The Fine Structure of Threefold Triple Systems: $v \equiv 5 \pmod 6$

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Abstract

The fine structure of a threefold triple system is (c_1, c_2, c_3) where c_i is the number of i-times repeated blocks. Necessary conditions for (c_1, c_2, c_3) to be the fine structure of a threefold triple system with $v \equiv 5 \pmod 6$ are determined, and shown to be sufficient for all $v \geq 17$.

1 Preliminaries

A *triple system* of *order* v and *index* λ, or $TS(v, \lambda)$, is a v-set V of *elements*, together with a collection $\mathcal{B}$ of 3-element subsets of V called *triples* or *blocks*; every 2-subset of V appears in precisely λ of the blocks. A $TS(v, \lambda)$ is permitted to have blocks that are identical as subsets, i.e. *repeated* blocks. The *fine structure* of a $TS(v, \lambda)$ is the vector $(c_1, c_2, \ldots, c_\lambda)$ where c_i is the number of blocks repeated exactly i times in the triple system.

In [4], necessary conditions for (c_1, c_2, c_3) to be the fine structure of a $TS(v, 3)$ with $v \equiv 1, 3 \pmod 6$ are determined, and shown to be sufficient for all $v \geq 19$. We adopt the same notation as that paper, and assume familiarity with the definitions and results therein. First, we extend the notation of [4]. Since any two of c_1, c_2, c_3 determine the third, we denote the fine structure as a pair (t, s) where $t = c_2$ and $s = \lfloor\frac{v(v-1)}{6}\rfloor - c_3$; for $v \equiv 1, 3 \pmod 6$, $c_1 = 3s - 2t$, while for $v \equiv 5 \pmod 6$, $c_1 = 3s + 1 - 2t$.

For $v \equiv 1, 3 \pmod 6$, let $Adm(v) = \{(t, s) : 0 \leq t \leq s \leq \frac{v(v-1)}{6}, s \notin \{1, 2, 3, 5\}, (t, s) \notin \{(0,4), (1,4), (2,4), (3,4), (1,6), (2,6), (3,6), (5,6), (2,7), (5,7), (1,8), (3,8), (5,8)\}\}$. Let $Fine(v) = \{(t, s) : \exists TS(v, 3)$ with t doubly repeated blocks and $\lfloor\frac{v(v-1)}{6}\rfloor - s$ triply repeated blocks$\}$. In [4], the following theorem is proved:

Theorem 1.1 *For $v \equiv 1, 3$ (mod 6), $v \geq 19$, $Fine(v) = Adm(v)$.* □

For $v \leq 15$, extensive partial results are known [4].

What pairs (t, s) can arise as fine structures for $TS(v, 3)$'s with $v \equiv 5$ (mod 6)? We define $Adm(v)$ for $v \equiv 5 \pmod 6$ to be $\{(t, s) : 0 \leq t < s \leq \lfloor\frac{v(v-1)}{6}\rfloor, s \notin \{1, 2, 4, 5\}, (t, s) \notin \{(1,3), (2,3), (0,6), (1,6), (2,6), (4,6), (5,6), (0,7), (1,7), (2,7), (3,7), (5,7), (6,7), (0,8), (1,8), (2,8), (6,8), (7,8), (1, 9)\}\}$. In this paper, we prove:

Theorem 1.2 *For $v \equiv 5$ (mod 6), $v \geq 17$, $Fine(v) = Adm(v)$.*

We first extend the necessary conditions from [4] to the case when $v \equiv 5$ (mod 6). In section 2, we then introduce recursive constructions that permit us to determine $Fine(v)$ for large v given the solution for smaller orders. In section 3, we outline the solution for smaller orders and in section 4, we apply the recursions from section 2 and from [4] to obtain the main theorem.

First we treat necessary conditions.

Lemma 1.3 *If $(t, s) \in Fine(v)$, then $0 \leq t < s \leq \lfloor\frac{v(v-1)}{6}\rfloor$.*

Proof: The only nontrivial inequality is $t < s$. To see that $t \leq s$, observe that the number of pairs *not* appearing in triply repeated blocks is $3s + 1$; since no two doubly repeated blocks share a pair, $3t \leq 3s + 1$ whence $t \leq s$. Consider the *set* of all blocks appearing as doubly or triply repeated blocks; these form a partial $TS(v, 1)$, whose leave (graph of uncovered pairs; see, e.g. [1]) must have all vertex degrees even. Thus there are at least four pairs not appearing in doubly or triply repeated blocks; this yields $3t \leq 3s - 3$, or equivalently $t < s$. □

In refining this elementary necessary condition, first observe that if $(t,s) \in Fine(v)$ for $v \equiv 5 \pmod 6$, $s \notin \{1,2,4,5\}$ as a consequence of Milici and Quattrocchi's determination of the possible numbers of triply repeated blocks in $TS(v,3)$'s [7].

It remains only to eliminate the nineteen sporadic values. We follow the same strategy as [4]. For $(t,s) \in Fine(v)$, there must exist a graph G on $3s+1$ edges for which $3G$ can be partitioned into t doubly repeated triples and $3s+1-2t$ singly repeated triples. Moreover, G must be the leave of a $PTS(v,1)$ for $v \equiv 5 \pmod 6$. It follows that G has all vertex degrees even, no vertex of degree two, and that the neighbours of each vertex of G induce a subgraph of minimum degree two (Lemma 2.3 of [4]). To eliminate the nineteen sporadic values, we generate all graphs on 10, 19, 22, 25, and 28 edges meeting these conditions. For each such graph G, we generate all partitions of $3G$ into triangles, and determine the number of doubly repeated triangles in the partition.

For $3s+1=10$, the only admissible graph is K_5, the complete graph on five vertices; $3K_5$ can be partitioned only into 10 singly repeated triples, leading to $(0,3) \in Adm(v)$. For $3s+1=19$, the only admissible graph is obtained by identifying a triangle of K_5 with a triangle of K_6-F, the unique 4-regular graph on six vertices. The only partition has three doubly repeated blocks, namely those arising from the edges of K_6-F omitting those in the triangle identified with K_5. For $3s+1=22$, the candidates are obtained as the disjoint union of K_5 and K_6-F, as K_5 and K_6-F identified at a single vertex, and identified at a nonadjacent pair in K_6-F. A further candidate is the unique graph with degree sequence $6^6 4^2$ having the degree four vertices nonadjacent. For all candidates, precisely four doubly repeated triples are required; in the first three, these are forced for the K_6-F. For $3s+1=25$, we generated all candidates by computer, using an orderly algorithm [5]. Identifying K_5 and $K_7 \setminus K_3$ on a triangle leads to $\{(3,8),(5,8)\} \subset Adm(v)$. Three candidates lead to $(4,8) \in Adm(v)$. All candidates lead to 3, 4 or 5 doubly repeated blocks.

For $3s+1=28$, the search for candidates was restricted by an easy observation. If the neighbourhood of a vertex has two or more vertices of degree two within the subgraph induced on the neighbourhood, any partition necessarily includes at least two doubly repeated triples. Hence we need only treat candidates having every neighbourhood inducing a subgraph with at most one degree two vertex in order to eliminate (1,9). It is a tedious computation to verify that no such partition exists.

The computational search undertaken, using the strategy of [4], establishes:

Lemma 1.4 *For all $v \equiv 5 \pmod 6$, $Adm(v) \subseteq Fine(v)$.* $\square$

2 Recursive Constructions

The proof of sufficiency proceeds in two stages. $Fine(v)$ is determined for "small" values of v, and then recursive constructions are used to obtain the solution for all $v \equiv 5 \pmod 6$. We develop the recursive constructions first, in part to determine which small orders must be settled independently, and in part to assist in the determination for small orders.

First, we note that the $2v+1$, $2v+7$ and $2v+19$ constructions from [4] carry over directly to the case $v \equiv 5 \pmod 6$. However, the tripling constructions used are of little value here. Instead, we develop a tripling construction tailored to the case when $v \equiv 5 \pmod 6$. The tripling construction we develop produces a $TS(3v-4,3)$ from a $TS(v,3)$; in addition to the smaller triple system, the construction employs a 3-factorization of $3K_{v-2,v-2}$. The *type* (t,s) of a 3-factorization $\{F_1,...,F_n\}$ of $3K_{n,n}$ has t equal to the number of edges repeated twice within a factor, and n^2-s equal to the number repeated three times within a factor. Colbourn [3] determined the possible types of 3-factorizations of $3K_{n,n}$ for all $n \geq 10$, along with substantial partial results for $5 \leq n \leq 9$; we refer to [4] for the quite lengthy statement of these results.

Lemma 2.1 *Let (t_0,s_0) be integers such that either $s_0 < (v-2)^2$ and (t_0,s_0) is the type of a 3-factorization of $3K_{v-2,v-2}$* or *$t_0 \geq 0$ and (t_0+1,s_0+1) is the type of a 3-factorization of $3K_{v-2,v-2}$. For $i=1,2,3$, let (t_i,s_i) be integers such that either $s_i < \lfloor\frac{v(v-1)}{6}\rfloor$ and $(t_i,s_i) \in Fine(v)$* or *$t_i \geq 0$ and $(t_i+1,s_i+1) \in Fine(v)$. Then if $v \equiv 1,3 \pmod 6$,*

$$(t_0+t_1+t_2+t_3, s_0+s_1+s_2+s_3+3) \in Fine(3v-4)$$

and if $v \equiv 5 \pmod 6$,

$$(t_0+t_1+t_2+t_3, s_0+s_1+s_2+s_3+4) \in Fine(3v-4)$$

Proof: We form a $TS(3v-4,3)$ on $(Z_{v-2} \times \{1,2,3\}) \cup \{\alpha,\beta\}$, denoting the element (j,i) as j_i. For $i=1,2,3$, if $(t_i,s_i) \in Fine(v)$ and $s_i < \lfloor\frac{v(v-1)}{6}\rfloor$, choose a $TS(v,3)$ of this type on $(Z_{v-2} \times \{i\}) \cup \{\alpha,\beta\}$ with a triply repeated block on $\{\alpha,\beta,0_i\}$. If instead $t_i \geq 0$ and $(t_i+1,s_i+1) \in Fine(v)$, choose a $TS(v,3)$ of type (t_i+1,s_i+1) on the same set with a doubly repeated block on $\{\alpha,\beta,0_i\}$. In either case, omit *two* copies of the block $\{\alpha,\beta,0_i\}$.

Now if (t_0, s_0) is the type of a 3-factorization of $3K_{v-2,v-2}$ and $s_0 < (v-2)^2$, place a 3-factorization $\{F_0, ..., F_{v-3}\}$ of this type on $Z_{v-2} \times \{1,2\}$ so that F_0 contains $\{0_1, 0_2\}$ as a triply repeated edge. If instead $t_0 \geq 0$ and (t_0+1, s_0+1) is the type of such a 3-factorization, choose $\{F_0, ..., F_{v-3}\}$ so that F_0 contains $\{0_1, 0_2\}$ as a doubly repeated edge. In either case, omit two copies of the edge $\{0_1, 0_2\}$ from F_0. The factors are then used to form triples as follows. For each occurrence of $\{x_1, y_2\}$ in factor F_k, form a triple $\{x_1, y_2, k_3\}$.

At this point, every pair is covered exactly three times *except for* the nine pairs on $\{\alpha, \beta, 0_1, 0_2, 0_3\}$ excepting $\{\alpha, \beta\}$; these nine pairs are covered once each. To complete the system, take triples $\{\alpha, 0_i, 0_j\}$ and $\{\beta, 0_i, 0_j\}$ for $i \neq j$.

To determine the fine structure of the resulting system, note that when $v \equiv 1, 3 \pmod 6$, $3(s_0 + s_1 + s_2 + s_3) + 10$ pairs do not appear in triply repeated blocks; for $v \equiv 5 \pmod 6$, $3s_0 + (3s_1 + 1) + (3s_2 + 1) + (3s_3 + 1) + 10$ pairs do not appear in triply repeated blocks. The count of doubly repeated blocks is routine. □

When $v \equiv 1, 3 \pmod 6$, Lemma 2.1 never yields a fine structure (t, s) with $s - t \leq 2$, and when $v \equiv 5 \pmod 6$, it never yields $s - t \leq 6$. Hence we require further constructions that need not produce a subdesign of order 5.

Lemma 2.2 *Let $n \geq 1$. For $i = 1, 2, 3$, let (t_i, s_i) be the type of a 3-factorization of $3K_{2n+2}$. Let $0 \leq b_1 \leq a_1 \leq 2n+2$ and $(b_1, a_1) \notin \mathcal{X} = \{(0,1), (0,2), (1,1), (1,2), (1,3)\}$. Let $0 \leq b_2 \leq a_2 \leq 2n-1$ and $(b_2, a_2) \notin \mathcal{X}$. Let $(t', s') \in Fine(6n-1)$. Then*

$$(t_1 + t_2 + t_3 + t' + b_1(4n+1) + b_2(2n+2), s_1 + s_2 + s_3 + s' + a_1(4n+1) + a_2(2n+2))$$

is in $Fine(12n+5)$.

Proof: We form a $TS(12n+5, 3)$ on $(Z_{4n+1} \times \{1,2,3\}) \cup \{\alpha, \beta\}$. First place a $TS(6n-1, 3)$ of fine structure (t', s') on $(Z_{2n-1} \times \{1,2,3\}) \cup \{\alpha, \beta\}$. For $i = 1, 2, 3$, form a 3-factorization of $3K_{2n+2}$ on $(Z_{4n+1} \setminus Z_{2n-1}) \times \{i\}$. The factorization has type (t_i, s_i). Index factors as $F_\alpha, F_\beta, F_0, ..., F_{2n-2}$. For $\{x_i, y_i\} \in F_z$, if $z \in \{\alpha, \beta\}$, form the triple $\{x_i, y_i, z\}$, and if $z \in Z_{2n-1}$, form instead $\{x_i, y_i, z_i\}$.

The remaining pairs to appear in triples are all of the form $\{x_i, y_j\}$ for $i \neq j$ and $\{x, y\} \not\subset Z_{2n-1}$. Consider the graph on $Z_{4n+1} \times \{1,2\}$ formed by these remaining pairs; it can be partitioned into a $(2n-1)$-regular subgraph H on $(Z_{4n+1} \setminus Z_{2n-1}) \times \{1,2\}$ and a $(2n+2)$-regular subgraph G on $Z_{4n+1} \times \{1,2\}$. $3H$ has a 3-factorization of type $(b_2(2n+2), a_2(2n+2))$ when $0 \leq b_2 \leq a_2 \leq 2n-1$, $(b_2, a_2) \notin \mathcal{X}$ [2]. Call the factors $C_0, \dots, C_{2n-2}$. Similarly $3G$ has a 3-factorization $C_{2n-1}, \dots, C_{4n}$ of type $(b_1(4n+1), a_1(4n+1))$. Finally, for $\{x_1, y_2\} \in C_k$, form the triple $\{x_1, y_2, k_3\}$. □

This construction appears to be a $3v-4$ construction; however, it is better viewed as a $2v+7$ construction, placing a subdesign of order $6n-1$ in a design of order $12n+5$. A variant of this construction follows.

Lemma 2.3 *Let $v \equiv 5 \pmod 6$, $v = 3n+2$. For $i = 1, 2, 3$, let (t_i, s_i) be the fine structure of a $TS(v, 3)$ missing a subdesign of order five. Let $(t_0, s_0) \in Fine(11)$. For $1 \le i < n^2$, let $(b_i, a_i) \in \{(0,0), (0,9), (4,9), (6,6), (6,8), (7,8), (6,9), (7,9), (9,9)\}$. Then*

$$(t_0 + t_1 + t_2 + t_3 + \sum_{i=1}^{n^2-1} b_i, s_0 + s_1 + s_2 + s_3 + \sum_{i=1}^{n^2-1} a_i) \in Fine(3v-4).$$

Proof: We form a $TS(9n+2, 3)$ on $(Z_{3n} \times Z_3) \cup \{\alpha, \beta\}$. For $i = 1, 2, 3$, place a $TS(3n+2, 3)$ with a hole of size 5 on $(Z_{3n} \times \{i\}) \cup \{\alpha, \beta\}$ with the hole on $\{\alpha, \beta, 0_i, 1_i, 2_i\}$. Place a $TS(11, 3)$ with fine structure (t_0, s_0) on

$$\{\alpha, \beta, 0_1, 0_2, 0_3, 1_1, 1_2, 1_3, 2_1, 2_2, 2_3\}.$$

The pairs that remain form $3(K_{3n,3n,3n} \setminus K_{3,3,3})$. Use a latin square of order n to partition this graph into $n^2 - 1$ disjoint $K_{3,3,3}$'s. For each such $K_{3,3,3}$, form a 3-factorization of $K_{3,3}$ of type (b_i, a_i); such a 3-factorization exists [3]. Form triples on $K_{3,3,3}$ in the usual way. □

This $3v-4$ construction introduces a subsystem of order 11, and permits $s - t \ge 1$; however, it has the drawback that it requires the use of fine structures for triple systems with a hole of size 5.

We also employ a $3v+2$ construction:

Lemma 2.4 *Let $v \ge 7$. Let $(t, s) \in Fine(v)$ and $s < \lfloor \frac{v(v-1)}{6} \rfloor$. Let $(t', s') \in Fine(11)$. Let $0 \le \mu \le \frac{v-3}{2}$. Then for $v \equiv 1, 3 \pmod 6$,*

$$(9t + t' + 6\mu, 9s + s' + 6\mu) \in Fine(3v+2),$$

and for $v \equiv 5 \pmod 6$,

$$(9t + t' + 6\mu, 9s + s' + 6\mu + 3) \in Fine(3v+2).$$

Proof: We form a $TS(3v+2, 3)$ on $(Z_v \times \{1, 2, 3\}) \cup \{\alpha, \beta\}$. Choose a permutation π on Z_v that fixes 0, 1 and 2, and maps $2i \mapsto 2i-1$ and $2i-1 \mapsto 2i$ for $i \ge 2$.

Let B be the blocks of a $TS(v,3)$ of fine structure (t_0, s_0) on Z_v, omitting a triply repeated block on $\{0,1,2\}$. For each $\{x,y,z\}$ in $\mathcal{B}$, form the nine triples $\{x_1,y_1,z_2\}$, $\{x_1,y_2,z_1\}$, $\{x_2,y_1,z_1\}$, $\{x_2,y_2,\pi(z)_3\}$, $\{x_2,\pi(y)_3,z_2\}$, $\{\pi(x)_3,y_2,z_2\}$, $\{x_3,y_3,z_1\}$, $\{x_3,y_1,z_3\}$ and $\{x_1,y_3,z_3\}$. At this point, all pairs $\{x_i,y_j\}$ are covered except when $\{x,y\} \subset \{0,1,2\}$, or $3 \le x < v$ and $y = x$ if $\{i,j\} = \{1,2\}$ or $\{1,3\}$, $y = \pi(x)$ if $\{i,j\} = \{2,3\}$.

Place a $TS(11,3)$ of fine structure (t',s') on $(Z_3 \times \{1,2,3\}) \cup \{\alpha,\beta\}$. At this point, the pairs remaining on $Z_v \times \{1,2,3\}$ induce $\frac{v-3}{2}$ triplicated hexagons. Each has a 3-factorization of type (0,0) and one of type (6,6); choose μ of them to be of type (6,6), the remainder of type (0,0). This yields two 3-factors that are used to form triples with α and β. □

Finally, for s small we use an embedding result. Mendelsohn and Rosa [6] establish the existence of a partial triple system with $\lambda = 1$ of order v having a hole of order w, provided $v,w \equiv 5 \pmod 6$ and $v \ge 2w+1$. Repeating each block of this partial triple system three times, and filling the hole with a $TS(w,3)$ establishes that

Lemma 2.5 *$Fine(w) \subseteq Fine(v)$ for all $v,w \equiv 5$ (mod 6), $v \ge 2w+1$.* □

3 Small Cases

Lemma 3.1 $Fine(5) = \{(0,3)\}$.

Proof: The trivial design $TS(5,3)$ has fine structure (0,3). □

Lemma 3.2 *If $(t,s) \in Adm(11) \setminus \{(4,7)\}$, $(t,s) \in Fine(11)$ except possibly for $(t,s) \in \{(3,8),(5,8),(2,9),(5,9),(7,9),(0,10),(1,10),(2,10),(3,10),(4,10)\}$.*

Proof: Milici and Quattrocchi [7] establish that (4,7) is impossible for $v = 11$. Now we apply the $2v+1$ construction, using the complete solution for $Fine(5)$ and the complete solution for types of 3-factorizations of $3K_6$ [2]. For $(t,s) \in Adm(11)$, this handles all cases with $s - t \ge 3$, except (3,6), (4,7), (3,8), (4,8), (5,8), (0,9), (2,9), (3,9), (4,9), (5,9), (0,10), (1,10), (2,10), (3,10), (4,10), (5,10), (6,10), (7,10), (0,11), (1,11), (2,11), (3,11), (4,11), (5,11), (1,12), (2,12), (3,12), (5,12), (8,12), (1,13), (2,13), (3,13), (4,13), (5,13), (0,14), (1,14), (3,14) and (11,14). This is a *complete* solution for fine structures of $TS(11,3)$'s having a sub-$TS(5,3)$. Hence to obtain further values, we must avoid a subsystem of order 5.

a subsystem of order 5. A 3-factorization of $3K_6$ on $\{0,1,2,3,4,5\}$ has five factors; index them by $\{a,b,c,d,e\}$. Then for $\{i,j\}$ in factor x, we form a triple $\{x,i,j\}$. We form a trade as follows. Let $\{i,j,k\}$ be three distinct elements from $\{0,1,2,3,4,5\}$, and let $\{x,y,z\}$ be three distinct symbols from $\{a,b,c,d,e\}$. Now if $\{i,j\}$, $\{i,k\}$ and $\{j,k\}$ occur in factors x, y and z with multiplicity α, β and γ respectively ($\alpha,\beta,\gamma > 0$), we can remove the four triples $\{a,i,j\}$, $\{b,i,k\}$, $\{c,j,k\}$ and $\{a,b,c\}$ once from the $TS(11,3)$. Then add the triples $\{i,j,k\}$, $\{i,a,b\}$, $\{j,a,c\}$ and $\{k,b,c\}$. Let n_i be the number of α, β, γ equal to i. Starting with a triple system of fine structure (t,s), we obtain a triple system with fine structure (t',s'), where $t' = t + n_3$, and $s' = s + n_3 - n_2$. We say that a trade has *class* $\{\alpha,\beta,\gamma\}$.

Using only the solutions displayed in [2], we list some trades that settle many of the remaining cases:

Fine Structure of Input $TS(11,3)$	3-factorization Type	Trade Class	Fine Structure of Final TS(11,3)
(0,3)	(0,0)	3,3,3	(3,6)
(6,9)	(6,6)	3,2,2	(5,10)
		3,1,1	(7,10)
(6,11)	(6,8)	2,2,1	(4,11)
		3,2,2	(5,12)
(7,11)	(7,8)	3,1,1	(8,12)
(8,11)	(8,8)	2,2,2	(5,11)
(0,12)	(0,9)	3,1,1	(1,13)
(4,12)	(4,9)	2,1,1	(3,12)
		3,2,2	(3,13)
		3,2,1	(4,13)
		3,1,1	(5,13)
(9,12)	(9,9)	3,3,1	(11,14)
(2,14)	(2,11)	2,1,1	(1,14)
(4,14)	(4,11)	2,1,1	(3,14)

It is possible to perform more than one trade, provided each trade performed employs a different subset of $\{a,b,c,d,e\}$ (remark that originally a $TS(5,3)$ resided on these five elements, and one trade on $\{x,y,z\}$ absorbs the singly repeated block on these three elements). In this way, (2,13) is easily produced from (3,12), and (0,14) from (1,14).

Similarly, a trade of class 3,3,2 on the (3,6) solution gives (4,8); a subsequent 3,2,2 trade gives (3,9), while a subsequent 3,3,1 trade gives (6,10). After this point, it becomes difficult to verify that a suitable sequence of trades exists, so we resort to displaying the required solutions that remain. Each is displayed as a collection of triples on 0-9,a followed by a T, D, or S to indicate a triply, doubly or singly repeated block.

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0  9: 047T 379T 27aT 136T 567T 178T 469T 026T 68aT 235S 589S 125S
  238S 015S 234S 348S 019S 058S 248S 34aS 14aS 259S 59aS 129S 035S
  089S 09aS 458S 01aS 124S 145S 03aS 19aS 289S 45aS 038S 35aS
4  9: 126T 03aT 456T 234T 359T 067T 69aT 368T 048D 28aD 025D 137T
  14aD 018S 27aS 15aS 019S 279S 479S 029S 289S 149S 015S 47aS 257S
  158S 57aS 789S 049S 58aS 578S 189S 478S
1 11: 579T 289T 023T 25aT 045T 158T 356T 139D 348S 126S 127S 347S
  378S 019S 37aS 124S 147S 07aS 38aS 468S 67aS 469S 017S 13aS 08aS
  48aS 069S 16aS 068S 078S 016S 49aS 247S 14aS 246S 678S 09aS 267S
  349S 69aS
1 12: 256T 36aT 129T 01aT 349T 24aT 689D 59aS 167S 467S 067S 046S
  059S 168S 028S 57aS 138S 045S 023S 79aS 357S 145S 079S 78aS 048S
  137S 158S 478S 458S 238S 035S 038S 579S 89aS 237S 027S 147S 146S
  069S 135S 58aS 278S
2 11: 279T 09aT 078T 023T 016T 248T 045T 256D 12aD 689S 189S 368S
  68aS 359S 149S 567S 467S 138S 159S 147S 35aS 134S 367S 125S 137S
  26aS 589S 37aS 57aS 46aS 58aS 469S 157S 349S 18aS 369S 358S 34aS
  47aS
2 12: 238T 13aT 369T 59aT 035T 049D 347T 257D 189S 026S 068S 126S
  24aS 179S 167S 014S 67aS 067S 08aS 089S 48aS 26aS 02aS 156S 158S
  129S 012S 07aS 78aS 245S 468S 456S 578S 148S 145S 568S 249S 46aS
  279S 017S 789S
3 11: 123T 367T 345T 038T 247T 48aT 17aD 39aT 168D 789D 569S 258S
  26aS 06aS 56aS 056S 07aS 269S 469S 289S 017S 15aS 158S 149S 02aS
  025S 578S 029S 268S 25aS 014S 146S 019S 579S 057S 046S 159S 049S
0 11: 014T 129T 13aT 178T 69aT 48aT 156T 238S 345S 27aS 02aS 479S
  247S 068S 237S 089S 367S 079S 579S 026S 05aS 058S 25aS 467S 349S
  357S 389S 368S 07aS 346S 589S 035S 039S 57aS 245S 246S 067S 459S
```

258S 268S 023S

8 9: 456T 279T 028T 67aT 48aD 19aD 359D 047T 015T 126D 069T 137D 234D 03aT 25aT 578D 368D 134S 49aS 589S 18aS 389S 489S 178S 357S 124S 168S 236S 149S

9 10: 18aT 248T 068T 389T 025T 578T 59aT 129D 23aD 46aD 049T 347D 679D 013D 356D 145D 07aD 236S 017S 146S 03aS 47aS 345S 267S 127S 156S 26aS 137S 169S 279S

10 11: 34aT 16aT 269T 235D 128D 489D 247D 59aT 379D 046D 058T 368D 013D 145D 02aT 78aT 567D 049S 245S 147S 189S 356S 067S 238S 139S 037S 157S 127S 019S 468S 079S

11 12: 156T 147T 128T 139T 267T 01aT 235D 037D 069D 024D 579D 29aD 36aD 78aD 489D 058D 45aD 234S 468S 378S 25aS 029S 68aS 057S 589S 046S 345S 038S 368S 79aS 34aS 469S

12 13: 135D 249D 236D 128T 19aD 27aD 378D 039T 34aD 08aT 017D 458D 025D 689T 146D 56aT 579D 045S 046S 47aS 014S 159S 479S 026S 247S 578S 235S 167S 367S 13aS 29aS 067S 348S

13 14: 479T 024T 23aT 039T 278D 134D 189D 368D 269D 59aD 067D 46aD 08aD 015D 357D 458D 17aD 14aS 56aS 126S 567S 169S 018S 136S 345S 378S 07aS 89aS 125S 259S 258S 127S 056S 468S

14 15: 58aT 01aD 039D 34aD 249D 159D 478D 045T 79aD 356T 137D 238D 26aD 146D 257D 067D 689D 467S 378S 027S 69aS 24aS 579S 349S 028S 026S 168S 123S 129S 125S 17aS 018S 03aS 089S 148S

15 16: 346D 148D 256D 024D 379D 358D 23aD 15aT 167D 08aD 459D 129D 69aD 278D 057T 013D 47aD 259S 127S 039S 789S 356S 09aS 068S 37aS 189S 458S 467S 68aS 026S 689S 016S 24aS 049S 238S 134S

16 17: 02aT 013D 058D 169D 35aD 678D 127D 259D 236D 248D 18aD 46aD 049D 145D 347D 389D 79aD 257S 16aS 047S 056S 45aS 579S 238S 135S 148S 246S 567S 349S 129S 036S 89aS 017S 568S 069S 37aS 078S

17 18: 08aD 47aD 26aD 029D 157D 35aD 037D 679D 234D 046D 389D 459D 136D 19aD 568D 148D 278D 256S 12aS 248S 147S 123S 129S 015S 89aS 018S 678S 025S 469S 07aS 36aS 016S 257S 034S 45aS 059S 379S 358S

8 10: 019T 267T 357T 24aT 479T 125T 07aT 036D 348D 239D 59aD 13aD 045D 028D 178T 146D 369S 023S 456S 048S 58aS 38aS 589S 689S 68aS 568S 69aS 289S 134S 056S 16aS

9 11: 349T 368T 267T 02aT 46aT 045T 037T 147D 248D 79aD 069D 156D 018D 235D 13aD 58aD 579S 78aS 478S 089S 123S 157S 578S 259S 016S

124S 289S 19aS 569S 189S 35aS 129S

10 12: 39aT 124T 01aT 137T 345T 25aT 056D 158D 48aD 038D 236D 579D 049D 169D 67aD 278D 467S 469S 68aS 189S 289S 368S 026S 47aS 156S 023S 078S 279S 047S 029S 057S 468S 589S

11 13: 478T 579T 169T 568T 07aT 015D 249D 267D 137D 39aD 036D 238D 18aD 25aD 345D 46aD 26aS 124S 018S 089S 025S 89aS 35aS 046S 028S 367S 029S 127S 049S 123S 145S 14aS 034S 389S

12 14: 236T 457T 278T 469T 358D 156D 034D 124D 068D 02aD 189D 48aD 259D 137D 39aD 67aD 015S 16aS 25aS 017S 024S 067S 35aS 079S 379S 05aS 348S 79aS 568S 129S 013S 059S 089S 14aS 18aS

13 15: 23aT 012T 258T 036D 468D 057D 345D 59aD 149D 18aD 269D 247D 67aD 04aD 156D 389D 078S 279S 367S 69aS 089S 17aS 049S 039S 246S 478S 08aS 579S 134S 056S 137S 45aS 179S 168S 135S 378S

14 16: 367T 257T 56aD 026D 146D 017D 24aD 129D 058D 18aD 689D 135D 459D 478D 03aD 238D 568S 15aS 029S 046S 126S 134S 389S 018S 248S 69aS 23aS 079S 349S 79aS 359S 78aS 034S 09aS 045S 179S 47aS

15 17: 12aT 056D 03aD 57aD 49aD 278D 179D 367D 029D 234D 138D 689D 359D 458D 047D 146D 36aS 69aS 235S 68aS 018S 026S 256S 048S 167S 159S 246S 079S 378S 015S 257S 08aS 289S 013S 145S 349S 47aS 58aS

16 18: 017D 129D 048D 138D 24aD 236D 146D 278D 025D 06aD 345D 59aD 689D 567D 37aD 039D 58aS 18aS 149S 079S 347S 459S 568S 27aS 479S 058S 467S 125S 135S 48aS 16aS 01aS 024S 238S 036S 157S 269S 789S 39aS

□

These missing values for $v = 11$ complicate the recursions, so we eliminate them for higher orders at the outset:

Lemma 3.3 *$\{(4,7),(3,8),(5,8),(2,9),(5,9),(7,9),(0,10),(1,10),(2,10),(3,10),(4,10)\} \subset Fine(v)$ for $v \equiv 5 \pmod 6$, $v \geq 17$.*

Proof: By Lemma 2.5, we need only treat $v \in \{17, 23, 29\}$. We give the triples repeated once (S) and twice (D) in the solutions. A suitable set of triply repeated triples can be found easily for $v \in \{17, 23, 29\}$ using a hill-climbing algorithm based on [8]. To do this, we observe that the pairs covered by the blocks repeated once and twice must form the leave L of a partial triple system of order v and index *one*. That is, $K_v - L$ must have a partition into triangles; when such a partition exists, Stinson's hill-climbing method appears to be very effective in finding one.

The required solutions are:

4 7: 146D 046S 056D 156S 347D 247S 257D 357S 034S 024S 124S 015S 013S 012S 023S 123S 135S 235S

3 8: 035D 025S 135S 125D 036S 026S 016S 024S 136S 126S 134S 234D 236S 018S 017S 048S 047S 078S 148S 147S 178S 478S

5 8: 035D 025S 135S 125D 026D 016S 034S 136D 124S 234D 236S 018S 017S 048S 047S 078S 148S 147S 178S 478S

2 9: 057D 027S 157S 127D 068S 028S 018S 168S 128S 268S 145S 135S 014S 013S 136S 146S 025S 036S 034S 046S 245S 235S 345S 236S 234S 246S

5 9: 057D 027S 157S 127D 068S 028S 018S 168S 128S 268S 145D 014S 013S 136D 035S 245S 235D 026S 034S 046S 234S 246S 346S

7 9: 057D 027S 157S 127D 068S 018D 128S 268D 145D 016S 136D 134S 035S 245S 235D 024S 023S 246S 034S 046S 346S

0 10: 067S 027S 017S 167S 127S 267S 068S 038S 018S 168S 138S 368S 036S 126S 236S 259S 249S 239S 359S 349S 459S 015S 025S 024S 034S 045S 124S 235S 135S 134S 145S

1 10: 067S 027S 017S 167S 127S 267S 068S 038S 018S 168S 138S 368S 016S 236D 025S 024S 035S 034S 045S 125S 124S 135S 134S 145S 239S 249S 259S 349S 359S 459S

2 10: 067S 027S 017S 167S 127S 267S 068S 038S 018S 168S 138S 368S 036S 126S 236S 259S 249S 239S 359S 349S 459S 015S 024D 035S 045S 125S 235S 134D 145S

3 10: 067D 027S 167S 127D 068S 038S 018S 168S 138S 368S 126S 236D 015S 014S 135S 134S 145S 025S 024S 035S 034S 239S 249S 245S 259S 349S 359S 459S

4 10: 067D 017S 127D 267S 068S 038S 018S 168S 138S 368S 136S 126S 236S 015S 134S 145D 025S 024D 035S 034S 239S 235S 249S 259S 349S 359S 459S

□

The case $v = 17$ is the most difficult, and it is here that Lemma 2.1 is critical.

Lemma 3.4 $Fine(17) = Adm(17)$.

Proof: Apply Lemma 2.1 using the complete determination of $Fine(7)$ in [4], and the partial determination of types of $3K_{5,5}$ [3]. This handles all values in $Adm(17)$

except those with $s - t \leq 2$ and the following: (4,8), (0,10), (1,10), (2,10), (3,10), (4,10), (5,10), (0,11), (1,11), (2,11), (3,11), (4,11), (5,11), (0,12), (1,12), (2,12), (4,12), (0,13), (1,13), (2,13), (0,14), (1,14), (2,14), (4,14), (1,15), (0,16), (1,16), (2,16), (3,16), (4,16), (5,16), (0,17), (1,17), (2,17), (4,17), (5,17), (1,18), (2,18), (2,19) and (0,20).

There exists a group divisible design *gdd* with blocks of size three having one group of size six, one group of size 4, and three groups of size two. An explicit construction for this *gdd* on groups $\{\{0,1,2,3,4,5\},\{6,7,8,9\},\{a,b\},\{c,d\},\{e,f\}\}$ is {ace, bdf, 0ad, 06b, 07c, 08e, 09f, 1cf, 16d, 17e, 18a, 19b, 2be, 26f, 27a, 28c, 29d, 3bc, 36a, 37d, 38f, 39e, 4de, 46c, 47f, 48b, 49a, 5af, 56e, 57b, 58d, 59c}. Replicate each block three times, and add a new element forming a $TS(7,3)$, a $TS(5,3)$ and three $TS(3,3)$'s with the groups. This gives $(t, s+3) \in Fine(17)$ for $(t,s) \in Fine(7)$, eliminating (0,10), (1,10), (3,10) and (4,10). Then removing one copy of the triples 08e, 09f, 38f, 39e and adding instead one copy of 08f, 09e, 38e, 39f gives $(t+4, s+7) \in Fine(17)$, eliminating (4,14). If we instead consider the triples of the *gdd* on 0,1,2,7,8,a,c,e, we find seven triples. Together with 012, they partition K_8 minus a 1-factor. If we choose $(t,s) \in Fine(7)$ with $s < 7$, we align the triply repeated block on 012. Then we can remove the eight triply repeated triples on 0,1,2,7,8,a,c,e and replace them with a partial triple system of type (0,8) to eliminate (0,11), (0,17) and (4,17). If we align the $TS(7,3)$ to omit a doubly repeated block on 012, omitting a singly repeated block on 012 in the partial triple system, we obtain $(t, s+11) \in Fine(17)$ for $(t+1, s+1) \in Fine(7)$. This eliminates (3,16), (5,16) and (2,17).

Next we use a $2v+3$ construction. On Z_{10}, we take the ten triples from the starter block $\{0,1,2\}$. The graph on differences 3,4 and 5 can be 1-factored, forming five 1-factors. Each is triplicated to form a 3-factor with only triplicated edges. Difference 1 remains once and difference 2 remains twice. This 6-regular multigraph can be partitioned into two simple 3-factors, or into two 3-factors containing two double edges in total. Hence for $(t,s) \in Fine(7)$, we have $(t, s+13)$ and $(t+2, s+13)$ in $Fine(17)$. This eliminates (0,13), (2,13), (2,19) and (0,20).

For the remaining cases, we again produce the singly and doubly repeated triples; the triply repeated triples can be found using hill-climbing.

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4  8: 137D 037S 078D 178S 048S 148D 035S 025S 015S 135S 125S 235S
 034S 234D 016S 012S 026S 046S 126S 146S 246S
5 10: 067D 027S 167S 127D 068S 038S 018S 168S 138S 368S 126S 236D
 015S 014S 135S 134S 145S 025S 023S 034S 045S 249D 245S 259S 349S
```

359D

1 11: 057S 037S 017S 157S 137S 357S 068S 048S 018S 168S 148S 468S 29aS 249S 239S 24aS 23aS 39aS 349S 34aS 49aS 134S 035S 023S 135S 046S 024S 146S 012S 056S 126S 125S 256D

2 11: 057S 037S 017S 157S 137S 357S 068S 048S 018S 168S 148S 468S 29aS 249S 239S 24aS 23aS 39aS 349S 34aS 49aS 134S 035D 123S 156S 125S 256D 012S 146S 026S 024S 046S

3 11: 057D 017S 137D 357S 068S 048S 018S 168S 148S 468S 29aS 249S 239S 24aS 23aS 39aS 349S 34aS 49aS 034S 035S 023S 135S 156S 125S 256D 012S 026S 046S 124S 146S

4 11: 057D 037S 157S 137D 068S 048S 018S 168S 148S 468S 29aS 249S 239S 24aS 23aS 39aS 349S 34aS 49aS 034S 035S 135S 235S 156S 256D 012D 026S 046S 124S 146S

5 11: 057D 017S 137D 357S 068D 018S 148D 468S 29aS 249S 239S 24aS 23aS 39aS 349S 34aS 49aS 034S 035S 023S 135S 156S 125S 256D 012S 024S 046S 126S 146S

0 12: 09aS 029S 019S 02aS 01aS 19aS 129S 12aS 29aS 236S 136S 036S 237S 137S 037S 038S 138S 238S 246S 146S 046S 056S 156S 256S 247S 147S 047S 048S 148S 248S 057S 157S 257S 012S 058S 158S 258S

1 12: 589D 489S 359S 057S 037S 017S 157S 137S 357S 29aS 249S 239S 24aS 23aS 39aS 34aS 49aS 134S 034S 035S 123S 246S 026S 025S 012S 125S 256S 156S 568S 018S 048S 046S 068S 148S 146S 168S

2 12: 589D 489S 359S 057S 037S 017S 157S 137S 357S 29aS 249S 239S 24aS 23aS 39aS 34aS 49aS 134S 034S 035S 123S 124S 026S 025S 012S 256D 158S 156S 018S 048S 046S 068S 146S 168S 468S

4 12: 589D 489S 359S 057D 037S 157S 137D 29aS 249S 239S 24aS 23aS 39aS 34aS 49aS 134S 034S 035S 235S 256S 125S 156S 568S 026S 024S 012S 126S 018D 046S 068S 148S 146S 468S

1 13: 057S 037S 017S 157S 137S 357S 068S 048S 018S 168S 148S 468S 29aS 249S 239S 24aS 23aS 39aS 349S 34aS 49aS 36bS 35bS 34bS 56bS 46bS 45bS 036S 035S 136S 123S 012S 026S 024S 045S 125S 256D 145S 146S

0 14: 057S 037S 017S 157S 137S 357S 068S 048S 018S 168S 148S 468S 29aS 249S 239S 39aS 349S 49aS 24aS 12aS 3abS 13aS 4abS 1abS 16bS 15bS 36bS 35bS 46bS 45bS 016S 125S 124S 134S 026S 025S 023S 034S 045S 456S 236S 256S 356S

1 14: 057S 037S 017S 157S 137S 357S 068S 048S 018S 168S 148S 468S 29aS 249S 239S 39aS 349S 49aS 24aS 23aS 3abS 14aS 1abD 16bS 35bS 34bS 46bS 45bS 56bS 013S 026S 025S 024S 036S 045S 126S 125S 123S 145S 256S 346S 356S

2 14: 057D 037S 157S 137D 068S 048S 018S 168S 148S 468S 29aS 249S 239S 39aS 349S 49aS 24aS 12aS 3abS 13aS 4abS 1abS 16bS 15bS 36bS 35bS 46bS 45bS 016S 014S 125S 124S 026S 025S 023S 034S 236S 256S 345S 356S 456S

1 15: 789D 379S 489S 178S 057S 037S 017S 157S 357S 068S 048S 018S 168S 468S 29aS 249S 239S 39aS 49aS 24aS 12aS 3abS 13aS 4abS 1abS 16bS 15bS 36bS 35bS 46bS 45bS 014S 026S 025S 023S 034S 056S 126S 123S 134S 145S 245S 256S 346S 356S

0 16: 067S 027S 017S 167S 127S 267S 068S 038S 018S 168S 138S 368S 036S 126S 236S 259S 249S 239S 359S 349S 459S 035S 045S 024S 014S 025S 135S 134S 234S 145S 125S 0adS 0acS 0abS 0bdS 0bcS 0cdS 1adS 1acS 1abS 1bdS 1bcS 1cdS 2adS 2acS 2abS 2bdS 2bcS 2cdS

1 16: 067S 027S 017S 167S 127S 267S 068S 038S 018S 168S 138S 368S 016S 236D 035S 034S 045S 024S 025S 135S 134S 145S 124S 125S 239S 249S 259S 349S 359S 459S 0adS 0acS 0abS 0bdS 0bcS 0cdS 1adS 1acS 1abS 1bdS 1bcS 1cdS 2adS 2acS 2abS 2bdS 2bcS 2cdS

2 16: 067S 027S 017S 167S 127S 267S 068S 038S 018S 168S 138S 368S 036S 126S 236S 259S 249S 239S 359S 349S 459S 035S 045S 024D 015S 145S 134D 125S 235S 0adS 0acS 0abS 0bdS 0bcS 0cdS 1adS 1acS 1abS 1bdS 1bcS 1cdS 2adS 2acS 2abS 2bdS 2bcS 2cdS

4 16: 067D 017S 127D 267S 068S 038S 018S 168S 138S 368S 136S 126S 236S 135S 145D 014S 035S 034S 024S 025D 239S 234S 249S 259S 349S 359S 459S 0adS 0acS 0abS 0bdS 0bcS 0cdS 1adS 1acS 1abS 1bdS 1bcS 1cdS 2adS 2acS 2abS 2bdS 2bcS 2cdS

1 17: 068S 048S 018S 168S 148S 468S 9acD 69cS 4acS 39aS 269S 249S 239S 349S 469S 24aS 23aS 12aS 13aS 14aS 6bcS 4bcS 0bcS 16cS 17cS 01cS 07cS 47cS 057S 047S 157S 137S 357S 347S 36bS 35bS 34bS 56bS 04bS 05bS 145S 245S 456S 013S 126S 125S 026S 025S 023S 036S 356S

5 17: 068D 048S 168S 148D 9acD 69cS 4acS 49aS 269S 249S 239S 369S 349S 24aS 23aS 12aS 13aD 6bcD 0bcS 17cS 14cS 01cS 07cS 47cS 157S 137S 016S 015S 126S 125S 025S 024S 023S 256S 047S 037S 03bS 05bS 357S 457S 34bS 346S 35bS 356S 46bS 456S 45bS

1 18: 468D 168S 178S 018S 489S 089S 078S 789S 79cS 379S 9acS 09cS 07cS 06cS 7bcS abcS 6bcS 6acS 057S 17bS 157S 37bS 357S 49aS 39aS 249S 239S 029S 26aS 24aS 23aS 3abS 14aS 1abS 16aS 15bS 34bS 46bS 45bS 56bS 045S 034S 014S 134S 245S 012S 026S 036S 035S 125S 123S 136S 256S 356S

2 18: 468D 168S 178S 148S 089D 078S 789S 79cS 379S 9acS 09cS 07cS 06cS 7bcS abcS 6bcS 6acS 057S 17bS 157S 37bS 357S 29aS 249S 239S 349S 49aS 26aS 24aS 3abS 36aS 13aS 14aS 1abS 15bS 34bS 46bS 45bS 56bS 045S 034S 024S 145S 016S 013S 012S 126S 123S 025S 036S 235S 256S 356S

It remains to handle the cases when $s-t \leq 2$, for $9 \leq s \leq 45$. All solutions of this type have been found using the hill-climbing algorithm directly to find $TS(17,3)$'s at random. The seventy-four remaining cases have all been found easily using the hill-climbing method; we omit the solutions here. □

Lemma 3.5 $Fine(23) = Adm(23)$.

Proof: Apply Lemma 3.1 of [4], the $2v+1$ construction, to $Fine(11)$, using the determination of types of 3-factorizations of $3K_{12}$ from [2]. This handles all values in $Adm(23)$ except for (4,7), (3,8), (5,8), (2,9), (5,9), (7,9), (0,10), (1,10), (2,10), (3,10) and (4,10). The cases (3,8) and (5,8) are handled by Lemma 2.1. Lemma 3.3 completes the proof. □

Lemma 3.6 $Fine(29) = Adm(29)$.

Proof: Apply Lemma 2.3, observing that the fine structures of $TS(11,3)$'s *missing* a subdesign of order 5 are precisely the types of 3-factorizations of $3K_6$. Using Lemma 3.2, we obtain all $(t,s) \in Fine(29)$ except for (4,7), (3,8), (5,8), (2,9), (5,9), (7,9), (0,10), (1,10), (2,10), (3,10) and (4,10). The cases (3,8) and (5,8) and handled by Lemma 2.1. Lemma 3.3 completes the proof. □

Lemma 3.7 $Fine(35) = Adm(35)$.

Proof: Since $Fine(17) = Adm(17)$, and the solution for 3-factorizations of $3K_{18}$ is complete [2], the $2v+1$ construction (Lemma 3.1 of [4]) suffices. □

Lemma 3.8 $Fine(41) = Adm(41)$.

Proof: We apply Lemma 2.2 with $n=3$. We have $Fine(17) = Adm(17)$. For 3-factorizations of $3K_8$, we use the partial determination in [2]. □

4 Applying the Recursions

In this section, we apply the recursions from section 2, using the solution for small orders in section 3 as base cases, to establish sufficiency for the characterization.

For $v \in \{17, 23, 29, 35, 41\}$, we have established $Fine(v) = Adm(v)$ in section 3. So assume that $v \geq 47$ and $v \equiv 5 \pmod 6$. Since $Fine(23) = Adm(23)$ and $v \geq 47$, by Lemma 2.5 we have that if $(t,s) \in Adm(v)$ and $s \leq 84$, $(t,s) \in Fine(v)$. We assume henceforth that $s > 84$.

Now set $z = \frac{v+4}{3}$. Since $v \geq 47$, $z \geq 17$. Hence $Fine(z) = Adm(z)$. Apply Lemma 2.1 to form a system of order $v = 3z - 4$ using the determination of $Fine(z)$ and the solution for types of 3-factorizations of $K_{z-2,z-2}$ (which is complete since $z - 2 \geq 10$ [3]). For $(t,s) \in Adm(v)$, this gives $(t,s) \in Fine(v)$ unless $s - t \leq 2$ and $z \equiv 1, 3 \pmod 6$, or $s - t \leq 6$ and $z \equiv 5 \pmod 6$. To complete the determination, if $v \equiv 11 \pmod{12}$, write $y = \frac{v-1}{2}$. Now $y \equiv 5 \pmod 6$ and $y \geq 23$; hence $Fine(y) = Adm(y)$. Moreover, the solution for types of 3-factorizations of K_{y+1} is complete [2]. Then the $2v+1$ construction (Lemma 3.1 of [4]) establishes $Fine(v) = Adm(v)$. The more complicated case is $v \equiv 5 \pmod{12}$. The $2v + 7$ construction of [4] applies only when $v \equiv 17 \pmod{24}$, and hence we use Lemma 2.2 of this paper instead. Write $y = \frac{v-7}{2}$. Again, since $v \geq 53$, we have $y \geq 23$. It is then easy to verify that $Fine(v) = Adm(v)$. We remark that since $s - t \leq 6$ is all that remained, Lemma 2.4 would also suffice in place of Lemma 2.2 here.

This completes the proof of sufficiency.

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