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Totally magic labelings of graphs

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Abstract

Totally magic labelings and totally magic injections of graphs have been studied in several recent papers by Exoo, Ling, McSorley, Phillips and Wallis. A *total labeling* of a graph with vertex set V and edge set E is a mapping from $V \cup E$ to the positive integers. An injective total labeling is said to be a *totally magic injection* if there are "magic constants" h and k such that the sum of any vertex label with the labels on the incident edges is h and the sum of any edge label with the labels on the incident vertices is k. The *total deficiency* of a totally magic injection with maximum label M is $M - |V| - |E|$. A totally magic injection with deficiency 0 is called a *totally magic labeling.*

In this paper, we solve two research problems from the book *Magic Graphs* by Wallis (Birkhäuser, Boston, 2001). We solve Research Problem 4.1 by showing that for any $m \geq 0$, although there is a totally magic injection of $K_{1,s} \cup mK_3$ for $s \geq 2$, there is no totally magic labeling for $s \geq 3$. We solve Research Problem 4.5 by showing, for all even $m \geq 2$, that 2 is the minimum total deficiency of totally magic injections of mK_3. This result has been obtained independently by J. P. McSorley (personal communication). In addition, we give a new recursive construction for totally magic labelings of mK_3 for m odd.

1 Introduction

There have been several graph labelings that generalize the concept of magic squares by requiring that sums of certain sets of labels be constant. We examine a rather restrictive type of labeling called a *totally magic labeling* and a less restrictive variation called a *totally magic injection.* Totally magic injections and labelings have been studied in [1, 5, 4, 6], from which our definitions are taken. Let $G = (V, E)$ be a finite, simple, and undirected graph, and let $v = |V|$ and $e = |E|$. A *total labeling* of G is a map from $V \cup E$ to the positive integers.

Definition 1.1. A one-to-one total labeling λ of G is said to be

(a) a *vertex-magic injection* [2, 4] if there is a constant h, called the *vertex sum*, such that for each vertex x,

$$\lambda(x) + \sum_{y \in N(x)} \lambda(xy) = h,$$

where $N(x)$ is the set of neighbors of x.

(b) an *edge-magic injection* [7, 4] if there is a constant k, called the *edge sum*, such that for each edge xy,

$$\lambda(x) + \lambda(y) + \lambda(xy) = k.$$

(c) a *totally magic injection* if λ is both a vertex-magic injection and an edge-magic injection.

The *total deficiency* of a totally magic injection with maximum label M is $M - v - e$. The *total deficiency* of a graph G is the least deficiency of all totally magic injections of G. We are particularly interested in totally magic injections which use the labels $1, 2, \ldots, v + e$.

Definition 1.2. A *totally magic labeling* is a totally magic injection with total deficiency 0.

We seek to identify the graphs that admit totally magic labelings and totally magic injections.

Definition 1.3.

(a) A graph G for which there exists a totally magic labeling is said to be *totally magic*. A totally magic graph is also referred to as a TM graph.

(b) A graph G for which there exists a totally magic injection is said to be a TMI graph.

The only known connected totally magic graphs are the isolated point K_1, the triangle K_3, and the star $K_{1,2}$. Note that a graph that is not a TMI graph cannot be a component of a TM graph. Exoo, Ling, McSorley, Phillips and Wallis [1, 6] have shown that no cycles, complete graphs, or trees others than stars are TMI except for K_1, K_3 and $K_{1,2}$. They have also shown that, although every star $K_{1,s}$ is TMI except for $K_{1,1}$, the only totally magic star is $K_{1,2}$.

In the following section, we give some basic constructions. In Section 3, we consider graphs of the form mK_3, a union of triangles. It has been shown in [1] that mK_3 is totally magic if and only if m is odd. In [5], the values of h and k that may be used in totally magic labelings mK_3 for odd m were determined. In Subsection 3.1 we give an alternate recursive approach to constructing totally magic labelings on mK_3 for m odd. In Subsection 3.2 we solve Research Problem 4.5 of [6] by showing that 2 is the total deficiency of mK_3 for all even $m \geq 2$. This result has been proved independently by J. P. McSorely [3]. In Section 4, we solve Research Problem 4.1 of [6] by showing that, although $K_{1,s} \cup mK_3$ is a TMI graph for $s \geq 2$, it is not totally magic for $s \geq 3$.

2 Basic Constructions

First we consider the magic labelings of the three known connected totally magic graphs. Totally magic labelings of the isolated point K_1 are trivial. The sole vertex must be labeled 1. All of the totally magic labelings of K_3 and $K_{1,2}$ are displayed in Figure 1. Of interest to us in totally magic labelings of K_3 with vertex sum h and

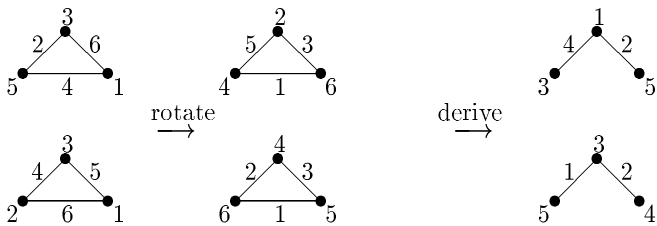

Figure 1: Connected Totally Magic Labelings

edge sum k will be the difference $d = k - h$. The total labelings of K_3 in the top row of Figure 1 have $d = \pm 1$ while those in the bottom row have $d = \pm 3$.

Figure 1 also exhibits some "new labelings from old" constructions. The *rotation* of a total labeling λ on a union of cycles is the total labeling obtained by rotating the labels on each cycle one step clockwise so that the vertex labels become edge labels and vice versa. The *derivative* of a total labeling λ of a graph G with an edge e' for which $\lambda(e') = 1$ is the total labeling $\lambda - 1$ for the graph $G - e'$.

An example of a TMI graph that is not a TM graph is shown in Figure 2. No

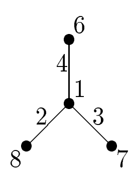

Figure 2: A Best Possible TMI

lower maximum label is possible for that graph, as we see in Theorem 4.1. The constructions shown in Figures 1 and 2 are generalized in the subsequent sections.

3 Labeling Unions of Triangles

Since only three connected totally magic graphs are known, research on totally magic graphs has primarily focused on disconnected graphs. As noted earlier, each component of a totally magic graph must be a TMI graph. Thus we will consider unions of triangles and stars. In this section, we study totally magic labelings of mK_3, the disjoint union of m copies of the triangle K_3, where m is a positive integer.

3.1 Odd Numbers of Triangles

In this subsection, we present an alternative proof of a theorem from [1, 5].

Theorem 3.1 ([1, 5]). *Let m be an odd positive integer. For every divisor d of $3m$, there is a totally magic labeling of mK_3 with a vertex sum h and an edge sum k such that $k - h = d$.*

Our proof is based on a pair of basic lemmas. The first explicitly handles the two smallest possible values for d.

Lemma 3.2. *For any odd positive integer t and for $d = 1$ or 3, the graph tK_3 has a totally magic labeling with vertex sum h and edge sum k such that $k - h = d$.*

Proof. The total labelings are specified in terms of Figure 3 and are easily seen to be totally magic labelings.

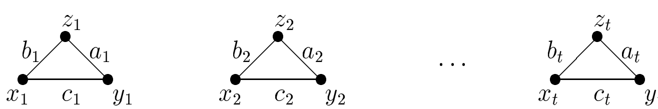

Figure 3: Totally Magic Labelings of tK_3 for odd t

Case 1: $k - h = 1$: For each $1 \le i \le t$, let $a_i = 2i - 1$,

$$b_i = \begin{cases} 3t - i & \text{if } i \text{ is even} \\ 4t - i & \text{if } i \text{ is odd} \end{cases}, \quad c_i = \begin{cases} 6t + 1 - i & \text{if } i \text{ is even} \\ 5t + 1 - i & \text{if } i \text{ is odd} \end{cases},$$

$x_i = a_i + 1$, $y_i = b_i + 1$, and $z_i = c_i + 1$. Here, we have $k = 9t + 2$ and $h = 9t + 1$.

Case 2: $k - h = 3$: For each $1 \le i \le t$, let $a_i = 6i - 5$,

$$b_i = \begin{cases} 3t - 1 - 3i & \text{if } i \text{ is even} \\ 6t - 1 - 3i & \text{if } i \text{ is odd} \end{cases}, \quad c_i = \begin{cases} 6t + 3 - 3i & \text{if } i \text{ is even} \\ 3t + 3 - 3i & \text{if } i \text{ is odd} \end{cases},$$

$x_i = a_i + 3$, $y_i = b_i + 3$, and $z_i = c_i + 3$. Here, we have $k = 9t + 3$ and $h = 9t$. □

The second lemma is an example of a "new labeling from old" result and enables us to blow up a totally magic labeling from a small number of triangles to a larger number.

Lemma 3.3. *Let s and t be odd positive integers, and suppose there is a totally magic labeling for tK_3 with vertex sum h', edge sum k', and difference $d' = k' - h'$. Then, there is a totally magic labeling for stK_3 with a vertex sum h and an edge sum k such that the difference $d = k - h$ satisfies $d = sd'$.*

Proof. Suppose that the totally magic labeling for tK_3 has been specified in terms of Figure 3. Note that there is a constant r such that $k' = r + 2d'$, $h' = r + d'$, and, for all i, $a_i + b_i + c_i = r$.

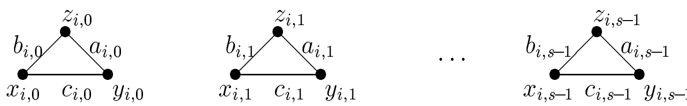

Figure 4: Totally Magic Labelings of sK_3 for odd s

For each $1 \le i \le t$, replace the i^{th} triangle in Figure 3 by the s triangles reflected in Figure 4. For each $0 \le j \le s-1$, let $a_{i,j} = sa_i - j$,

$$b_{i,j} = \begin{cases} sb_i - \frac{2s-j-2}{2} & \text{if } j \text{ is even} \\ sb_i - \frac{s-j-2}{2} & \text{if } j \text{ is odd} \end{cases}, \quad c_{i,j} = \begin{cases} sc_i - \frac{s-j-1}{2} & \text{if } j \text{ is even} \\ sc_i - \frac{2s-j-1}{2} & \text{if } j \text{ is odd} \end{cases},$$

$x_{i,j} = a_{i,j} + sd'$, $y_{i,j} = b_{i,j} + sd'$, and $z_{i,j} = c_{i,j} + sd'$. It is straightforward to verify that this gives a totally magic labeling for stK_3 with edge sum $k = sr - \frac{3}{2}(s-1) + 2sd'$, vertex sum $h = sr - \frac{3}{2}(s-1) + sd'$, and difference $d = k - h = sd'$. □

Proof of Theorem 3.1. Let $q = \frac{3m}{d}$. Clearly, q, d, and $3m$ are all odd.

Case 1: $3 \mid q$.

Here, $d \mid m$. By Lemma 3.2, there is a totally magic labeling of $\frac{m}{d}K_3$ with a vertex sum h' and an edge sum k' such that $k' - h' = 1$. Since d is odd, it follows from Lemma 3.3 that there is a totally magic labeling of mK_3 with a vertex sum h and an edge sum k such that $k - h = d$.

Case 2: $3 \nmid q$.

It must be that $3 \mid d$. By Lemma 3.2, there is a totally magic labeling of $\frac{3m}{d}K_3$ with a vertex sum h' and an edge sum k' such that $k' - h' = 3$. Since q and $\frac{d}{3}$ are odd and $m = \frac{d}{3}q$, it follows from Lemma 3.3 that there is a totally magic labeling of mK_3 with a vertex sum h and an edge sum k such that $k - h = d$. □

Note that each of the totally magic labelings constructed in Theorem 3.1 has an edge labeled 1. Thus by applying the derive operation, each of these totally magic labeling of mK_3 for m odd also generates a totally magic labeling of $(m-1)K_3 \cup K_{1,2}$.

The totally magic labelings guaranteed by Theorem 3.1 for certain values of m and d are not unique. Figure 5 shows two distinct totally magic labelings for $m = 3$ triangles and difference $d = 1$. The second labeled graph is not isomorphic to the first and is also not isomorphic to a rotation of the first.

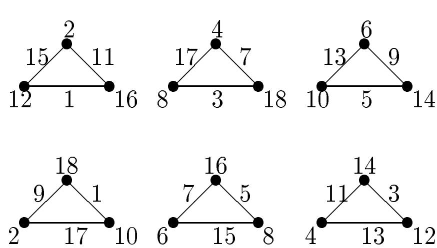

Figure 5: Distinct Totally Magic Labelings

3.2 Even Numbers of Triangles

While an even number of triangles is not totally magic, it is straightforward to give a construction that shows the total deficiency is at most 2. Our first lemma gives such a construction.

Lemma 3.4. *For any even positive integer* m *the graph* mK_3 *admits a totally magic injection with maximum label* $6m + 2$.

Proof. The total labeling is specified in terms of Figure 3 and is easily seen to be a totally magic injection. For each $1 \le i \le m$, let $x_i = 2i - 1$,

$$y_i = \begin{cases} 4m + 1 - i & \text{if } i \text{ is even} \\ 3m - i & \text{if } i \text{ is odd} \end{cases}, \qquad z_i = \begin{cases} 5m + 1 - i & \text{if } i \text{ is even} \\ 6m + 2 - i & \text{if } i \text{ is odd} \end{cases},$$

$a_i = x_i + 1$, $b_i = y_i + 1$, and $c_i = z_i + 1$. Here, we have $h = 9m + 3$ and $k = 9m + 2$. □

Theorem 3.1 and Lemma 3.4 together give the following result.

Corollary 3.5. *For any positive integer* m *the graph* mK_3 *is a TMI graph.*

We now show that the result in Lemma 3.4 cannot be improved, in other words that the total deficiency is 2. The proof that the deficiency is not 1 is somewhat complicated. John McSorley [3] has independently obtained the same result with a different proof.

Theorem 3.6. *Let* m *be an even positive integer. Any totally magic injection for* mK_3 *has maximum label at least* $6m + 2$.

Proof. For a proof by contradiction, assume that there is a totally magic injection for mK_3 with largest label $6m + 1$ or less. Since $6m$ distinct labels are used, there is exactly one positive integer $g \le 6m + 1$ that is not used as a label. If $g = 1$ then we can obtain a totally magic injection for mK_3 with $g = 6m + 1$ by subtracting one from each label. Hence, we may assume that $1 < g \le 6m + 1$.

Let k be the edge-magic number and h be the vertex-magic number. We may assume $k \ge h$ since a rotation can interchange the values of k and h. Also, let $d = k - h \ge 0$. We use several lemmas to complete the proof.

Lemma 3.7 ([6]). *If* v_1, v_2, v_3 *are the three vertices in a component of* mK_3 *then* $\lambda(v_1) - \lambda(v_2 v_3) = d$.

Proof. From the edge-magic equation $\lambda(v_1) + \lambda(v_1 v_2) + \lambda(v_2) = k$ subtract the vertex-magic equation $\lambda(v_1 v_2) + \lambda(v_2) + \lambda(v_2 v_3) = h$. □

Lemma 3.8. $d > 0$.

Proof. This follows from Lemma 3.7 since $\lambda(v_1) \ne \lambda(v_2 v_3)$. □

Lemma 3.9. *For some* $a \in \{1, 2, 3, 4, 5, 6\}$, $g = am + 1$.

Proof. The sum of all the labels is

$$1 + 2 + \cdots 6m + 1 - g = (6m + 1)(6m + 2)/2 - g = 18m^2 + 9m + 1 - g.$$

Also, the sum of the labels on each triangle is $k + h$. Since

$$k + h = 18m + 9 - (g - 1)/m,$$

it follows that $m \mid (g - 1)$. □

For the value of a from Lemma 3.9, let $T = 9m + (9 - a + 3d)/2$.

Lemma 3.10. *If v_1, v_2, v_3 are the three vertices in a component of mK_3, then $\lambda(v_1) + \lambda(v_2) + \lambda(v_3) = T$.*

Proof. By Lemma 3.7 and the proof of Lemma 3.9, the sum of the labels in any one component of mK_3 is $18m + 9 - a = 2(\lambda(v_1) + \lambda(v_2) + \lambda(v_3)) - 3d$. Thus $\lambda(v_1) + \lambda(v_2) + \lambda(v_3) = (18m + 9 - a + 3d)/2 = T$. □

Lemma 3.11. *d is even if and only if a is odd.*

Proof. By Lemma 3.10, since T is an integer, $9 - a + 3d$ is even. □

Lemma 3.12. *Let L_E be the set of labels on the edges of mK_3. For any integer x, define $\tilde{x} = \begin{cases} x - d & \text{if } x > g \text{ and } x \equiv g \pmod{d}, \\ x & \text{otherwise.} \end{cases}$*
Then $x \in L_E$ if and only if $1 \le x \le 6m + 1$, $x \ne g$, and $1 \le \tilde{x} \bmod 2d \le d$.

Proof. Our proof is by induction on x. First, suppose $1 \le x \le d$ and $x \ne g$. Since $x - d < 1$, it follows from Lemma 3.7 that x is not the label on a vertex. Therefore, $x \in L_E$. Now suppose $d < y \le 6m + 1$, $y \ne g$ and the lemma holds for all $x < y$.

Case 1: $y = g + d$. Note that $\tilde{y} = y - d$, and since g is not a label, $y \in L_E$ and $y - 2d = g - d \notin L_E$. It follows that $1 \le \tilde{y} \bmod 2d \le d$. So the lemma holds for y.

Case 2: $y \ne g + d$ and $1 \le \tilde{y} \bmod 2d \le d$. Then $y - d \notin L_E$. By Lemma 3.7, y is not the label on a vertex. Therefore $y \in L_E$.

Case 3: Any remaining y value. Then $y - d \in L_E$. By lemma 3.7, y is the label on a vertex. Therefore $y \notin L_E$. □

Lemma 3.13. *If $a \in \{1, 2, 4, 5\}$, then $2d \mid m$. If $a \in \{3, 6\}$, then $2d \mid 3m$.*

Proof. By Lemma 3.7, any label y such that $y \ne g$ and $y > 6m + 1 - d$ must be on a vertex. Let x be the largest number less than or equal to $6m + 1$ such that $x \equiv g \pmod{d}$. If $a = 6$, then $x = 6m+1$. Otherwise, since $x+d > 6m+1$, x is on a vertex. If $6m + 2 - d < x < 6m + 1$, then, since $(6m + 1) - (6m + 2 - d) = d - 1$, Lemma 3.12 tells us that $(6m + 2 - d) \bmod 2d = d + 1$ and $(6m + 1) \bmod 2d = 0$. However, this is impossible since $1 \le x \bmod 2d \le d$. Therefore $x = 6m + 1$ or $x = 6m + 2 - d$. In the latter case $a \ne 6$.

Case 1: $x = 6m + 1$. By Lemma 3.12, $6m \bmod 2d = 0$. So $2d \mid 6m$, and hence $d \mid 3m$. If $a = 6$ then d is odd by Lemma 3.11. Since m is even, this implies $2d \mid 3m$.

Now assume $a \neq 6$. By definition of x, $6m+1 \equiv am+1 \pmod{d}$. Thus $d \mid (6-a)m$. It follows that $d \mid am$. If $am \bmod 2d = d$ then $(am+1-d) \bmod 2d = 1$. But then $g-d$ is an edge label, which is impossible. Therefore, $2d \mid am$. So $2d \mid \gcd(a,6)m$. If $a=1$ or $a=5$ then $2d \mid m$. If $a=3$ then $2d \mid 3m$. If $a=2$ or $a=4$ then $2d \mid 2m$, or $d \mid m$. Since d is odd for a even, we can conclude that $2d \mid m$.

Case 2: $x = 6m+2-d$ and $a \neq 6$. It follows from the definition of x that $6m+2-d \equiv am+1 \pmod{d}$. So $d \mid (6-a)m+1$. Furthermore, by Lemma 3.12, $(6m+2-d) \bmod 2d = d$. So $2d \mid (6m+2)$, and hence $d \mid (3m+1)$. It follows that $d \mid (am+1)$. If $(am+1) \bmod 2d = 0$ then $(am+1-d) \bmod 2d = d$. But then $g-d$ is an edge label, which is impossible. Therefore, $(am+1) \bmod 2d = d$. Note that since m is even, the previous equation implies d is odd. By Lemma 3.11, a is even. If $a=2$ then $(2m+1) \bmod 2d = d$. Hence $6m+2 \equiv 3d-1 \equiv d-1 \equiv 0 \pmod{2d}$. Thus $d=1$. Similarly, if $a=4$, we get $(4m+1) \bmod 2d = d$. Now,

$$6(4m+1) - 4(6m+2) = 2 \equiv 6d - 4 \cdot 0 \equiv 0 \pmod{2d}.$$

Thus again $d=1$. Therefore, we may conclude $2d \mid m$ in this case. □

Lemma 3.14. $d \neq 1$

Proof. Suppose $d=1$. By Lemma 3.11, a is even.

Case 1: $a=2$. So $T = 9m+5 \equiv 1 \pmod{2}$. By Lemma 3.12, the labels on the vertices are $\{2,4,6,\ldots,2m\} \cup \{2m+3, 2m+5, \ldots, 6m+1\}$. Consider the component that has a vertex labeled 2. Since T is odd, exactly one of the other two vertex labels is odd. Thus the largest possible sum for the three vertex labels is $2+2m+(6m+1) = 8m+3 < 9m+5 = T$. This is a contradiction.

Case 2: $a=4$. So $T = 9m+4 \equiv 0 \pmod{2}$. By Lemma 3.12, the labels on the vertices are $\{2,4,6,\ldots,4m\} \cup \{4m+3, 4m+5, \ldots, 6m+1\}$. Consider the component that has a vertex labeled $6m+1$. Since T is even, exactly one of the other two vertex labels is odd. Thus the smallest possible sum for the three vertex labels is $(6m+1)+(4m+3)+2 = 10m+6 > 9m+4 = T$. This is a contradiction.

Case 3: $a=6$. So $T = 9m+3 \equiv 1 \pmod{2}$. However, all of the vertex labels are even by Lemma 3.12. This is a contradiction. □

Lemma 3.15. *The number of vertex labels congruent to $i \pmod{2d}$ is $\frac{cm}{2d}$ where*

$$c = \begin{cases} 6 & \text{if } i = 0 \text{ or } d+1 < i < 2d, \\ 6-a & \text{if } i = 1, \\ a & \text{if } i = d+1, \\ 0 & \text{if } 2 \leq i \leq d. \end{cases}$$

Proof. This follows from Lemmas 3.12 and 3.13. □

Lemma 3.16. *If $a=1$ then $d \neq 2$.*

Proof. Suppose $a = 1$ and $d = 2$. Note that $4 \mid m$, by Lemma 3.13. By Lemma 3.15, the vertex labels are distributed in the four congruence classes modulo 4 as follows: $6m/4$ are congruent to 0, $5m/4$ are congruent to 1, 0 are congruent to 2, and $m/4$ are congruent to 3. Since $T = 9m + 7 \equiv 3 \pmod 4$, any component that has a vertex label congruent to 0 modulo 4 must have another vertex label congruent to 0 modulo 4 and the third vertex congruent to 3 modulo 4. Hence, there are $3m/4$ such components. However, this is impossible since there are only $m/4$ vertex labels congruent to 3 modulo 4. □

Lemma 3.17. $0 \le (T-3) \bmod 2d \le d-1$.

Proof. Toward a contradiction, suppose $d \le (T-3) \bmod 2d \le 2d-1$.

Case 1: $(T-3) \bmod 2d \le 2d-3$. So $d+2 \le (T-1) \bmod 2d \le 2d-1$. By Lemma 3.15, for any component with a vertex label congruent to $(T-1)$ modulo $2d$, the other two vertices must have labels congruent to 0 and 1 modulo $2d$. There must be $\frac{6m}{2d}$ such components. However, there are only $\frac{(6-a)m}{2d}$ vertex labels congruent to 1 modulo $2d$.

Case 2: $2d-2 \le (T-3) \bmod 2d$. Then $(T-2) \bmod 2d \in \{0, 2d-1\}$. By Lemma 3.15, for any component with a vertex label congruent to $T-2$ modulo $2d$, the other two vertex labels must either both be congruent to 1 or $d+1$ modulo $2d$. There must be $\frac{6m}{2d}$ such components. However, there are only $\frac{6-a}{2d}$ vertex labels congruent to 1 modulo $2d$ and $\frac{a}{2d}$ congruent to $d+1$ modulo $2d$. This provides enough labels for at most $\frac{6m}{4d}$ such components. □

Completing the proof of Theorem 3.6. By Lemmas 3.10 and 3.17,

$$0 \le (9m + \tfrac{9-a+3d}{2} - 3) \mod 2d \le d-1.$$

Since $2d \mid 3m$ by Lemma 3.13, we have

$$0 \le (\tfrac{3-a}{2} + \tfrac{3d}{2}) \mod 2d \le d-1.$$

Note that $d > 1$ by Lemma 3.14. It follows that $\frac{3-a}{2} + \frac{3d}{2} \ge 0$.

Case 1: $\frac{3-a}{2} + \frac{3d}{2} \le d-1$. Then $d \le a-5 \le 1$, contradicting Lemmas 3.8 and 3.14.

Case 2: $\frac{3-a}{2} + \frac{3d}{2} \ge 2d$. Then $3-a \ge d$. This can only be true if $a = 1$ and $d = 2$, contradicting Lemma 3.16.

□

One consequence of Theorem 3.6 is a theorem from [1].

Theorem 3.18 ([1]). *Let m be an even positive integer. Then, mK_3 has no totally magic labeling.*

Additionally, a stronger result holds.

Corollary 3.19. *Let m be an even positive integer. The total deficiency of mK_3 is 2.*

Proof. This follows from Theorem 3.6 and Lemma 3.4. □

4 Triangles and a Star

In this section we show that no additional totally magic graphs can be obtained by taking unions of stars and triangles. Although it was shown in [1] that no star other than $K_{1,2}$ is totally magic, the following theorem of J. P. McSorley shows that every star other than $K_{1,1}$ is TMI and determines the total deficiency of each of these stars.

Theorem 4.1 ([4, 6]). *The star $K_{1,s}$ has a totally magic injection provided $s > 1$. The total deficiency when $s > 2$ is $\binom{s+2}{2} - 2s - 3$.*

Since stars other than $K_{1,1}$ are TMI, it is possible that they could be components of totally magic graphs. It has been shown in [1, 6] that a TMI graph cannot have more than one star as a component, and that the only totally magic graphs with K_1 as a component are $K_1 \cup K_{1,2}$ and K_1 itself. Furthermore, as mentioned previously, $K_{1,2} \cup mK_3$ is totally magic for m even. The theorems in the next two subsections show that these are the only totally magic graphs that can be formed as a union of a star and some number of triangles.

4.1 Totally Magic Labelings

Theorem 4.2. *The graph $K_{1,2} \cup mK_3$ is a totally magic graph if and only if m is even.*

Proof. Suppose m is even. From 3.1 we know that there is a totally magic labeling of $(m+1)K_3$. If we take the derivative (as defined in Section 2) of this labeling of $(m+1)K_3$, it gives a totally magic labeling of $K_{1,2} \cup mK_3$.

Suppose that m is odd, and that $K_{1,2} \cup mK_3$ has a totally magic labeling. If we take the reverse of the derivative, we would have a totally magic labeling of $(m+1)K_3$. This is impossible. □

So we see that the union of mK_3 with the star $K_{1,2}$ is totally magic if and only if m is odd. Previous results do not rule out the possibility of forming a totally magic graph as the union of a larger star and some number of triangles. This suggests Wallis' Research Problem 4.1: Is the graph $K_{1,s} \cup mK_3$ ever totally magic for $s > 2$? We answer this question in the negative in the following theorem. Since the only cycle that is TMI is K_3 and the only trees that are TMI are stars and K_1, this theorem completes the characterization of all totally magic graphs with maximum degree 2 or less.

Theorem 4.3. *For any $m \geq 0$ and $s \geq 1$, suppose G is a totally magic graph isomorphic to $K_{1,s} \cup mK_3$. Then $s = 2$.*

Let λ be a totally magic labeling for G with vertex sum h and edge sum k. Let $d = k - h$. Let c be the central vertex of the star and $b_1, b_2, \ldots, b_s$ be the other vertices of the star. We use several lemmas to prove the theorem.

Lemma 4.4. *$\lambda(c) = d$.*

Proof. From the edge-magic equation $\lambda(b_1) + \lambda(b_1 c) + \lambda(c) = k$ subtract the vertex-magic equation $\lambda(b_1) + \lambda(b_1 c) = h$. □

Lemma 4.5. *Let M be the maximum label. Then $M = 6m + 2s + 1$.*

Proof. The number of vertices of G is $3m + s + 1$. The number of edges is $3m + s$. The sum of these is the maximum label. □

Lemma 4.6. $h = \frac{M(M+1)/2-(m+1)d}{2m+s}$

Proof. The sum of all the labels is $M(M + 1)/2$. Since the sum of the labels on each K_3 is $h + k$ and for each i, $\lambda(b_i) + \lambda(b_i c) = h$ we can also find the sum of the labels as $d + sh + m(h + k) = d + sh + m(2h + d) = (2m + s)h + (m + 1)d$. Setting these expressions equal and solving for h gives the result. □

Lemma 4.7. $h(s - 1) \leq s(M - (s - 1)/2) - d$.

Proof. Start with the vertex-magic equation $d + \Sigma\lambda(b_i c) = h$. This yields $h - d = \Sigma\lambda(b_i c)$. Now, notice that since M is the largest label,

$$\begin{aligned}\Sigma\lambda(b_i) &\leq M + (M-1) + \cdots + (M - s + 1)\\ &= sM - s(s-1)/2 = s(M - (s-1)/2).\end{aligned}$$

By the vertex-magic property $\lambda(b_i c) = h - \lambda(b_i)$ for $i = 1, \ldots, s$. Hence $h - d \geq sh - s(M - (s - 1)/2)$. The result follows. □

Lemma 4.8. $(s - 3)(6m^2 + 3(s + 1)m + s(s + 2)/2) + s - 1 + d \leq (s - 3)md$

Proof. First, substitute the expression for h from Lemma 4.6 into the inequality from Lemma 4.7 to get

$$\frac{(s-1)(M(M+1)/2 - (m+1))d}{2m+s} \leq s(M - (s-1)/2) - d.$$

Now substitute the expression for M from lemma 4.5 and multiply both sides by $2m + s$ to get

$$\begin{aligned}&(s-1)((6m + 2s + 1)(3m + s + 1) - (m+1)d) \leq\\ &\qquad (2m+s)(s(6m + 3s/2 + 3/2) - d).\end{aligned}$$

Expanding we get

$$\begin{aligned}&(s-1)(18m^2 + 12ms + 2s^2 + 9m + 3s + 1 - (m+1)d) + 2md \leq\\ &\qquad s(12m^2 + 9ms + 3s^2/2 + 3m + 3s/2 - d).\end{aligned}$$

This can be rearranged to obtain

$$6(s-3)m^2 + 3(s-3)(s+1)m + s(s-3)(s+2)/2 + s - 1 + d \leq (s-3)dm.$$

Factoring $(s - 3)$ from the first three terms gives the result. □

Lemma 4.9. *If $s \geq 3$, then $(s-3)(6m^2 + 3(s+1)m) < (s-3)md$.*

Proof. If $s \geq 3$, then $(s-3)s(s+2)/2 + s - 1 + d > 0$. The result follows from Lemma 4.8. □

Lemma 4.10. $s = 2$

Proof. If $s = 3$ or if $s > 3$ and $m = 0$, then Lemma 4.9 implies $0 < 0$. If $s > 3$ and $m > 0$, then Lemmas 4.5 and 4.9 imply $M = 6m + 2s + 1 < 6m + 3(s+1) < d$. But by Lemma 4.4, $M \geq d$. Finally, if $s = 1$ then $\lambda(b_1) = d = \lambda(c)$, so G is not totally magic. The only remaining possibility is $s = 2$. □

4.2 Totally Magic Injections

We have shown that $K_{1,s} \cup mK_3$ is totally magic if and only if $s = 2$ and m is even. However, this leaves open the question of whether $K_{1,s} \cup mK_3$ is a TMI graph. We will show in Theorem 4.12 below that $K_{1,s} \cup mK_3$ is TMI for all $s > 1$ and all $m \geq 0$. First we prove a lemma that will be useful in constructing a totally magic injection for $K_{1,s} \cup mK_3$.

Lemma 4.11. *For all positive integers m and s, there is a totally magic injection of mK_3 with vertex magic number h and edge magic number k such that*

$$\frac{s(s+1)}{2} \leq 2h - k$$

and every label is greater than $d = k - h$.

Proof. By Corollary 3.5 there is a totally magic injection of mK_3, say with $h = a$ and $k = b$. We get a new totally magic injection by adding c to every vertex label, where $c = \max\{d, (s(s+1)/2 - 2a + b)/3\}$. The result is a totally magic injection with $h = a + 3c$ and $k = b + 3c$. Note that d is unchanged. Also, the least label used in the new labeling is at least $1 + c > d$ and

$$2h - k = 2(a + 3c) - (b + 3c) \geq 2a - b + 3(s(s+1)/2 - 2a + b)/3 = s(s+1)/2.$$

□

Theorem 4.12. *For any $m \geq 0$ and $s > 1$, $K_{1,s} \cup mK_3$ is a TMI graph.*

Proof. The cases $m = 0$ and $s = 2$ are handled in [6], so we assume $m \geq 1$ and $s \geq 3$. Consider a total labeling of the triangles that satisfies Lemma 4.11, say with $h = a$ and $k = b$. We may assume $a < b$. Multiply each label by n, where $n = s(s-1)/2+1$. This gives us a totally magic injection of the triangles with $h = na$, $k = nb$, and $d = nb - na$. Note that by the conditions of Lemma 4.11 each label is greater than d. Now label the central vertex of the star d. Label $s - 1$ of the edges $1, 2, \ldots, s-1$ and label the corresponding vertices $na - 1, na - 2, \ldots, na - s + 1$. Label the last edge $n(2a - b) - s(s-1)/2$, and label the corresponding vertex $n(b - a) + s(s-1)/2$. Note that none of these labels are divisible by n, so they have not been used in the labeling of the triangles or the central vertex. Also, by the conditions of Lemma 4.11,

$$n(2a-b)-s(s-1)/2 \geq ns(s+1)/2 - s(s-1)/2 \geq s.$$

Hence the labels on the edges are all distinct. Since $na+s-1 < nb+s(s-1)/2$, we also have that $n(2a-b)-s(s-1)/2 < na-s+1$. Clearly,

$$s-1 < n(b-a)+s(s-1)/2.$$

Furthermore, by the conditions of Lemma 4.11,

$$n(2a-b) \geq 2a-b \geq s(s+1)/2 > s(s-1)/2+s-1.$$

So $na-s+1 > n(b-a)+s(s-1)/2$. It follows that all the labels are distinct except possibly $n(2a-b)-s(s-1)/2$ and $n(b-a)+s(s-1)/2$. Suppose we had $n(2a-b)-s(s-1)/2 = n(b-a)+s(s-1)/2$. Then $(s(s-1)/2+1)(3a-2b) = s(s-1)$ or equivalently $(1+\frac{2}{s(s-1)})(3a-2b) = 2$. Thus $0 < 3a-2b < 2$. Since $3a-2b$ is an integer, $3a-2b=1$. But then $s=2$, so this case cannot occur. We have shown all the labels are distinct. It is now easy to verify that the magic equations hold. So we have a totally magic injection of the graph. □

The previous proof shows the existence of a totally magic injection of $K_{1,s} \cup mK_3$ for $m \geq 0$ and $s > 1$. However, no attempt has been made to achieve the minimum total deficiency. This leaves open the following question.

Question 4.13. *What is the total deficiency of $K_{1,s} \cup mK_3$?*

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