aligned}$$

Since $f(x) = x(2^{n-1}+1-x)$ increases as $x \leq \frac{2^{n-1}+1}{2}$ and decreases as $x \geq \frac{2^{n-1}+1}{2}$, we have

$$\begin{aligned} &\textstyle\sum_{i=1}^{k} |\mathcal{A}_i| + 2[(\sum_{i\in I}|\mathcal{A}_i|)(\sum_{i\in[k]\setminus I} |\mathcal{A}_i|)]^{\frac{1}{2}} \\ &> 2^{n-1}+1+2[\tfrac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2}\cdot\tfrac{2^{n-1}+1+2^{\frac{n+1}{2}}}{2}]^{\frac{1}{2}} = 2^n \end{aligned}$$

which contradicts to Proposition 1. ∎

## 2.1 Proof of Lemma 5

**Proof.** We divide our proof into two cases:

**Case 1.** Suppose that $|\mathcal{A}_2| < 2^{\frac{n+1}{2}}$. If

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2},$$

then Lemma 5 is proved. Also note that we can assume that $|\mathcal{A}_1| \leq \frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2}$ since otherwise $\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1}$ holds by combining Lemma 9 and the assumption for $|\mathcal{A}_1|$. So we can assume that there exists an integer $1 \leq l < k$ such that

$$\sum_{i=1}^{l} |\mathcal{A}_i| \leq \frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2} \tag{1}$$

and

$$\sum_{i=1}^{l+1} |\mathcal{A}_i| > \frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2}. \tag{2}$$

Due to $|\mathcal{A}_{l+1}| \leq |\mathcal{A}_2| < 2^{\frac{n+1}{2}}$ and (1), we have

$$\sum_{i=1}^{l+1} |\mathcal{A}_i| < \frac{2^{n-1}+1+2^{\frac{n+1}{2}}}{2}. \tag{3}$$

Combining (2), (3) and Lemma 9, we conclude that $\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1}$.

**Case 2.** Suppose that $|\mathcal{A}_2| \geq 2^{\frac{n+1}{2}}$ and $|\mathcal{A}_3| < 2^{\frac{n+1}{2}}$.

If $\dfrac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2} < |\mathcal{A}_i| < \dfrac{2^{n-1}+1+2^{\frac{n+1}{2}}}{2}$ where $i=1$ or 2, then by Lemma 9, $\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1}$ holds.

Now we assume that $2^{\frac{n+1}{2}} \leq |\mathcal{A}_i| \leq \dfrac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2}$ where $i=1,2$ and $|\mathcal{A}_3| < 2^{\frac{n+1}{2}}$.

If $\sum_{i=2}^{k} |\mathcal{A}_i| \leq \frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2}$, then it is easy to see that

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1},$$

and Lemma 5 is proved. Otherwise, we could find $l$, where $2 \leq l < k$, such that

$$\sum_{i=2}^{l} |\mathcal{A}_i| \leq \frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2}$$

and

$$\sum_{i=2}^{l+1} |\mathcal{A}_i| > \frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2}.$$

Since $|\mathcal{A}_{l+1}| \leq |\mathcal{A}_3| < 2^{\frac{n+1}{2}}$,

$$\sum_{i=2}^{l+1} |\mathcal{A}_i| < \frac{2^{n-1}+1+2^{\frac{n+1}{2}}}{2}.$$

Then by Lemma 9, $\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1}$ holds. ∎

## 2.2 Proof of Lemma 6

**Proof.** (i) By Lemma 9, we can assume that $2^{\frac{n+1}{2}} \leq |\mathcal{A}_1| \leq \frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2}$. Similar to the proof of Lemma 5, we can also assume that there exists an integer $l < k$ such that

$$\sum_{i=1}^{l} |\mathcal{A}_i| \leq \frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2} \tag{4}$$

and

$$\sum_{i=1}^{l+1} |\mathcal{A}_i| > \frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2}. \tag{5}$$

Since $|\mathcal{A}_{l+1}| \leq |\mathcal{A}_1| \leq \dfrac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2}$, by (4), we obtain

$$\sum_{i=1}^{l+1} |\mathcal{A}_i| \leq 2^{n-1} + 1 - 2^{\frac{n+1}{2}}. \tag{6}$$

Let $s = \sum_{i=1}^{k} |\mathcal{A}_i|$ and $a = \sum_{i=1}^{l+1} |\mathcal{A}_i|$. Inequalities (5) and (6) imply that

$$\frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2} < a \leq 2^{n-1} + 1 - 2^{\frac{n+1}{2}}. \tag{7}$$

Now we are going to apply Proposition 1 to estimate $s$. Proposition 1 implies that

$$s + 2[a(s-a)]^{\frac{1}{2}} \leq 2^n.$$

This is equivalent to

$$s^2 - (2^{n+1} + 4a)s + 2^{2n} + 4a^2 \geq 0.$$

Solving this quadratic equation for $s$, we have

$$s \leq \frac{2^{n+1} + 4a - \sqrt{2^{n+4}a}}{2} = f(a) \tag{8}$$

or

$$s > \frac{2^{n+1} + 4a + \sqrt{2^{n+4}a}}{2}. \tag{9}$$

Due to the fact that $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ are incomparable, $s \leq 2^n$, hence (9) will not happen, consequently (8) always holds.

Since $f(a)$ increases as $a \geq 2^{n-2}$ and decreases as $a \leq 2^{n-2}$, by the range of $a$ from (7), we have

$$\begin{aligned} s \leq f(a) &\leq max\{f(\frac{2^{n-1}+1-2^{\frac{n+1}{2}}}{2}), f(2^{n-1}+1-2^{\frac{n+1}{2}})\} \\ &< max\{f(2^{n-2}-2^{\frac{n-1}{2}}), f(2^{n-1})\}. \end{aligned}$$

Now we estimate $f(2^{n-2}-2^{\frac{n-1}{2}})$ and $f(2^{n-1})$. By direct calculation,

$$f(2^{n-1}) = (4 - 2\sqrt{2})2^{n-1},$$

and

$$f(2^{n-2}-2^{\frac{n-1}{2}}) = (3 - 4 \cdot 2^{\frac{-n-1}{2}} - \sqrt{4 - 2^{\frac{-n+7}{2}}}) \cdot 2^{n-1}. \tag{10}$$

When $n \geq 9$, $3 - 4 \cdot 2^{\frac{-n-1}{2}} - \sqrt{4 - 2^{\frac{-n+7}{2}}} < 3 - \sqrt{7/2} < 4 - 2\sqrt{2}$. When $n = 7, 8$, by direct calculation, $3 - 4 \cdot 2^{\frac{-n-1}{2}} - \sqrt{4 - 2^{\frac{-n+7}{2}}} < 4 - 2\sqrt{2}$.

(ii). The proof is similar to the proof of part (i). Let $s = \sum_{i=1}^{k} |\mathcal{A}_i|$ and $a = |\mathcal{A}_1|$. In this case, the range of $a$ (i.e. inequality (7)) becomes

$$\frac{2^{n-1}+1+2^{\frac{n+1}{2}}}{2} \leq a \leq 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil} + 2.$$

As in part (i), we have

$$s \le f(a) = \frac{2^{n+1} + 4a - \sqrt{2^{n+4}a}}{2}.$$

Since $f(a)$ increases as $a \ge 2^{n-2}$,

$$\begin{aligned} s \le f(a) &\le f(2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil} + 2) \\ &< f(2^{n-1}) = (4 - 2\sqrt{2})2^{n-1}. \ \blacksquare \end{aligned}$$

### 2.3 Proof of Theorem 3

We apply Lemmas 5, 7 and 9 to prove Theorem 3.

**Proof.** Assume that $\sum_{i=1}^{k} |\mathcal{A}_i| \ge 2^{n-1} + 1$ and $|\mathcal{A}_1| \ge |\mathcal{A}_2| \ge \cdots \ge |\mathcal{A}_k|$. By Lemma 7,

$$|\mathcal{A}_1| \le 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil} + 2. \tag{11}$$

When $n \le 5$,

$$2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil} + 2 < 2^{\frac{n+1}{2}}.$$

Therefore by Lemma 5 Case 1, $\sum\limits_{i=1}^{k} |\mathcal{A}_i| \le 2^{n-1}$.

When $n = 6$, by (11), $|\mathcal{A}_1| \le 18$. If $|\mathcal{A}_1| \le 11 < 2^{7/2}$, by Lemma 5 Case 1,

$$\sum_{i=1}^{k} |\mathcal{A}_i| \le 2^{n-1} = 32$$

holds. If $12 \le |\mathcal{A}_1| \le 18$, by Lemma 9, $\sum\limits_{i=1}^{k} |\mathcal{A}_i| \le 2^{n-1} = 32$ holds as well. This completes the proof of Theorem 3. ∎

## 3 Proof of Theorem 4

Theorem 4 follows directly from Theorems 10 and 11.

**Theorem 10** *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ be incomparable collections of distinct subsets of set $[n]$. For every $c \in (0, 1]$, if $max_{1 \le i \le k}\{|\mathcal{A}_i|\} \le c2^n$, then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \le (1 + c^2 k)2^{n-1}.$$

**Theorem 11** *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ be incomparable collections of distinct subsets of set $[n]$. For every $c \in (0, \frac{3}{16 \ln 2}]$, if $max_{1 \le i \le k}\{|\mathcal{A}_i|\} \le c2^n$, then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \le \left(\frac{2}{1 + \sqrt{1 - (\frac{1}{2} + \frac{8 \ln 2}{3} c)^2}}\right)2^{n-1}.$$

Proof of Theorems 10 and 11 are given in Sections 3.1 and 3.2 respectively. Both proofs are based on a probabilistic approach.

### 3.1 Proof of Theorem 10

**Proof.** Define a random subset $I \subset [k]$ by setting

$$\text{Prob}[i \in I] = \frac{1}{2}, \quad i \in [k],$$

these choices are mutually independent. Set $X = \sum_{i \in I} |\mathcal{A}_i|$ and let $k$ independent random variables $X_1, X_2, \ldots, X_k$ be defined as

$$\text{Prob}(X_i = |\mathcal{A}_i|) = \text{Prob}(X_i = 0) = \frac{1}{2}.$$

Then $X = \sum_{i=1}^{k} X_i$. Let $s = \sum_{i=1}^{k} |\mathcal{A}_i|$.

Since for every $i \in [k]$, $\text{Prob}(i \in I) = \frac{1}{2}$, then the expectation of $X$ is

$$E(X) = \sum_{i=1}^{k} E(X_i) = \frac{\sum_{i=1}^{k} |\mathcal{A}_i|}{2} = \frac{s}{2}.$$

The variance of $X$ is

$$\begin{aligned} \sigma^2 &= E(X^2) - [E(X)]^2 \\ &= E(\sum_{i \in [k]} X_i^2 + 2 \sum_{1 \le i < j \le k} X_i X_j) - (\frac{s}{2})^2 \\ &= \frac{\sum_{i \in [k]} |\mathcal{A}_i|^2 + \sum_{1 \le i < j \le k} |\mathcal{A}_i||\mathcal{A}_j|}{2} - (\frac{\sum_{i \in [k]} |\mathcal{A}_i|}{2})^2 \\ &= \sum_{i=1}^{k} \frac{|\mathcal{A}_i|^2}{4}. \end{aligned}$$

Applying the Chebyshev inequality, we have

$$\text{Prob}(|X - E(X)| > \sigma) < \frac{\sigma^2}{\sigma^2} = 1.$$

Then it follows that

$$\text{Prob}(|X - E(X)| \le \sigma) > 0.$$

Therefore there exists an $I_0 \subset [k]$ such that

$$|\sum_{i \in I_0} |\mathcal{A}_i| - E(X)| \le \sigma;$$

this is equivalent to

$$E(X) - \sigma \le \sum_{i \in I_0} |\mathcal{A}_i| \le E(X) + \sigma. \tag{12}$$

Let $a = \sum_{i \in I_0} |\mathcal{A}_i|$ and recall $s = \sum_{i=1}^{k} |\mathcal{A}_i|$ and $E(X) = \frac{s}{2}$; then

$$\frac{s}{2} - \sigma \le a \le \frac{s}{2} + \sigma. \tag{13}$$

Using Proposition 1, we have

$$s + 2[a(s-a)]^{1/2} \leq 2^n.$$

Since $f(a) = s + 2[a(s-a)]^{1/2}$ increases as $a \leq \frac{s}{2}$ and decreases as $a \geq \frac{s}{2}$, by (13),

$$\begin{aligned} 2^n \geq f(a) &\geq min\{f(\frac{s}{2}+\sigma), f(\frac{s}{2}-\sigma)\} \\ &= s + 2[(\frac{s}{2}+\sigma)(\frac{s}{2}-\sigma]^{1/2}. \end{aligned}$$

Therefore,

$$s \leq 2^{n-1}(1+\frac{4\sigma^2}{2^{2n}}).$$

Since $\max_{1\leq i\leq k}\{|\mathcal{A}_i|\} \leq c2^n$, $4\sigma^2 = \sum_{i=1}^k |\mathcal{A}_i|^2 \leq c^2k2^{2n}$, thus

$$s = \sum_{i=1}^k |\mathcal{A}_i| \leq (1+c^2k)2^{n-1}. \quad \blacksquare$$

### 3.2 Proof of Theorem 11

The following lemma will be the main tool in proving Theorem 11.

**Lemma 12** *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ be incomparable collections of distinct subsets of set $[n]$. For every $c' \in (\frac{1}{2}, 1]$, if $max_{1\leq i\leq k}\{|\mathcal{A}_i|\} \leq \frac{3(2c'-1)}{16\ln 2}2^n$, then there exists $I_0 \subset [k]$ such that*

$$\frac{1-c'}{2}\sum_{i=1}^k |\mathcal{A}_i| \leq \sum_{i\in I_0} |\mathcal{A}_i| \leq \frac{1+c'}{2}\sum_{i=1}^k |\mathcal{A}_i|.$$

*Remark.* $c'$ in Lemma 12 and $c$ in Theorem 11 are related as $c = \frac{3(2c'-1)}{16\ln 2}$ and $c' = \frac{1}{2} + \frac{8\ln 2}{3}c$.

We delay the proof of Lemma 12 until we finish the proof of Theorem 11.

**Proof of Theorem 11.** For $c \in (0, \frac{3}{16\ln 2}]$, let $c' = \frac{1}{2} + \frac{8\ln 2}{3}c$, then $c' \in (\frac{1}{2}, 1]$. Let $s = \sum_{i=1}^k |\mathcal{A}_i|$ and $a = \sum_{i\in I_0} |\mathcal{A}_i|$ where $I_0 \subset [k]$ is the set from Lemma 12. Then $\frac{1-c'}{2}s \leq a \leq \frac{1+c'}{2}s$. Using Proposition 1,

$$f(a) = s + 2[a\cdot(s-a)]^{\frac{1}{2}} \leq 2^n.$$

Since $f(a)$ increases as $a \leq \frac{s}{2}$ and decreases as $a \geq \frac{s}{2}$,

$$\begin{aligned} f(a) &\geq min\{f(\frac{1-c'}{2}s), f(\frac{1+c'}{2}s)\} \\ &= s + 2[\frac{1-c'}{2}s\cdot\frac{1+c'}{2}s]^{\frac{1}{2}}. \end{aligned}$$

So,

$$s + 2[\frac{1-c'}{2}s\cdot\frac{1+c'}{2}s]^{\frac{1}{2}} \leq 2^n;$$

solving for $s$,

$$s \le \left(\frac{2}{1+\sqrt{1-c'^2}}\right)2^{n-1} = \left(\frac{2}{1+\sqrt{1-(\frac{1}{2}+\frac{8\ln 2}{3}c)^2}}\right)2^{n-1}. \blacksquare$$

What remains is to prove Lemma 12.

**Proof of Lemma 12.** As in the proof of Theorem 10, define a random subset $I \subset [k]$ by setting

$$\text{Prob}[i \in I] = \frac{1}{2}, \quad i \in [k];$$

these choices are mutually independent. Set $X = \sum_{i \in I} |\mathcal{A}_i|$ and let $k$ independent random variables $X_1, X_2, \ldots, X_k$ be defined as

$$\text{Prob}(X_i = |\mathcal{A}_i|) = \text{Prob}(X_i = 0) = \frac{1}{2}.$$

Then $X = \sum_{i=1}^{k} X_i$. Let $s = \sum_{i=1}^{k} |\mathcal{A}_i|$.

Since for every $i \in [k]$, $\text{Prob}(i \in I) = \frac{1}{2}$, then the expectation of $X$ is

$$E(X) = \sum_{i=1}^{k} E(X_i) = \frac{\sum_{i=1}^{k} |\mathcal{A}_i|}{2}.$$

Let $u > 0$; applying Markov's inequality to $E(e^{uX})$ (see [4] Page 26 or [1] Page 266), then

$$\begin{aligned}
\text{Prob}(X > E(X)+t) &= \text{Prob}(e^{uX} > e^{u(E(X)+t)}) \\
&< e^{-u(E(X)+t)} E(e^{uX}) \\
&= e^{-u(E(X)+t)} \prod_{i=1}^{k} E(e^{uX_i}) \\
&= e^{-u(E(X)+t)} \prod_{i=1}^{k} \frac{1}{2}(1+e^{u|\mathcal{A}_i|}) \\
&= e^{-u(E(X)+t)} e^{u\sum_{i=1}^{k}|\mathcal{A}_i|} \prod_{i=1}^{k} \frac{1}{2}(1+\frac{1}{e^{u|\mathcal{A}_i|}}) \\
&= e^{u(E(X)-t)} \prod_{i=1}^{k} \frac{1}{2}(1+e^{-u|\mathcal{A}_i|}).
\end{aligned}$$

Now for $c' \in (\frac{1}{2}, 1]$, take $\frac{2ln2}{(2c'-1)E(X)} \le u \le \frac{3}{2max_{i\in[k]}\{|\mathcal{A}_i|\}}$. We note that the choice of $u$ is reasonable since $\frac{3}{2max_{i\in[k]}\{|\mathcal{A}_i|\}} \ge \frac{\ln 2}{(2c'-1)2^{n-3}} \ge \frac{2ln2}{(2c'-1)E(X)}$. This is because we may assume that $E(X) \ge 2^{n-2}$, otherwise $E(X) < 2^{n-2}$ and it implies that $\sum_{i=1}^{k} |\mathcal{A}_i| < 2^{n-1}$ and the conclusion of Theorem 11 holds. Now we have $u|\mathcal{A}_i| \le \frac{3}{2}$ for every $i \in [k]$. Notice that $e^{-x} \le 1 - \frac{x}{2}$ when $0 \le x \le \frac{3}{2}$. This is because

$f(x) = e^{-x} - (1 - \frac{x}{2})$ is concave upward and $f(0), f(\frac{3}{2}) \leq 0$. Thus

$$\begin{aligned}\mathrm{Prob}(X > E(X)+t) &< e^{u(E(X)-t)} \prod_{i=1}^{k} \frac{1}{2}(1 + 1 - \frac{u|\mathcal{A}_i|}{2}) \\ &= e^{u(E(X)-t)} \prod_{i=1}^{k} (1 - \frac{u|\mathcal{A}_i|}{4}). \qquad (14)\end{aligned}$$

Since the geometric mean is no more than the arithmetic mean, we have

$$\begin{aligned}\mathrm{Prob}(X > E(X)+t) &< e^{u(E(X)-t)} (1 - \frac{u\sum_{i=1}^{k}|\mathcal{A}_i|}{4k})^k \\ &= e^{u(E(X)-t)} (1 - \frac{uE(X)}{2k})^k.\end{aligned}$$

Since the sequence $\{(1 - \frac{uE(X)}{2k})^k\}$ is increasing and $\lim_{k\to\infty}(1 - \frac{uE(X)}{2k})^k = e^{-\frac{uE(X)}{2}}$,

$$\begin{aligned}\mathrm{Prob}(X > E(X)+t) &< e^{u(E(X)-t)} e^{-\frac{uE(X)}{2}} \\ &= e^{\frac{u}{2}(E(X)-2t)}.\end{aligned}$$

Let $t = c'E(X)$; then

$$\mathrm{Prob}(X > E(X)+c'E(X)) < e^{-\frac{u(2c'-1)E(X)}{2}} \leq \frac{1}{2},$$

since $u \geq \frac{2\ln 2}{(2c'-1)E(X)}$.

Similarly, we will prove that $\mathrm{Prob}(X < E(X) - c'E(X)) < \frac{1}{2}$. Let $u > 0$; notice that

$$\begin{aligned}\mathrm{Prob}(X < E(X)-t) &= \mathrm{Prob}(-uX > -u(E(X)-t)) \\ &= \mathrm{Prob}(e^{-uX} > e^{-u(E(X)-t)}).\end{aligned}$$

Applying Markov's inequality to $E(e^{-uX})$, we have

$$\begin{aligned}\mathrm{Prob}(X < E(X)-t) &< e^{u(E(X)-t)} E(e^{-uX}) \\ &= e^{u(E(X)-t)} \prod_{i=1}^{k} E(e^{-uX_i}) \\ &= e^{u(E(X)-t)} \prod_{i=1}^{k} \frac{1}{2}(1 + e^{-u|\mathcal{A}_i|}).\end{aligned}$$

Again, since $u|\mathcal{A}_i| \leq \frac{3}{2}$ for every $i \in [k]$ and $e^{-x} \leq 1 - \frac{x}{2}$ when $0 \leq x \leq \frac{3}{2}$, we get

$$\begin{aligned}\mathrm{Prob}(X < E(X)-t) &< e^{u(E(X)-t)} \prod_{i=1}^{k} \frac{1}{2}(1 + 1 - \frac{u|\mathcal{A}_i|}{2}) \\ &= e^{u(E(X)-t)} \prod_{i=1}^{k} (1 - \frac{u|\mathcal{A}_i|}{4}).\end{aligned}$$

Now we have a similar situation as inequality (14) and $\text{Prob}(X < E(X) - t) < \frac{1}{2}$ follows exactly the same lines after inequality (14) as in proving $\text{Prob}(X > E(X) + t) < \frac{1}{2}$. Thus

$$\text{Prob}((1 - c')E(X) \leq X \leq (1 + c')E(X)) > 0,$$

and this implies that there exists an $I_0 \subset [k]$ such that $\frac{1-c'}{2} \sum_{i=1}^{k} |\mathcal{A}_i| \leq \sum_{i \in I_0} |\mathcal{A}_i| \leq \frac{1+c'}{2} \sum_{i=1}^{k} |\mathcal{A}_i|$. ∎

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