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Complementary cycles in regular multipartite tournaments

Lutz Volkmann

Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen
Germany
volkm@math2.rwth-aachen.de

Abstract

A c-partite tournament is an orientation of a complete c-partite graph. A digraph D is cycle complementary if there exist two vertex disjoint cycles C and C' such that $V(D) = V(C) \cup V(C')$. In 1999, Yeo conjectured that each regular c-partite tournament D with $c \geq 4$ and $|V(D)| \geq 6$ has a pair of vertex disjoint cycles of length t and $|V(D)| - t$ for all $t \in \{3, 4, \ldots, |V(D)| - 3\}$. In this paper we prove that this conjecture is valid for the case $t = 3$, unless D is isomorphic to T_7, $D_{4,2}$, or $D^*_{4,2}$, where T_7 is a 3-regular tournament with 7 vertices and $D_{4,2}$ and $D^*_{4,2}$ are 3-regular 4-partite tournaments such that there are exactly two vertices in each partite set.

1. Terminology

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. By a *cycle* (*path*) we mean a directed cycle (directed path).

We shall assume that the reader is familiar with standard terminology on directed graphs (see, e.g., Bang-Jensen and Gutin [1]). In this paper all digraphs are finite without loops or multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y. If X and Y are two disjoint subsets of $V(D)$ or subdigraphs of D such that each vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the property that there is no arc from Y to X. The number of arcs going from X to Y is denoted by $d^+(X, Y)$.

The *out-neighborhood* $N^+_D(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x, and the *in-neighborhood* $N^-_D(x) = N^-(x)$ is the set of vertices dominating x. For a set of vertices X in D, we define $D[X]$ as the subdigraph induced by X.

The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are the *out-degree* and *indegree* of x, respectively. The *minimum outdegree* and the *minimum indegree* of D are denoted by $\delta^+(D)$ and $\delta^-(D)$, and the *maximum outdegree* and the *maximum indegree* of D are denoted by $\Delta^+(D)$ and $\Delta^-(D)$, respectively.

The *global irregularity* of a digraph D is defined by

$$i_g(D) = \max\{\max(d^+(x), d^-(x)) - \min(d^+(y), d^-(y)) \,|\, x, y \in V(D)\},$$

and the *local irregularity* by $i_l(D) = \max |d^+(x) - d^-(x)|$ over all vertices x of D. If $i_g(D) = 0$, then D is *regular*.

A cycle of length m is an *m-cycle*. A cycle in a digraph D is *Hamiltonian* if it includes all the vertices of D. A set $X \subseteq V(D)$ of vertices is *independent* if the induced subdigraph $D[X]$ has no arcs. The *independence number* $\alpha(D) = \alpha$ is the maximum size among the independent sets of vertices of D. A digraph D is *strongly connected* or *strong* if, for each pair of vertices u and v, there is a path from u to v in D. A digraph D with at least $k+1$ vertices is *k-connected* if for any set A of at most $k-1$ vertices, the subdigraph $D - A$ obtained by deleting A is strong. The *connectivity* of D, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k-connected. A *cycle-factor* of a digraph D is a spanning subdigraph consisting of disjoint cycles. A cycle-factor with the minimum number of cycles is called a *minimal cycle-factor*. If x is a vertex of a cycle C, then the *predecessor* and the *successor* of x on C are denoted by x^- and x^+, respectively.

2. Introduction and preliminary results

A digraph D is *cycle complementary* if there exist two vertex disjoint cycles C and C' such that $V(D) = V(C) \cup V(C')$. The problem of complementary cycles in tournaments was almost completely solved by Reid [7] in 1985 and by Z. Song [10] in 1993. They proved that every 2-connected tournament T on at least 8 vertices has complementary cycles of length t and $|V(T)|-t$ for all $t \in \{3, 4, \ldots, |V(T)|-3\}$. Some years later, Guo and Volkmann [4], [5] extended this result to locally semicomplete digraphs. A digraph is *locally semicomplete* if for each vertex x, the set of in-neighbors as well as the set of out-neighbors of x induce semicomplete digraphs, where a digraph is called *semicomplete* if any two vertices are adjacent. The more general problem of partitioning a highly connected tournament into k vertex-disjoint cycles was posed by Bollobás (see Reid [8]). Recently, Chen, Gould, and Li [3] proved that every k-connected tournament T with $|V(T)| \geq 8k$ contains k vertex-disjoint cycles spanning the vertex set.

In addition, there are some results on complementary cycles in bipartite tournaments by Z. Song [9], K. Zhang and Z. Song [19], K. Zhang, Manoussakis and Z. Song [18], and K. Zhang and J. Wang [20]. However, there is nothing known on complementary cycles in c-partite tournaments for $c \geq 3$. There exist only the following two conjectures.

Conjecture 2.1 (Yeo [16] 1999) A regular c-partite tournament D with $c \geq 4$ and $|V(D)| \geq 6$ has a pair of vertex disjoint cycles of length t and $|V(D)| - t$ for all $t \in \{3, 4, \ldots, |V(D)| - 3\}$.

Conjecture 2.2 (Volkmann, [11] 2002) Let D be a multipartite tournament. If $\kappa(D) \geq \alpha(D) + 1$, then D is cycle complementary, unless D is a member of a finite family of multipartite tournaments.

Our first three examples will show that Conjecture 2.1 by Yeo is not valid in general when $t = 3$.

Example 2.3 (Reid [7] 1985) Let T_7 be the 3-regular tournament presented in Figure 1. Then it is straightforward to verify that T_7 does not contain a 3-cycle C_3 and a 4-cycle C_4 such that $V(T_7) = V(C_3) \cup V(C_4)$.

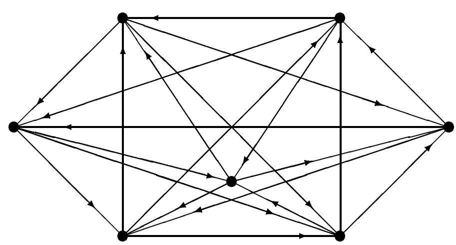

Figure 1: The 3-regular tournament T_7

Example 2.4 Let $V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2\}$, $V_3 = \{u_1, u_2\}$, and $V_4 = \{v_1, v_2\}$ be the partite sets of the 3-regular 4-partite tournament $D_{4,2}$ presented in Figure 2. Then it is straightforward to verify that $D_{4,2}$ does not contain a 3-cycle C_3 and a 5-cycle C_5 such that $V(D_{4,2}) = V(C_3) \cup V(C_5)$.

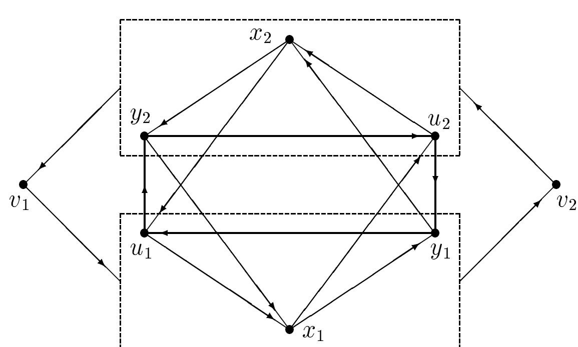

Figure 2: The 3-regular 4-partite tournament $D_{4,2}$

Example 2.5 Let $D^*_{4,2}$ be the 3-regular 4-partite tournament presented in Figure 3. Then it is straightforward to verify that $D^*_{4,2}$ does not contain a 3-cycle C_3 and a 5-cycle C_5 such that $V(D^*_{4,2}) = V(C_3) \cup V(C_5)$.

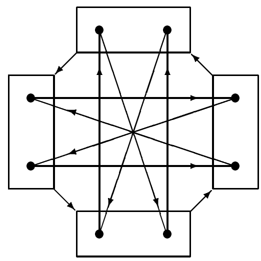

Figure 3: The 3-regular 4-partite tournament $D^*_{4,2}$

However, we will show in this paper that Conjecture 2.1 is true for $t = 3$, unless D is isomorphic to T_7, $D_{4,2}$, or $D^*_{4,2}$ (cf. Examples 2.3, 2.4, and 2.5).

The following results play an important role in our investigations. We start with a well-known fact about regular multipartite tournaments.

Lemma 2.6 *If D is a regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$, then $\alpha(D) = |V_1| = |V_2| = \ldots = |V_c|$.*

Theorem 2.7 (Bondy [2] 1976) *Each strong c-partite tournament with $c \geq 3$ contains an m-cycle for each $m \in \{3, 4, \ldots, c\}$.*

Theorem 2.8 (Reid [7] 1985) *If T is a 2-connected tournament with $|V(T)| \geq 6$, then T contains two complementary cycles of length 3 and $|V(T)| - 3$, unless T is the tournament T_7 described in Example 2.3.*

Theorem 2.9 (Yeo [15] 1998) *If D is a multipartite tournament, then*

$$\kappa(D) \geq \frac{|V(D)| - \alpha(D) - 2i_l(D)}{3}.$$

Theorem 2.10 (Yeo [14] 1997) *Let D be a $(\lfloor q/2 \rfloor + 1)$-connected multipartite tournament such that $\alpha(D) \leq q$. If D has a cycle-factor, then D is Hamiltonian.*

Theorem 2.11 (Yeo [17] 1999) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If*

$$i_g(D) \leq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 2}{2},$$

then D is Hamiltonian.

Lemma 2.12 (Yeo [17] 1999, Gutin, Yeo [6] 2000) *A digraph D has no cycle-factor if and only if its vertex set $V(D)$ can be partitioned into four subsets Y, Z, R_1, and R_2 such that*

$$R_1 \leadsto Y \quad \text{and} \quad (R_1 \cup Y) \leadsto R_2,$$

where Y is an independent set and $|Y| > |Z|$.

Theorem 2.13 (Yeo [14] 1997) *Let D be a multipartite tournament having a cycle-factor but no Hamiltonian cycle. Then there exists a partite set V^* of D and an indexing $C_1, C_2, \dots, C_t$ of the cycles of some minimal cycle-factor of D such that for all arcs yx from C_j to C_1 for $2 \le j \le t$, it holds $\{y^+, x^-\} \subseteq V^*$.*

3. Main results

Theorem 3.1 *Let D be a regular c-partite tournament with $c \ge 4$ and $|V(D)| \ge 6$. Then D contains two complementary cycles of length 3 and $|V(D)| - 3$, unless D is isomorphic to T_7, $D_{4,2}$, or $D^*_{4,2}$.*

Proof. If $V'_1, V'_2, \dots, V'_c$ are the partite sets of D, then Lemma 2.6 leads to $|V'_1| = |V'_2| = \ldots = |V'_c| = \alpha(D)$. Let $r = \alpha(D)$; hence $|V(D)| = cr$. According to Theorem 2.9, we have

$$\kappa(D) \ge \frac{|V(D)| - \alpha(D) - 2i_l(D)}{3} = \frac{(c-1)r}{3}. \tag{1}$$

If $c \ge 6$ and $r = 1$ (that is, D is a tournament), then (1) yields $\kappa(D) \ge 2$, and the desired results follows from Theorem 2.8.

Therefore, it remains the case that $r \ge 2$. In view of Theorem 2.7, there exist a 3-cycle C_3 in D. If we define the c-partite tournament H by $H = D - V(C_3)$, then $i_g(H) \le 3$ and $|V(H)| = cr - 3$. If $V_1, V_2, \dots, V_c$ are the partite sets of H such that $|V_1| \le |V_2| \le \ldots \le |V_c|$, then $|V_1| = r - 1$, $|V_c| = r$, and $|V_3| = |V_{c-1}| = r - 1$ in the case that $c = 4$. With exception of the cases $c = 6$ and $r = 2$, $c = 5$ and $r \le 3$, and $c = 4$ and $r \le 5$, the hypothesis leads to

$$i_g(H) \le 3 \le \frac{|V(H)| - |V_{c-1}| - 2|V_c| + 2}{2}.$$

Applying Theorem 2.11, we conclude that H has a Hamiltonian cycle C, and we obtain the desired result that $V(D) = V(C_3) \cup V(C)$. Since there is no regular c-partite tournament for $c = 4$ and $r = 3, 5$, there remain the cases $c = 6$ and $r = 2$, $c = 5$ and $r = 2, 3$, and $c = 4$ and $r = 2, 4$.

Case 1. Suppose that $c = 6$ and $r = 2$.

Then D is 5-regular and $\alpha(H) = 2$. In addition, inequality (1) yields $\kappa(D) \ge 4$, and thus $\kappa(H) \ge 1$.

Subcase 1.1. Assume that H has a cycle-factor.

Let $C'_1, C'_2, \dots, C'_t$ be a minimal cycle-factor with the properties described in Theorem 2.13. If $t = 1$, then H has the Hamiltonian cycle C'_1 and so $V(D) =$

$V(C_3) \cup V(C_1')$. If not, then $|V(H)| = 9$ implies $t \le 3$. Because of $|V^*| \le 2$, it follows from Theorem 2.13 that there is at most one arc from $C_2' \cup C_3'$ to C_1'. If C_1' is a 3-cycle, then it is easy to see that there is a vertex x in C_1' such that $d_H^+(x) \ge 6$, a contradiction to the 5-regularity of D. If C_1' is a 4-cycle, then $t = 2$ and C_2' is a 5-cycle. If C_1' induces a tournament in H, then, since there is only one arc from C_2' to C_1', we obtain

$$\begin{aligned}\sum_{x \in V(C_1')} d_H^+(x) &= \sum_{x \in V(C_1')} d_{D[V(C_1')]}^+(x) + d^+(C_1', C_2') \\ &\ge 6 + (17 - 1) = 22,\end{aligned}$$

a contradiction to the 5-regularity of D. The same holds true if C_1' induces a 3-partite or a bipartite tournament. If C_1' is a 5-cycle and C_2' is a 4-cycle, then we arrive analogously at the contradiction $\sum_{x \in V(C_2')} d_H^-(x) \ge 22$. Finally, if C_1' is a 6-cycle, then it is easy to see that there is a vertex x in C_2' such that $d_H^-(x) \ge 6$, a contradiction to the 5-regularity of D.

Subcase 1.2. Assume that H has no cycle-factor.

Then, with respect to Lemma 2.12, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \leadsto Y$, $(R_1 \cup Y) \leadsto R_2$, $|Y| > |Z|$, and Y is an independent set. Since $\kappa(H) \ge 1$ and $\alpha(H) = 2$, we see that $1 = |Z| < |Y| = 2$. Let $V_1 = \{a\}, V_2 = \{b\}, V_3 = \{c\}, V_4 = \{u_1, u_2\}, V_5 = \{v_1, v_2\}$, and $V_6 = \{w_1, w_2\}$, and, without loss of generality, $Y = V_4$. Since D is 5-regular, we see that $d_H^+(x), d_H^-(x) \ge 2$ for all $x \in V(H)$ and $d_H^+(x), d_H^-(x) \ge 3$ for $x = a, b, c$.

If $R_1 = \emptyset$, then $V_4 = Y \leadsto R_2$ leads to the contradiction $d_H^-(u_1) \le 1$. If $R_2 = \emptyset$, then $R_1 \leadsto V_4$ leads to the contradiction $d_H^+(u_1) \le 1$. If $1 \le |R_1| \le 2$, then there exists a vertex $x \in R_1$ such that $d_H^-(x) \le 1$, a contradiction. If $1 \le |R_2| \le 2$, then there exists a vertex $x \in R_2$ such that $d_H^+(x) \le 1$, a contradiction. In the remaining case that $|R_1| = |R_2| = 3$, let $R_1 = \{x, y, z\}$ and let, without loss of generality, $d_H^-(x) \ge 3$. This implies $\{y, z\} \to x$ and so, $d_H^-(y) \le 1$ or $d_H^-(z) \le 1$, a contradiction.

Case 2. Suppose that $c = 5$ and $r = 3$.

Then D is 6-regular and $\alpha(H) = 3$. Furthermore, inequality (1) yields $\kappa(D) \ge 4$. Suppose that there exists a separating set S of D with $|S| = 4$. Let $D_1, D_2, \ldots, D_t$ be the strong components of $D - S$ such that $D_i \leadsto D_j$ for $1 \le i < j \le t$. Then we can assume, without loss of generality, that $|V(D_1)| \le 5$. If there exists a vertex $x \in V(D_1)$ with $d_{D_1}^-(x) \le 1$, then $d_D^-(x) \le 5$, a contradiction. Otherwise, D_1 is necessarily a 2-regular tournament. But then there are vertices $x \in S$ and $y \in V(D_1)$ which are not adjacent. This leads to the contradiction $d_D^-(y) \le 5$. Consequently, we even observe that $\kappa(D) \ge 5$ and thus, $\kappa(H) \ge 2$.

Assume that H has a cycle-factor. Applying Theorem 2.10 with $q = 3$, we deduce that H has a Hamiltonian cycle C_{12} and so $V(D) = V(C_3) \cup V(C_{12})$.

Suppose now that H has no cycle-factor. Then, with respect to Lemma 2.12, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \leadsto Y$, $(R_1 \cup Y) \leadsto R_2$, $|Y| > |Z|$, and Y is an independent set. Since $\kappa(H) \ge 2$ and $\alpha(H) = 3$, we see that $2 = |Z| < |Y| = 3$. Let $V_1 = \{u_1, u_2\}, V_2 = \{v_1, v_2\}, V_3 =$

$\{w_1, w_2\}, V_4 = \{x_1, x_2, x_3\}$, $V_5 = \{y_1, y_2, y_3\}$, and, without loss of generality, $Y = V_4$. Since D is 6-regular, we see that $d_H^+(x), d_H^-(x) \geq 3$ for all $x \in V(H)$ and $d_H^+(x), d_H^-(x) \geq 4$ for $x \in V_1 \cup V_2 \cup V_3$.

If $R_1 = \emptyset$, then $V_4 \rightsquigarrow R_2$ leads to the contradiction $d_H^-(x_1) \leq 2$. If $R_2 = \emptyset$, then $R_1 \rightsquigarrow V_4$ leads to the contradiction $d_H^+(x_1) \leq 2$. If $1 \leq |R_1| \leq 2$, then there exists a vertex $x \in R_1$ such that $d_H^-(x) \leq 2$, a contradiction. If $1 \leq |R_2| \leq 2$, then there exists a vertex $x \in R_2$ such that $d_H^+(x) \leq 2$, a contradiction. In the remaining, assume, without loss of generality, that $|R_1| = 3$. If $R_1 = V_5$, then $d_H^-(x) \leq 2$ for every vertex $x \in R_1$, a contradiction. Otherwise, let $R_1 = \{x, y, z\}$ and let, without loss of generality, $d_H^-(x) \geq 4$. This implies $\{y, z\} \to x$ and so, $d_H^-(y) \leq 2$ or $d_H^-(z) \leq 2$, a contradiction.

Case 3. Suppose that $c = 5$ and $r = 2$.

Then D is 4-regular and $\alpha(H) = 2$. Furthermore, inequality (1) implies $\kappa(D) \geq 3$. Let $V_1' = \{x_1, x_2\}, V_2' = \{y_1, y_2\}, V_3' = \{u_1, u_2\}, V_4' = \{v_1, v_2\}$, and $V_5' = \{w_1, w_2\}$. Suppose that there exists a separating set S of D with $|S| = 3$. Let $D_1, D_2, \ldots, D_t$ be the strong components of $D - S$ such that $D_i \rightsquigarrow D_j$ for $1 \leq i < j \leq t$. Then we can assume, without loss of generality, that $|V(D_1)| \leq 3$. If there exists a vertex $x \in V(D_1)$ with $d_{D_1}^-(x) = 0$, then $d_D^-(x) \leq 3$, a contradiction. Otherwise, D_1 is a 1-regular tournament. Assume, without loss of generality, that D_1 is the 3-cycle $x_1y_1u_1x_1$. It follows that $S \to V(D_1)$ and $S \subseteq V_4' \cup V_5'$, say $S = \{w_1, w_2, v_1\}$. Hence, there remain the two cases that $D_2 = v_2$ and D_3 is, without loss of generality, the 3-cycle $x_2y_2u_2x_2$ and $V(D_2) = \{x_2, y_2, u_2, v_2\}$. In the first case, we observe that $V(D_3) \to S$ and we arrive at the two complementary cycles $C_3' = v_1x_1y_2v_1$ and $w_1y_1u_2w_2u_1v_2x_2w_1$. In the second case we assume, without loss of generality, that $u_2 \to v_2 \to \{x_2, y_2\}$ and then $x_2 \to y_2$. This implies $v_2 \to \{w_1, w_2\}$ and $y_2 \to (\{u_2\} \cup S)$. If $x_2 \to u_2$, then $u_2 \to v_1$ and we arrive at the two complementary cycles $C_3' = v_1x_1u_2v_1$ and $w_1y_1x_2y_2w_2u_1v_2w_1$. If $u_2 \to x_2$, then $x_2 \to v_1$ and we arrive at the two complementary cycles $C_3' = v_1y_1x_2v_1$ and $w_1x_1u_2v_2w_2u_1y_2w_1$.

Consequently, it remains the case that $\kappa(D) \geq 4$, and thus $\kappa(H) \geq 1$.

Firstly, assume that H has a cycle-factor. Let $C_1', C_2', \ldots, C_t'$ be a minimal cycle-factor with the properties described in Theorem 2.13. If $t = 1$, then H has the Hamiltonian cycle C_1' and so $V(D) = V(C_3) \cup V(C_1')$. If not, then $|V(H)| = 7$ implies $t = 2$. Because of $|V^*| \leq 2$, it follows from Theorem 2.13 that there is at most one arc from C_2' to C_1'. Let $V_1 = \{x_1\}, V_2 = \{y_1\}, V_3 = \{u_1\}, V_4 = \{v_1, v_2\}, V_5 = \{w_1, w_2\}$, $C_3 = x_2y_2u_2x_2$, and, without loss of generality, C_1' a 3-cycle and C_2' a 4-cycle. If C_1' contains at least two vertices of $\{x_1, y_1, u_1\}$, then it follows that

$$\sum_{x \in V(C_1')} d_H(x) \geq 3 + (11 - 1) = 13,$$

a contradiction to the 4-regularity of D. It remains, without loss of generality, the case that $C_1' = x_1v_1w_1x_1$ and $C_2' = y_1u_1v_2w_2y_1$. According to Theorem 2.13, there only exists the arc v_2x_1 from C_2' to C_1'. Thus, the 4-regularity of D implies $C_3 \rightsquigarrow C_1'$, $C_2' \rightsquigarrow \{y_2, u_2\}$, and $y_1 \rightsquigarrow C_3$. Consequently, we have the two complementary cycles $y_1x_2v_1y_1$ and $u_1v_2w_2y_2u_2w_1x_1u_1$.

Secondly, assume that H has no cycle-factor. Then, with respect to Lemma 2.12, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \rightsquigarrow Y$, $(R_1 \cup Y) \rightsquigarrow R_2$, $|Y| > |Z|$, and Y is an independent set. Since $\kappa(H) \geq 1$ and $\alpha(H) = 2$, we see that $1 = |Z| < |Y| = 2$. Let $V_1 = \{x_1\}, V_2 = \{y_1\}, V_3 = \{u_1\}, V_4 = \{v_1, v_2\}, V_5 = \{w_1, w_2\}$, $C_3 = x_2y_2u_2x_2$, and, without loss of generality, $Y = V_5$. Since D is 4-regular, we see that $\delta^+(H), \delta^-(H) \geq 1$ and $d_H^+(x), d_H^-(x) \geq 2$ for $x \in \{x_1, y_1, u_1\}$.

Subcase 3.1. Suppose that $Z \subset V_4$, say $Z = \{v_1\}$.

It is easy to see that $1 \leq |R_1|, |R_2| \leq 2$ is impossible. Therefore, it remains the case that, without loss of generality, $R_1 = \emptyset$ and $R_2 = \{x_1, y_1, u_1, v_2\}$. Because of $Y \to R_2$, it follows that $V(C_3) \to V_5$. Furthermore, $v_1 \to V_5 = \{w_1, w_2\}$ and v_2 has at least one positive neighbor in R_2, say $v_2 \to x_1$. Assume, without loss of generality, that $y_1 \to u_1$.

Subcase 3.1.1. Suppose that $u_1 \to x_1$.

Then $x_1 \to \{y_1, v_1\}$. If $u_1 \to v_1$ and, without loss of generality, $y_1 \to u_2$, then there exists the 3-cycle $C_3' = v_1w_1u_1v_1$ and the complementary cycle of length 7 in D: $w_2v_2x_1y_1u_2x_2y_2w_2$. Otherwise, $v_1 \to u_1$ and thus, $u_1 \to v_2$. If, without loss of generality, $v_2 \to u_2$, then there exists the 3-cycle $C_3' = v_1w_1x_1v_1$ and the complementary cycle of length 7 in D: $w_2y_1u_1v_2u_2x_2y_2w_2$.

Subcase 3.1.2. Suppose that $x_1 \to u_1$.

Then $u_1 \to \{v_1, v_2\}$. If $x_1 \to v_1$ and, without loss of generality, $v_2 \to u_2$, then there exists the 3-cycle $C_3' = v_1w_1x_1v_1$ and the complementary cycle $w_2y_1u_1v_2u_2x_2y_2w_2$ of length 7 in D. Otherwise, $v_1 \to x_1$ and thus, $x_1 \to y_1$. If, without loss of generality, $y_1 \to u_2$, then there exists the 3-cycle $C_3' = v_1w_1u_1v_1$ and the complementary cycle $w_2v_2x_1y_1u_2x_2y_2w_2$ of length 7 in D.

Subcase 3.2. Suppose that $Z \subset V_1 \cup V_2 \cup V_3$, say $Z = \{x_1\}$.

We can assume, without loss of generality, that $|R_1| \leq 2$.

Subcase 3.2.1. Suppose that $|R_1| = 2$.

Then we can assume, without loss of generality, that $R_1 = \{u_1, v_1\}$ and $R_2 = \{y_1, v_2\}$ such that $v_1 \to u_1$, $y_1 \to v_2$ and $\{y_1, v_2\} \to x_1 \to \{u_1, v_1\}$. This implies $v_2 \to V(C_3) \to v_1$. If $w_1 \to x_1$, then there exists the 3-cycle $C_3' = x_1u_1w_1x_1$ and the complementary cycle $v_1w_2y_1v_2u_2x_2y_2v_1$ of length 7 in D. If $x_1 \to w_1$, then there exists the 3-cycle $C_3' = x_1w_1y_1x_1$ and the complementary cycle $v_1u_1w_2v_2u_2x_2y_2v_1$ of length 7 in D.

Subcase 3.2.2. Suppose that $|R_1| = 1$.

Then we can assume, without loss of generality, that $R_1 = \{v_1\}$ and $R_2 = \{u_1, y_1, v_2\}$ such that $x_1 \to v_1$. If we assume, without loss of generality, that $y_1 \to u_1$, then it follows that $u_1 \to \{x_1, v_2, x_2, y_2\}$ and $V(C_3) \to v_1$.

Subcase 3.2.2.1. Suppose that $v_2 \to y_1$.

Then $y_1 \to \{x_1, x_2, u_2\}$.

If $x_1 \to v_2$, then $v_2 \to V(C_3)$, $x_2 \to \{w_1, w_2\}$, and we can assume, without loss of generality, that $u_2 \to w_1$. Now there exists the 3-cycle $C_3' = x_1v_1y_1x_1$ and the complementary cycle $u_2w_1u_1x_2w_2v_2y_2u_2$ of length 7 in D.

If $v_2 \to x_1$, then we can assume, without loss of generality, that $x_1 \to w_1$. Now there exists the 3-cycle $C_3' = x_1w_1v_2x_1$ and the complementary cycle $v_1w_2y_1u_1x_2y_2u_2v_1$

of length 7 in D.

Subcase 3.2.2.2. Suppose that $y_1 \to v_2$.

Then $v_2 \to x_1$ and $v_2 \to V(C_3)$.

If $x_1 \to w_1$ or $x_1 \to w_2$, say $x_1 \to w_1$, then there exists the 3-cycle $C_3' = x_1w_1u_1x_1$ and the complementary cycle $v_1w_2y_1v_2x_2y_2u_2v_1$ of length 7 in D. Otherwise, $\{w_1, w_2\} \to x_1$. This implies $V(C_3) \to \{w_1, w_2\}$, $x_1 \to y_1$ and thus, $y_1 \to \{x_2, u_2\}$. Hence, there exists the 3-cycle $C_3' = x_1v_1u_1x_1$ and the complementary cycle $u_2w_1y_1x_2w_2v_2y_2u_2$ of length 7 in D.

Subcase 3.2.3. Suppose that $|R_1| = 0$.

Then $R_2 = \{u_1, y_1, v_1, v_2\}$ and we can assume, without loss of generality, that $y_1 \to u_1$. Since $Y = \{w_1, w_2\} \to R_2$, we deduce that $Z = \{x_1\} \to \{w_1, w_2\}$ and $V(C_3) \to \{w_1, w_2\}$.

Subcase 3.2.3.1. Suppose that $v_1 \to u_1$.

This implies $u_1 \to \{x_1, x_2, y_2, v_2\}$. Furthermore, there exists an arc from v_2 to C_3, say $v_2 \to y_2$. Because of $d_H^-(x_1) \geq 2$, we conclude that $y_1 \to x_1$ or $v_1 \to x_1$ or $v_2 \to x_1$.

If $y_1 \to x_1$, then there exists the 3-cycle $C_3' = x_1w_1y_1x_1$ and the complementary cycle $w_2v_1u_1v_2y_2u_2x_2w_2$ of length 7 in D.

If $v_1 \to x_1$, then there exists the 3-cycle $C_3' = x_1w_1v_1x_1$ and the complementary cycle $w_2y_1u_1v_2y_2u_2x_2w_2$ of length 7 in D.

If $v_2 \to x_1$ and $y_1 \to v_1$, then there exists the 3-cycle $C_3' = x_1w_1v_2x_1$ and the complementary cycle $w_2y_1v_1u_1y_2u_2x_2w_2$ of length 7 in D. If $v_2 \to x_1$ and $v_1 \to y_1$, then there exists the 3-cycle $C_3' = x_1w_1v_2x_1$ and the complementary cycle $w_2v_1y_1u_1y_2u_2x_2w_2$ of length 7 in D.

Subcase 3.2.3.2. Suppose that $u_1 \to v_1$ and $v_2 \to u_1$.

This implies $u_1 \to \{x_1, x_2, y_2\}$. Furthermore, there exists an arc from v_1 to C_3 as well as an arc from v_2 to C_3, say $v_1 \to y_2$ and $v_2 \to y_2$. Because of $d_H^-(x_1) \geq 2$, we conclude that $y_1 \to x_1$ or $v_1 \to x_1$ or $v_2 \to x_1$.

If $y_1 \to x_1$, then there exists the 3-cycle $C_3' = x_1w_1y_1x_1$ and the complementary cycle $w_2v_2u_1v_1y_2u_2x_2w_2$ of length 7 in D.

If $v_2 \to x_1$, then there exists the 3-cycle $C_3' = x_1w_1v_2x_1$ and the complementary cycle $w_2y_1u_1v_1y_2u_2x_2w_2$ of length 7 in D.

If $v_1 \to x_1$ and $v_2 \to y_1$, then there exists the 3-cycle $C_3' = x_1w_1v_1x_1$ and the complementary cycle $w_2v_2y_1u_1y_2u_2x_2w_2$ of length 7 in D. If $v_1 \to x_1$ and $y_1 \to v_2$, then there exists the 3-cycle $C_3' = x_1w_1v_1x_1$ and the complementary cycle $w_2y_1v_2u_1y_2u_2x_2w_2$ of length 7 in D.

Subcase 3.2.3.3. Suppose that $u_1 \to v_1$ and $u_1 \to v_2$.

Then there exists an arc from v_1 to C_3 as well as an arc from v_2 to C_3, say $v_1 \to y_2$ and $v_2 \to y_2$.

If $v_1 \to x_1$, then there exists the 3-cycle $C_3' = x_1w_1v_1x_1$ and the complementary cycle $w_2y_1u_1v_2y_2u_2x_2w_2$ of length 7 in D.

If $v_2 \to x_1$, then there exists the 3-cycle $C_3' = x_1w_1v_2x_1$ and the complementary cycle $w_2y_1u_1v_1y_2u_2x_2w_2$ of length 7 in D.

In the remaining case that $x_1 \to \{v_1, v_2\}$, we see that $\{u_1, y_1\} \to x_1$, $v_1 \to y_1$, and $v_2 \to y_1$. Furthermore, we have $\{v_1, v_2\} \to C_3$ and $\{y_2, u_2\} \to x_1$. Now there exists the 3-cycle $C_3' = u_2w_1v_1u_2$ and the complementary cycle $x_1w_2y_1u_1v_2x_2y_2x_1$ of length 7 in D.

Case 4. Suppose that $c = 4$ and $r = 4$.

Then D is 6-regular, and inequality (1) yields $\kappa(D) \geq 4$. Suppose that there exists a separating set S of D with $|S| = 4$. Let $D_1, D_2, \ldots, D_t$ be the strong components of $D - S$ such that $D_i \rightsquigarrow D_j$ for $1 \leq i < j \leq t$. Then we can assume, without loss of generality, that $|V(D_1)| \leq |V(D_2)|$ and $|V(D_1)| \leq 6$. If there exists a vertex $x \in V(D_1)$ with $d^-_{D_1}(x) \leq 1$, then $d^-_D(x) \leq 5$, a contradiction. If $|V(D_1)| \leq 5$, then, since H is 4-partite, there exists a vertex $x \in V(D_1)$ with $d^-_{D_1}(x) \leq 1$, a contradiction. Consequently, $|V(D_1)| = |V(D_2)| = 6$, $\delta^-(D_1) \geq 2$, $\Delta^-(D_1) \leq 3$, and there is at most one vertex $v \in V(D_1)$ such that $d^-_{D_1}(v) = 3$. If D_1 is exactly 4-partite, then $H[S]$ is at least 2-partite, and thus, there exists a vertex $w \in V(D_1)$ such that $d^-_H(w) \leq 5$, a contradiction. Hence, D_1 as well as D_2 are 3-partite such that $d^-_{D_i}(x) = d^+_{D_i}(x) = 2$ for $i = 1, 2$ and for each $x \in V(D_i)$, and S consists of one partite set. In addition, $S \rightarrow V(D_1) \rightsquigarrow V(D_2) \rightarrow S$. Now let $V_1' = \{x_1, x_2, x_3, x_4\}, V_2' = \{u_1, u_2, u_3, u_4\}, V_3' = \{v_1, v_2, v_3, v_4\}$, $V_4' = \{w_1, w_2, w_3, w_4\}$, and assume, without loss of generality, that $S = V_1'$, $V(D_1) = \{u_1, u_2, v_1, v_2, w_1, w_2\}$, $V(D_2) = \{u_3, u_4, v_3, v_4, w_3, w_4\}$, and $u_1 \rightarrow v_1$. We distinguish firstly the two cases that $w_3 \rightarrow u_3$ or $w_3 \rightarrow v_3$, and secondly the two cases that $u_2 \rightarrow v_2$ or $u_2 \rightarrow w_2$.

If $w_3 \rightarrow u_3$ and $u_2 \rightarrow v_2$, then $w_4 \rightarrow u_4$ (or, without loss of generality, $w_4 \rightarrow v_4$). In this situation we arrive at the two desired complementary cycles:

$$x_1w_1v_4x_1 \quad \text{and} \quad x_2u_1v_1w_3u_3x_3u_2v_2w_4u_4x_4w_2v_3x_2 \text{ or}$$
$$(x_1w_1v_3x_1 \quad \text{and} \quad x_2u_1v_1w_3u_3x_3u_2v_2w_4v_4x_4w_2u_4x_2).$$

If $w_3 \rightarrow u_3$ and, without loss of generality, $u_2 \rightarrow w_2$, then $w_4 \rightarrow u_4$ (or, without loss of generality, $w_4 \rightarrow v_4$). Now we have the desired complementary cycles

$$x_1w_1v_4x_1 \quad \text{and} \quad x_2u_1v_1w_3u_3x_3u_2w_2v_3x_4v_2w_4u_4x_2 \text{ or}$$
$$(x_1w_1v_3x_1 \quad \text{and} \quad x_2u_1v_1w_3u_3x_3u_2w_2u_4x_4v_2w_4v_4x_2).$$

If $w_3 \rightarrow v_3$ and $u_2 \rightarrow v_2$, then $w_4 \rightarrow v_4$ (or, without loss of generality, $w_4 \rightarrow u_4$). Now we have the desired complementary cycles

$$x_1w_1u_3x_1 \quad \text{and} \quad x_2u_1v_1w_3v_3x_3u_2v_2w_4v_4x_4w_2u_4x_2 \text{ or}$$
$$(x_1w_1u_3x_1 \quad \text{and} \quad x_2u_1v_1w_3v_3x_3u_2v_2w_4u_4x_4w_2v_4x_2).$$

If $w_3 \rightarrow v_3$ and, without loss of generality, $u_2 \rightarrow w_2$, then $w_4 \rightarrow v_4$ (or, without loss of generality, $w_4 \rightarrow u_4$). Now we have the desired complementary cycles

$$x_1w_1u_3x_1 \quad \text{and} \quad x_2u_1v_1w_3v_3x_3u_2w_2u_4x_4v_2w_4v_4x_2 \text{ or}$$
$$(x_1w_1u_3x_1 \quad \text{and} \quad x_2u_1v_1w_3v_3x_3u_2w_2v_4x_4v_2w_4u_4x_2).$$

Altogether, we see that D contains two complementary cycles of length 3 and $|V(D)| - 3 = 13$, when $\kappa(D) = 4$. Therefore, we investigate next the case that $\kappa(D) \geq 5$, that means that $\kappa(H) \geq 2$.

Firstly, assume that H has a cycle-factor. Let $C_1', C_2', \ldots, C_t'$ be a minimal cycle-factor with the properties described in Theorem 2.13. If $t = 1$, then H has the Hamiltonian cycle C_1' and so $V(D) = V(C_3) \cup V(C_1')$. If not, then, because of $|V^*| \leq 4$, it follows from Theorem 2.13 that there are at most four arcs from $V(H) - V(C_1')$ to C_1'. Furthermore, if $|V^* \cap V(C_1')| = 1$ or $|V^* \cap (V(H) - V(C_1'))| = 1$, then we arrive at the contradiction $\kappa(H) \leq 1$. Hence, $|V^*| = 4$ and $|V^* \cap V(C_1')| = 2$ and the remaining three parite sets of H have cardinality three. If $|V(C_1')| \leq 5$, then it follows from Theorem 2.13 that $d_H^+(w) \geq 7$ for $w \in V^* \cap V(C_1')$, a contradiction to the 6-regularity of D. If $|V(C_1')| = 6$ and C_1' induces a 4-parite tournament, then we obtain

$$\begin{aligned} \sum_{x \in V(C_1')} d_H^+(x) &= \sum_{x \in V(C_1')} d_{D[V(C_1')]}^+(x) + d^+(C_1', H - V(C_1')) \\ &\geq 13 + (32 - 4) = 41, \end{aligned}$$

a contradiction to the 6-regularity of D. The same holds true, if C_1' induces a 3-partite tournament. If $|V(C_1')| = 7$ and C_1' induces a 4-parite tournament, then we obtain

$$\begin{aligned} \sum_{x \in V(C_1')} d_H^+(x) &= \sum_{x \in V(C_1')} d_{D[V(C_1')]}^+(x) + d^+(C_1', H - V(C_1')) \\ &\geq 18 + (32 - 4) = 46, \end{aligned}$$

or

$$\begin{aligned} \sum_{x \in V(C_1')} d_H^+(x) &= \sum_{x \in V(C_1')} d_{D[V(C_1')]}^+(x) + d^+(C_1', H - V(C_1')) \\ &\geq 17 + (34 - 4) = 47, \end{aligned}$$

a contradiction to the 6-regularity of D. The same holds true, if C_1' induces a 3-partite tournament. If $|V(C_1')| \geq 8$, then $t = 2$ and $|V(C_2')| \leq 5$. Hence, it follows from Theorem 2.13 that $d_H^-(w) \geq 7$ for $w \in V^* \cap V(C_2')$, a contradiction to the 6-regularity of D.

Secondly, assume that H has no cycle-factor. Then, with respect to Lemma 2.12, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \leadsto Y$, $(R_1 \cup Y) \leadsto R_2$, $|Y| > |Z|$, and Y is an independent set. Since $\kappa(H) \geq 2$ and $\alpha(H) = 4$, we see that $2 \leq |Z| < |Y| \leq 4$. Let $V_1 = \{x_1, x_2, x_3, \}, V_2 = \{u_1, u_2, u_3\}, V_3 = \{v_1, v_2, v_3, \}$, $V_4 = \{w_1, w_2, w_3, w_4\}$, and $C_3 = x_4u_4v_4x_4$. Since D is 6-regular, we have $\delta^+(H), \delta^-(H) \geq 3$ and $d_H^+(x), d_H^-(x) \geq 4$ for all $x \in V_1 \cup V_2 \cup V_3$.

Subcase 4.1. Suppose that $|Z| = 2$ and $|Y| = 3$.

It is easy to see that $0 \leq |R_1|, |R_2| \leq 2$ is impossible. If $|R_1| = 3$, then $\delta^-(H) \geq 3$ implies that $H[R_1]$ is a 3-ycle. Thus, $d_H^-(x) = 3$ for all $x \in R_1$, a contradiction to the fact that R_1 contains a vertex of $V_1 \cup V_2 \cup V_3$. Similarly, one can show that $|R_2| = 3$ is not possible. Therefore, it remains the case that $|R_1| = |R_2| = 4$.

Firstly, let $Y = \{w_1, w_2, w_3\}$. This leads to the contradiction $15 \leq \sum_{x \in R_1} d_H^-(x) \leq 14$. Secondly, assume, without loss of generality, that $Y = V_1$. This implies that

$H[R_1]$ is 3-partite. If R_1 contains at least three vertices of V_4, then we obtain the contradiction $12 \le \sum_{x \in R_1} d_H^-(x) \le 11$. If R_1 contains at most two vertices of V_4, then we obtain the contradiction $14 \le \sum_{x \in R_1} d_H^-(x) \le 13$.

Subcase 4.2. Suppose that $|Z| = 2$ and $|Y| = 4$.

This implies that $Y = V_4$. The cases $0 \le |R_1|, |R_2| \le 3$ easily lead to a contradiction. However, $|R_1|, |R_2| \ge 4$ is also impossible, since $|V(H)| = 13$.

Subcase 4.3. Suppose that $|Z| = 3$, $|Y| = 4$, and $|R_1|, |R_2| \ge 1$.

This implies $Y = V_4$. The cases $1 \le |R_1|, |R_2| \le 2$ easily lead to a contradiction. It remain the cases that $H[R_1]$ and $H[R_2]$ are 3-cycles. But then we obtain the contradiction $12 \le \sum_{x \in R_1} d_H^-(x) \le 11$.

Subcase 4.4. Suppose that $|Z| = 3$, $|Y| = 4$, and assume, without loss of generality, that $|R_1| = 0$.

This implies $Y = V_4$, $Z \to Y \to R_2$, $V(C_3) \to Y$, and $\delta^+(H[R_2]) \ge 1$.

Subcase 4.4.1. Suppose that Z be an independent set, say $Z = V_1$.

Then $H[R_2]$ is bipartite and every vertex of R_2 has at least one positive neighbor in V_1. Furthermore, there are at least 3 vertices a_1, a_2, a_3 in R_2 such that $d^+_{H[R_2]}(a_i) = 1$ for $i = 1, 2, 3$.

Subcase 4.4.1.1. Suppose that $d^+_{H[R_2]}(u_i) = 1$ for $i = 1, 2, 3$.

This implies $V_2 = \{u_1, u_2, u_3\} \to V_1 = Z$. If we assume, without loss of generality, that $u_1 \to v_1 \to u_2$, then we deduce that $\{v_2, v_3\} \to u_1$. If we assume next, without loss of generality, that $u_2 \to v_2$, then we deduce that $v_3 \to u_2$. It follows that $V_2 \to \{x_4, v_4\}$. Since v_1 has at least one positive neighbor in V_1, say x_1, we now choose the 3-cycle $C_3' = x_1 w_1 v_1 x_1$ instead of C_3, and we have the complementary cycle of length 13 in D: $w_2 v_2 u_1 x_4 u_4 v_4 w_3 v_3 u_2 x_2 w_4 u_3 x_3 w_2$.

Subcase 4.4.1.2. Suppose that $d^+_{H[R_2]}(u_i) = 1$ for $i = 1, 2$ and $d^+_{H[R_2]}(v_1) = 1$.

This implies $\{u_1, u_2, v_1\} \to V_1 = Z$.

Firstly, we assume that $v_1 \to u_1$. It follows that $\{u_2, u_3\} \to v_1$ and thus $\{v_2, v_3\} \to u_2$. If we assume next, without loss of generality, that $u_1 \to v_2$, then we deduce that $v_3 \to u_1$. It follows that $u_1 \to x_4$. Since u_3 has at least one positive neighbor in V_1, say x_1, we now choose the 3-cycle $C_3' = x_1 w_1 u_3 x_1$ instead of C_3, and we have the complementary cycle $w_2 v_3 u_1 x_4 u_4 v_4 w_3 v_2 u_2 x_2 w_4 v_1 x_3 w_2$ of length 13 in D.

The remaining case that $v_1 \to u_3$ is analogous to the last case.

Subcase 4.4.2. Suppose that $H[Z]$ is 3-partite, say $Z = \{x_1, u_1, v_1\}$.

Then $H[R_2]$ is 2-regular and 3-partite such that $R_2 \leadsto Z$ and $R_2 \leadsto V(C_3)$. Now we assume, without loss of generality, that $u_2 \to v_2$. If $u_3 \to v_3$, then we choose the 3-cycle $C_3' = u_1 w_1 x_2 u_1$, and we have the complementary cycle of length 13 in D: $w_2 u_2 v_2 x_4 u_4 v_4 w_3 u_3 v_3 x_1 w_4 x_3 v_1 w_2$. If $u_3 \to x_3$, then we choose the 3-cycle $C_3' = u_1 w_1 x_2 u_1$, and we have the complementary cycle $w_2 u_2 v_2 x_4 u_4 v_4 w_3 u_3 x_3 v_1 w_4 v_3 x_1 w_2$ of length 13 in D.

Subcase 4.4.3. Suppose that $H[Z]$ is 2-partite, say $Z = \{x_1, x_2, u_1\}$.

Then $d^+_{H[R_2]}(x_3) \ge 3$, $d^+_{H[R_2]}(u_i) \ge 2$ for $i = 1, 2$, and $d^+_{H[R_2]}(v_i) \ge 1$ for $i = 1, 2, 3$.

Subcase 4.4.3.1. Suppose that $x_3 \to \{v_1, v_2, v_3\}$.

Assume, without loss of generality, that $\{v_1, v_2\} \to u_2$ and $v_3 \to u_3$. It follows that $u_2 \to \{x_3, v_3\}$. Furthermore, we can assume, without loss of generality, that $u_3 \to v_1$. This leads to $\{v_1, v_3\} \to Z$. In addition, v_2 has at least one positive neighbor in C_3, say $v_2 \to x_4$. Since u_3 has at least one positive neighbor in Z, say x_1, there exists the 3-cycle $C_3' = x_1w_1u_3x_1$ and the complementary cycle $w_2x_3v_2x_4u_4v_4w_3u_2v_3x_2w_4v_1u_1w_2$ of length 13 in D.

Subcase 4.4.3.2. Suppose that $v_3 \to x_3 \to \{v_1, v_2, u_2\}$.

Assume, without loss of generality, that $u_2 \to v_1$. It follows that $v_1 \to u_3$.

The case $v_2 \to u_3$ implies $u_3 \to \{v_3, x_3\}$, $\{v_1, u_3\} \rightsquigarrow Z$, and $\{x_3, v_1, u_3\} \rightsquigarrow V(C_3)$. Since u_2 has at least one positive neighbor in Z, say x_1, there is the 3-cycle $C_3' = x_1w_1u_2x_1$ and the complementary cycle $w_2v_3x_3u_4v_4x_4w_3v_2u_3x_2w_4v_1u_1w_2$ of length 13 in D.

The case $u_3 \to v_2$ implies $v_2 \to u_2$, and thus, $u_2 \to v_3$. Consequently, we see that $\{v_1, v_2, u_2\} \rightsquigarrow (Z \cup V(C_3))$. Since v_3 has at least two positive neighbors in Z, say $v_3 \to x_1$, there exists the 3-cycle $C_3' = x_1w_1v_3x_1$ and the complementary cycle $w_2u_3v_2x_4u_4v_4w_3x_3u_2x_2w_4v_1u_1w_2$ of length 13 in D.

Subcase 4.4.3.3. Suppose that $\{v_2, v_3\} \to x_3 \to \{v_1, u_2, u_3\}$.

It follows that $x_3 \to \{u_1, u_4, v_4\}$. Assume, without loss of generality, that $u_2 \to v_2$.

The case $u_2 \to v_1$ implies $v_1 \to \{x_1, x_2, u_1, u_3, x_4, u_4\}$, $u_3 \to \{x_1, x_2, v_2, v_3, x_4, v_4\}$, and $v_2 \to u_1$. Hence, there is the 3-cycle $C_3' = u_1w_1v_2u_1$ and the complementary cycle $w_2v_3x_3u_4v_4x_4w_3u_2v_1x_1w_4u_3x_2w_2$ of length 13 in D.

The case $v_1 \to u_2$ implies $u_2 \to \{x_1, x_2, v_3, x_4, u_4\}$. If we assume, without loss of generality, that $u_3 \to v_3$, then we deduce that $v_3 \rightsquigarrow (Z \cup V(C_3))$. Since v_2 has at least one positive neighbor in C_3, say $v_2 \to u_4$, there exists the 3-cycle $C_3' = u_1w_1x_3u_1$, and the complementary cycle $w_2v_2u_4v_4x_4w_3u_3v_3x_1w_4v_1u_2x_2w_2$ of length 13 in D.

Case 5. Suppose that $c = 4$ and $r = 2$.

This implies that D is 3-regular and $\alpha(H) = 2$. Let now $V_1' = \{x_1, x_2\}, V_2' = \{y_1, y_2\}, V_3' = \{u_1, u_2\}, V_4' = \{v_1, v_2\}$, and $C_3 = x_2y_2u_2x_2$. Since D is 3-regular, we see that $d_H^+(x), d_H^-(x) \geq 1$ for $x \in \{x_1, y_1, u_1\}$.

Subcase 5.1. Assume that H is not strong, and let $D_1, D_2, \ldots, D_t$ be the strong components of H such that $D_i \rightsquigarrow D_j$ for $1 \leq i < j \leq t$.

Subcase 5.1.1. Suppose that $|V(D_i)| = 1$ for $1 \leq i \leq t$.

Then, because of $d_H^+(x), d_H^-(x) \geq 1$ for $x \in \{x_1, y_1, u_1\}$, we deduce, without loss of generality, that $D_1 = v_1$, $D_5 = v_2$, $D_2 = x_1$, $D_3 = y_1$, and $D_4 = u_1$. This implies $v_2 \to V(C_3) \to v_1$ and $u_1 \rightsquigarrow V(C_3) \rightsquigarrow x_1$, and hence D contains the 3-cycle $C_3' = y_2x_1u_1y_2$ and the complementary cycle $u_2x_2v_1y_1v_2u_2$.

Subcase 5.1.2. Suppose that $|V(D_2)| = 3$ and $|V(D_1)| = |V(D_3)| = 1$.

Then, because of $d_H^+(x), d_H^-(x) \geq 1$ for $x \in \{x_1, y_1, u_1\}$, we deduce, without loss of generality, that $D_1 = v_1$, $D_3 = v_2$, and D_2 is the 3-cycle $x_1y_1u_1x_1$. This implies $v_2 \to V(C_3) \to v_1$.

If $x_2 \to y_1$, then $y_1 \to u_2$ and $u_1 \to x_2$. This yields $u_2 \to x_1 \to y_2 \to u_1$. Therefore, D contains the 3-cycle $C_3' = x_2y_1u_2x_2$ and the complementary cycle $y_2v_1u_1x_1v_2y_2$.

If $y_1 \to x_2$, then $x_2 \to u_1$ and $u_2 \to y_1$. This yields $u_1 \to y_2 \to x_1 \to u_2$. But now we observe that D is isomorphic to $D_{4,2}$ in Example 2.4.

Subcase 5.1.3. Suppose that $|V(D_1)| = 3$ or $|V(D_3)| = 3$, say $|V(D_1)| = 3$.

Then D_1 is a 3-cycle and, without loss of generality, $D_3 = v_2$.

Subcase 5.1.3.1. Suppose that $D_2 = v_1$.

Assume, without loss of generality, that D_1 consists of the 3-cycle $x_1y_1u_1x_1$. This implies $\{v_1, v_2\} \to V(C_3) \rightsquigarrow V(D_1)$, and hence D contains the 3-cycle $C_3' = y_2x_1v_1y_2$ and the complementary cycle $u_2x_2y_1u_1v_2u_2$.

Subcase 5.1.3.2. Suppose, without loss of generality, that $D_2 = u_1$.

Assume, without loss of generality, that D_1 consists of the 3-cycle $x_1y_1v_1x_1$. This implies $\{u_1, v_2\} \rightsquigarrow V(C_3) \rightsquigarrow \{x_1, y_1\}$. If $u_2 \to v_1$, then D contains the 3-cycle $C_3' = x_2y_1u_1x_2$ and the complementary cycle $y_2u_2v_1x_1v_2y_2$.

If $v_1 \to u_2$, then D contains the 3-cycle $C_3' = u_2y_1v_1u_2$ and the complementary cycle $x_2y_2x_1u_1v_2x_2$.

Subcase 5.1.4. Suppose that $|V(D_1)| = 4$ or $|V(D_2)| = 4$, say $|V(D_1)| = 4$.

This implies, without loss of generality, that $D_2 = v_2$, and D_1 is a strong tournament. It follows that $v_2 \to V(C_3)$ and there exists a 3-cycle C_3' in D_1 such that $v_1 \in V(C_3')$. Assume, without loss of generality, that u_1 is not a vertex of C_3'. Since u_1 has at least one negative neighbor in C_3, say y_2, we have the complementary cycle $u_2x_2y_2u_1v_2u_2$ in D.

Subcase 5.2. Assume that H is strong.

If H has a cycle-factor, then, because of $|V(H)| \leq 5$, it must be a Hamiltonian cycle C_5 and so $V(D) = V(C_3) \cup V(C_5)$.

Suppose now that H has no cycle-factor. Then, with respect to Lemma 2.12, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \rightsquigarrow Y$, $(R_1 \cup Y) \rightsquigarrow R_2$, $|Y| > |Z|$, and Y is an independent set. Since $\alpha(H) = 2$, we see that $1 = |Z| < |Y| = 2$, and thus $Y = \{v_1, v_2\}$. We assume, without loss of generality, that $Z = \{u_1\}$ and $|R_1| \leq |R_2|$.

Subcase 5.2.1. Suppose that $|R_1| = |R_2|$ such that, without loss of generality, $R_1 = \{x_1\}$ and $R_2 = \{y_1\}$.

This implies $y_1 \to u_1 \to x_1$, $y_1 \rightsquigarrow V(C_3) \rightsquigarrow x_1$, and we can assume, without loss of generality, that $v_2 \to u_2 \to v_1$.

Subcase 5.2.1.1. Assume that $y_2 \to v_2$.

If $v_1 \to u_1$, then D contains the 3-cycle $C_3' = u_1x_1v_1u_1$ and the complementary cycle $y_2v_2y_1u_2x_2y_2$. If $u_1 \to v_1$, then D contains the 3-cycle $C_3' = u_1v_1y_1u_1$ and the complementary cycle $v_2u_2x_2y_2x_1v_2$.

Subcase 5.2.1.2. Assume that $v_2 \to y_2$ and $u_1 \to v_1$.

This implies $\{u_1, x_2\} \to v_2$ and thus $\{x_2, y_2\} \to u_1$. Hence, D contains the 3-cycle $C_3' = v_2u_2x_1v_2$ and the complementary cycle $y_2u_1v_1y_1x_2y_2$.

Subcase 5.2.1.3. Assume that $v_2 \to y_2$, $v_1 \to u_1$, and $y_2 \to v_1$.

This implies $\{u_1, x_2\} \to v_2$ and $v_1 \to x_2 \to u_1 \to y_2$. Hence, D contains the 3-cycle $C_3' = u_2x_1v_2u_2$ and the complementary cycle $y_2v_1y_1x_2u_1y_2$.

Subcase 5.2.1.4. Assume that $v_2 \to y_2$, $v_1 \to u_1$, and $v_1 \to y_2$.

This implies $\{u_1, x_2\} \to v_2$ and $y_2 \to u_1 \to x_2 \to v_1$. Altogether, we have

$$\{v_1, v_2\} \to \{y_1, y_2\} \to \{u_1, u_2\} \to \{x_1, x_2\} \to \{v_1, v_2\},$$

$y_1 \to x_2 \to y_2 \to x_1 \to y_1$, and $v_1 \to u_1 \to v_2 \to u_2 \to v_1$. Consequently, D is isomorphic to $D_{4,2}^*$ in Example 2.5.

Subcase 5.2.2. Suppose that $|R_1| = 0$ and $R_2 = \{x_1, y_1\}$ such that, without loss of generality, $x_1 \to y_1$.

This implies $u_1 \to \{v_1, v_2\}$ and $y_1 \to \{u_1, u_2, x_2\}$.

Subcase 5.2.2.1. Assume that $u_1 \to x_1$.

This implies $\{x_2, y_2\} \to u_1$, $x_1 \to \{y_2, u_2\}$, and thus $u_2 \to \{v_1, v_2\}$. Then D contains the 3-cycle $C_3' = u_1v_1y_1u_1$. If $x_2 \to v_2$, then D contains the complementary cycle $v_2x_1y_2u_2x_2v_2$. If $v_2 \to x_2$, then $y_2 \to v_2$, and D contains the complementary cycle $v_2x_1u_2x_2y_2v_2$.

Subcase 5.2.2.2. Assume that $x_1 \to u_1$ and $u_2 \to v_1$ or $u_2 \to v_2$, say $u_2 \to v_1$.

In this case, D contains the 3-cycle $C_3' = u_1v_2x_1u_1$ and the complementary cycle $u_2v_1y_1x_2y_2u_2$.

Subcase 5.2.2.3. In the remaining case that $x_1 \to u_1$ and $\{v_1, v_2\} \to u_2$, we obtain the contradiction $d_D^-(u_2) \geq 4$. □

Since $D_{4,2}$ and $D_{4,2}^*$ contain two complementary cycles of length 4, Theorem 3.1 immediately leads to the following result.

Corollary 3.2 *Let D be a regular c-partite tournament with $c \geq 4$ and $|V(D)| \geq 6$. Then D is cycle complementary unless D is isomorphic to T_7.*

As an application of Corollary 3.2, the author [12] recently derived the following result.

Theorem 3.3 (Volkmann [12] 2004) *Each regular multipartite tournament D of order $|V(D)| \geq 8$ is cycle complementary.*

In [13], Volkmann constructed an example showing that Yeo's Conjecture 2.1 is not true in general for $t = 4$ and regular 4-partite tournaments with two vertices in each partite set. However, in all the remaining cases Volkmann [13] could prove Conjecture 2.1 for $t = 4$.

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