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The total chromatic numbers of joins of sparse graphs

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Abstract

In this paper, we investigate the total colorings of the join graphs $G = G_1 + G_2$, where G_1 and G_2 are graphs with maximum degree at most two. We prove that

(1) when both G_1 and G_2 are bipartite graphs with maximum degree at most two, then G is Type 1 if and only if G is not isomorphic to $K_{n,n}$ $(n = 1, 2, \ldots)$ or to K_4, and

(2) $C_m + C_n$ is Type 2 if and only if $m = n$ and n is odd.

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1 Introduction

All graphs in this paper will be finite simple graphs. Given two graphs G_1 and G_2, we define their *join graph*, denoted by $G_1 + G_2$, to be the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $\{uv \mid uv \in E(G_1) \cup E(G_2)$ where $u \in V(G_1), v \in V(G_2)\}$. We note that $G_1 + G_2$ is a complete bipartite graph if both G_1 and G_2 are sets of independent vertices. Let C_n and P_n be the cycle and path of n vertices, respectively.

An *edge coloring* of a graph G is a map $f : E(G) \to \mathcal{C}$, where $\mathcal{C}$ is the set of colors, such that no two edges with the same color are incident with the same vertex. The chromatic index or edge chromatic number $\chi'(G)$ of G is the least value of $|\mathcal{C}|$ for which G has an edge coloring. In [16] Vizing showed that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$ for every graph G with maximum degree $\Delta(G)$. A fairly long-standing problem has been to classify which graphs G are *Class one* ($\chi'(G) = \Delta(G)$) and which are *Class two* ($\chi'(G) = \Delta(G) + 1$).

A *total coloring* of a graph G is a coloring of the vertices and edges of G such that no two edges incident with the same vertex receive the same color, no two adjacent vertices receive the same color, and no incident edge and vertex receive the same color. The *total chromatic number* $\chi''(G)$ of a graph G is the least number of colors needed in a total coloring of G.

For the total chromatic number, there is no known analogue of Vizing's theorem about the chromatic index. Instead we have the Total Chromatic Number Conjecture (TCC) of Behzad [1] and Vizing [17] that

$$\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2.$$

The lower bound here is very easy to prove. This conjecture, now more than thirty years old, has been verified when $\Delta(G) \geq \frac{3}{4}|V(G)|$ by Hilton and Hind [9], and when $\Delta(G) \leq 5$ by Kostochka [13]. It has recently been shown by Molloy and Reed [14] that there is a constant c such that $\chi''(G) \leq \Delta(G) + c$.

A graph G is called *Type 1* if $\chi_T(G) = \Delta(G) + 1$ and *Type 2* otherwise. To classify Type 1 and Type 2 graphs, Chetwynd and Hilton [4], introduced the following concepts. They defined the *deficiency* of a graph G, denoted by $\mathrm{def}(G)$, to be

$$\mathrm{def}(G) = \sum_{v \in V(G)} (\Delta(G) - d_G(v)).$$

A vertex coloring of a graph G with $\Delta(G) + 1$ colors is called *conformable* if the number of color classes of parity different from that of $|V(G)|$ is at most $\mathrm{def}(G)$. Note that empty color classes are permitted in this definition. A graph G is *conformable* if it has a conformable vertex coloring. It is not very hard to see that G is Type 2 if G is non-conformable. The Conformability Conjecture of Chetwynd and Hilton [4], modified by Hamilton, Hilton and Hind [6] is:

Conjecture 1 *Let G be a graph such that $\Delta(G) \geq \frac{1}{2}(|V(G)| + 1)$. Then G is Type 2 if and only if G contains a subgraph H with $\Delta(G) = \Delta(H)$ which is either non-conformable, or, when $\Delta(G)$ is even, consists of $K_{\Delta(G)+1}$ with one edge subdivided.*

Conjecture 1 has been verified for several cases when $\Delta(G)$ is big and close to the order of G (see [8], [3], [19], [18], [10], [6] and [11]). It would be interesting to provide nontrivial evidence for Conjecture 1 when $\Delta(G)$ is close to one half of the order of G.

A good characterization of all Type 1 graphs is unlikely as Sanchez-Arroyo [15] showed that the problem of determining the total chromatic number of a graph is NP-hard. Not only that, there are few results about the total chromatic numbers of even very nice graphs, for example, the complete multipartite graphs. In [2], it was determined which complete bipartite graphs are Type 1. It is natural to ask which graphs G, obtained by adding edges to a complete bipartite graph, are Type 1. Such a graph G can be represented as a join of two graphs.

In this paper, we determine the total chromatic numbers of graphs of the form $G_1 + G_2$, where G_1 and G_2 are graphs of maximum degrees at most two. Our results generalize Behzad, Chartrand and Cooper's classical result [2] and also provide evidence to support Conjecture 1 when $\Delta(G)$ is close to $\frac{1}{2}|V(G)|$, as we note that the maximum degrees of these graphs are close to half of the order of the graphs.

2 Useful lemmas

The following results will be used in this paper.

Lemma 2.1 (Behzad, Chartrand and Cooper [2]) *Let $K_{m,n}$ be the complete bipartite graph. Then $K_{m,n}$ is Type 1 if $m \neq n$, and Type 2 otherwise.*

Lemma 2.2 (König's Theorem) *If G is bipartite graph with maximum degree $\Delta(G)$, then $\chi'(G) = \Delta(G)$.*

We use $\bar{G}$ and $\alpha'(G)$ to denote the complement and the edge independence number of G, respectively.

Lemma 2.3 (see [5]) *Let G be a graph of even order $2n$. If $|E(\bar{G})| + \alpha'(\bar{G}) \leq n(2n - \Delta(G)) - 1$, then G is non-conformable and therefore Type 2.*

Let B_n denote a copy of $K_{n,n}$ with partite sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$. Let H_n be the graph obtained from B_n by adding the edges

$$x_1x_2, x_3x_4, \ldots, x_{k-1}x_k \text{ and } y_1y_2, y_3y_4, \ldots, y_{k-1}y_k,$$

where $k = n$ if n is even, and $k = n - 1$ otherwise. Note that if $n \geq 2$ then $\Delta(H_n) = n + 1$.

Lemma 2.4 *(a) For $n = 3, 4, 5$, H_n has a totally 4-colorable spanning subgraph F_n such that $H_n - E(F_n)$ is an $(n-2)$-regular bipartite subgraph of B_n, the edges of which are properly colorable with $n - 2$ colors. Moreover the two colorings together give a total coloring of H_n which is Type 1.*

(b) H_6 has a totally 6-colorable 5-regular spanning subgraph F_6 such that $H_6 - E(F_6)$ is a 2-regular bipartite subgraph of B_6. Again the two colorings together give a total coloring of H_6 which is Type 1.

Proof. (a) For $n = 5$, assign colors to the vertices and edges of $H_5 - E(B_5)$ as follows:

vertices $x_1, \ldots, x_5$	get colors $1, 2, 1, 2, 1$,
vertices $y_1, \ldots, y_5$	get colors $3, 4, 3, 4, 3$,
edges x_1x_2, x_3x_4	get colors $3, 4$,
edges y_1y_2, y_3y_4	get colors $1, 2$.

For $n = 3, 4$, give each vertex and edge of $H_n - E(B_n)$ the same color as the element with the same label in H_5. Color the remaining edges of F_n as in the following tables. (Here a ϕ means there is no edge joining the two vertices, a * means that the edge remains uncolored, that is, it is not in F_n.)

	x_1	x_2	x_3	y_1	y_2	y_3
x_1	1	3	ϕ	*	2	4
x_2	3	2	ϕ	4	*	1
x_3	ϕ	ϕ	1	2	3	*
y_1	*	4	2	3	1	ϕ
y_2	2	*	3	1	4	ϕ
y_3	4	1	*	ϕ	ϕ	3

Table F_3

	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4
x_1	1	3	ϕ	ϕ	*	2	4	*
x_2	3	2	ϕ	ϕ	4	*	*	1
x_3	ϕ	ϕ	1	4	2	*	*	3
x_4	ϕ	ϕ	4	2	*	3	1	*
y_1	*	4	2	*	3	1	ϕ	ϕ
y_2	2	*	*	3	1	4	ϕ	ϕ
y_3	4	*	*	1	ϕ	ϕ	3	2
y_4	*	1	3	*	ϕ	ϕ	2	4

Table F_4

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5
x_1	1	3	ϕ	ϕ	ϕ	4	*	*	*	2
x_2	3	2	ϕ	ϕ	ϕ	*	*	*	1	4
x_3	ϕ	ϕ	1	4	ϕ	*	2	*	3	*
x_4	ϕ	ϕ	4	2	ϕ	*	3	1	*	*
x_5	ϕ	ϕ	ϕ	ϕ	1	2	*	4	*	*
y_1	4	*	*	*	2	3	1	ϕ	ϕ	ϕ
y_2	*	*	2	3	*	1	4	ϕ	ϕ	ϕ
y_3	*	*	*	1	4	ϕ	ϕ	3	2	ϕ
y_4	*	1	3	*	*	ϕ	ϕ	2	4	ϕ
y_5	2	4	*	*	*	ϕ	ϕ	ϕ	ϕ	3

Table F_5

It is easy to check that this defines total colorings of appropriate graphs F_n. In view of Lemma 2.2, the edges of the $(n-2)$-regular bipartite graphs $H_n - E(F_n)$ can be properly colored with the $n-2$ colors $5, \ldots, n+2$, thereby giving a total coloring of H_n with $n+2 = \Delta(H_n)+1$ colors, for $n = 3, 4$ and 5.

(b) The argument for $n = 6$ is very similar. Assign colors to the vertices and edges of $H_6 - E(B_6)$ as follows:

vertices $x_1, \ldots, x_6$	get colors $1, 2, 1, 3, 2, 3$,
vertices $y_1, \ldots, y_6$	get colors $4, 5, 4, 6, 5, 6$,
edges x_1x_2, x_3x_4, x_5x_6	get colors $4, 5, 6$,
edges y_1y_2, y_3y_4, y_5y_6	get colors $3, 2, 1$.

Color the remaining edges of F_6 as in the following table.

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	y_5	y_6
x_1	1	4	ϕ	ϕ	ϕ	ϕ	*	*	3	5	6	2
x_2	4	2	ϕ	ϕ	ϕ	ϕ	1	*	6	*	3	5
x_3	ϕ	ϕ	1	5	ϕ	ϕ	2	6	*	*	4	3
x_4	ϕ	ϕ	5	3	ϕ	ϕ	6	1	*	4	2	*
x_5	ϕ	ϕ	ϕ	ϕ	2	6	5	4	1	3	*	*
x_6	ϕ	ϕ	ϕ	ϕ	6	3	*	2	5	1	*	4
y_1	*	1	2	6	5	*	4	3	ϕ	ϕ	ϕ	ϕ
y_2	*	*	6	1	4	2	3	5	ϕ	ϕ	ϕ	ϕ
y_3	3	6	*	*	1	5	ϕ	ϕ	4	2	ϕ	ϕ
y_4	5	*	*	4	3	1	ϕ	ϕ	2	6	ϕ	ϕ
y_5	6	3	4	2	*	*	ϕ	ϕ	ϕ	ϕ	5	1
y_6	2	5	3	*	*	4	ϕ	ϕ	ϕ	ϕ	1	6

Table F_6

It is easy to check that this defines a total coloring of an appropriate graph F_6, and the conclusion that H_6 is Type 1 follows as in (a). □

Lemma 2.5 *H_n is Type 1 for all $n \geq 3$.*

Proof. We have seen in Lemma 2.4 that this holds if $n \leq 6$, so suppose $n \geq 7$. Let $k = \lfloor (n+1)/4 \rfloor$.

If $n \not\equiv 2 \pmod 4$, then H_n has a spanning subgraph that is the union of vertex-disjoint subgraphs $S_1, \ldots, S_k$ that are all isomorphic to F_4, except that, if n is odd, exactly one of them is isomorphic to F_3 or F_5. By Lemma 2.4, these graphs are all totally 4-colorable. If they are all colored with colors $1, \ldots, 4$ as in Lemma 2.4, then the graph of the uncolored edges of H_n is an $(n-2)$-regular bipartite graph whose edges can be colored with the $n-2$ colors $5, \ldots, n+2$ by Lemma 2.2. Since the sets of colors used on $x_1, \ldots, x_n$ and on $y_1, \ldots, y_n$ are disjoint, the result is a total coloring of H_n with $n+2 = \Delta(H_n)+1$ colors, showing that H_n is Type 1.

If $n \equiv 2 \pmod 4$, then H_n has a spanning subgraph that is the union of vertex-disjoint subgraphs $S_1, \ldots, S_k$ that are all isomorphic to H_4, except for S_k that is isomorphic to H_6. We totally color a 5-regular spanning subgraph F_6 of S_k using colors $1, 2, 3, 4, 5, 6$ and totally color a 3-regular spanning subgraph F_4 of S_i ($i = 1, \ldots, k-1$) using colors $1, 2, 5, 4$ as we did in Lemma 2.4 (note that the color 3 has been changed to color 5). We then color two matchings in S_i using colors 3 and 6 obtaining a totally 6-colored 5-regular spanning subgraph F'_4 of S_i for $i = 1, \ldots, k-1$. The remaining uncolored edges in H_n form an $(n-4)$-regular bipartite graph which can be colored by the $n-4$ colors $7, 8, \ldots, n+2$.

This completes the proof. □

The following lemma is an easy observation.

Lemma 2.6 *Let G be a Type 1 graph and H a spanning subgraph of G such that $\Delta(H) = \Delta(G)$. Then H is Type 1.*

3 Type 1 join graphs

One of our main results is the following.

Theorem 3.1 *Let $G = G_1 + G_2$ where both G_1 and G_2 are bipartite graphs with maximum degree at most two. Then G is Type 1 if and only if G is not isomorphic to $K_{n,n}$ ($n = 1, 2, \ldots$) or to K_4.*

To prove Theorem 3.1, we need a number of lemmas.

Lemma 3.2 *Let $G = G_1 + G_2$, where the maximum degree of G_i is at most one for $i = 1, 2$. Then G is Type 1 if and only if G is not isomorphic to K_4 or $K_{n,n}$ for any positive integer n.*

Proof. Without loss of generality, let $|V(G_1)| \leq |V(G_2)| = n$. Furthermore, we assume that we cannot add any vertices or edges to G without increasing its maximum degree or violating the hypotheses of the lemma. Then $|V(G_1)| = |V(G_2)| = n$, or $|V(G_2)| = n$, $|V(G_1)| = n-1$ and G_2 contains $\lfloor \frac{n}{2} \rfloor$ independent edges.

If $\Delta(G_1) = \Delta(G_2) = 0$, then $G \cong K_{n,n}$, which is known to be a Type 2 graph. If $\Delta(G_1) = 0$ and $\Delta(G_2) = 1$, then $|V(G_1)| = n-1$ and $|V(G_2)| = n$, and we may color the vertices of G_1 and the edges of G_2 with color 1. The edges of the $K_{n-1,n}$ with vertex sets $V(G_1)$ and $V(G_2)$ may be colored with the colors in $2, \ldots, n+1$ by Lemma 2.2, and each vertex v of G_2 may be colored with the color in $\{2, \ldots, n+1\}$ that is not used on edges incident with v. This gives a total coloring of G with $\Delta(G) + 1 = n+1$ colors, so G is Type 1. Finally, suppose that $\Delta(G_1) = \Delta(G_2) = 1$. If $n = 2$ then $G \cong K_4$, which is Type 2. If $n \geq 3$, then we must have $G \cong H_n$ since we have assumed that we cannot add any further edges to G without increasing $\Delta(G)$. Therefore, G is Type 1 by Lemma 2.5.

This completes the proof of the lemma. □

Lemma 3.3 *Let $G = G_1 + G_2$ where both G_1 and G_2 are bipartite graphs with maximum degree two. Then G is Type 1.*

Proof. Without loss of generality, let $|V(G_1)| \leq |V(G_2)| = n$. Then $\Delta(G) = n+2$, where evidently $n \geq 3$. Assume that we cannot add any vertices or edges to G without increasing its maximum degree or violating the hypotheses of the lemma. Then $|V(G_1)| = |V(G_2)| = n$, and each of G_1 and G_2 is a disjoint union of even cycles and at most one path. Let $V(G_1) = \{x_1, \ldots, x_n\}$ and $V(G_2) = \{y_1, \ldots, y_n\}$, and assume that each cycle or path occupies a consecutive set of vertices of G_1 or G_2 in the obvious way, with the path first, then cycles according to their lengths, shorter cycles following longer ones. There exist matchings M_1 and M_2 of G_1 and G_2, respectively, such that the graph $G' = G - (M_1 \cup M_2)$ is isomorphic to H_n.

If $n \not\equiv 2 \pmod 4$, then we can totally $(n+2)$-color G', as in Lemma 2.5, so that the vertices of G_1 are colored alternately 1 and 2, and the vertices of G_2 are colored alternately 3 and 4. Since no edge in $M_1 \cup M_2$ joins two vertices with the same color, if we give all these edges the color $n+3$ then we obtain a total coloring of G with $n+3 = \Delta(G)+1$ colors; thus G is Type 1.

If $n \equiv 2 \pmod 4$, then we can totally $(n+2)$-color G', as in Lemma 2.5, so that the vertices $\{x_1, x_2, \ldots,\}$ of G_1 have colors $1,2,1,3,2,3,1,2,1,2,\ldots$ and the vertices $\{y_1, y_2, \ldots,\}$ of G_2 have colors $4,5,4,6,5,6,4,5,4,5,\ldots$ (in increasing order of subscript). It is easy to check that no edges of M_1 (M_2) join two vertices of X (Y) with the same color. We color the edges in $M_1 \cup M_2$ with the color $n+3$, the result is a total $(n+3)$-coloring of a graph isomorphic to G. This shows that G is Type 1. □

Lemma 3.4 *Let $G = G_1 + G_2$ where G_1 and G_2 are bipartite graphs with $n-1$ and n vertices respectively, and $\Delta(G_1) = 1$ and $\Delta(G_2) = 2$. Then G is Type 1.*

Proof. Let $V(G_1) = \{x_1, \ldots, x_{n-1}\}$ and $V(G_2) = \{y_1, \ldots, y_n\}$, where evidently $n \geq 3$. Without loss of generality, we may assume that G_1 has edges $x_1x_2, x_3x_4, \ldots, x_{k-1}x_k$, where $k = n-1$ if $n-1$ is even, and $k = n-2$ otherwise, and G_2 is a disjoint union of even cycles and, possibly, one path. Note that $\Delta(G) = n+1$, and so we must totally color G with $n+2$ colors. Let $B = G - (E(G_1) \cup E(G_2))$.

Claim. *Using just colors $1,2,3,4$, one can totally color a subset of $V(G) \cup E(G)$ consisting of all vertices and edges of G_1, all edges of G_2, two vertices a, b of G_2, and a set of $2n-2$ edges of B forming a subgraph in which every vertex of G has degree 2 except for a and b, which both have degree 1.*

If one can do this, then the proof of the lemma is easily completed as follows. Form G^* from G by deleting all the colored edges and adding a new vertex x^* that is adjacent to all vertices of $V(G_2) \setminus \{a,b\}$. G^* is an $(n-2)$-regular bipartite graph, and so its edges can be properly colored with the $n-2$ colors $5, \ldots, n+2$ by Lemma 2.2. This causes colors to be assigned to all vertices and edges of G except for the vertices in $V(G_2) \setminus \{a,b\}$; give each such vertex y the color of the edge x^*y. This gives the required total coloring of G with $n+2$ colors, and this will complete the proof of the lemma.

Proof of the Claim. We shall color the edges of G_2 with colors 1, 2 and 3. A vertex y of G_2 will be said to have *type* c ($c \in \{1,2,3\}$) if there is no edge of color

c incident with y. (A vertex with degree less than 2 in G_2 will have more than one type.) If y_j, y_k are vertices of G_2 of different types, say of types 2 and 3 respectively, and $x_i x_{i+1} \in E(G_1)$, then the vertices and edges $y_j x_i, x_i, x_i x_{i+1}, x_{i+1}, x_{i+1} y_k$ can be colored with colors $2, 3, 1, 2, 3$, respectively; this is called the *standard coloring method.*

There are three cases to consider.

Case 1: $n = 4k$.

For $n = 4$, the graph G_2 is a 4-cycle $y_1 y_2 y_3 y_4$ and G_1 is a K_2 with vertices x_1, x_2 together with a single vertex x_3. A total coloring of G is shown by the following table.

	y_1	y_2	y_3	y_4	x_1	x_2	x_3
y_1	6	1	ϕ	3	2	4	5
y_2	1	4	2	ϕ	5	6	3
y_3	ϕ	2	5	1	4	3	6
y_4	3	ϕ	1	4	6	5	2
x_1	2	5	4	6	3	1	ϕ
x_2	4	6	3	5	1	2	ϕ
x_3	5	3	6	2	ϕ	ϕ	1

Suppose now $n \geq 8$. We will color $E(G_2)$ so that $2k$ vertices of G_2 have type 2 and $2k$ have type 3. To do this, for each cycle whose length is a multiple of 4, and for the path component of G_2 if there is one, color the edges $1, 2, 1, 3, 1, 2, 1, 3, \ldots$. If G_2 has cycles of length 2 (mod 4), then the number of these cycles is even if G_2 does not have a path component. Color half of the cycles with colors $1, 2, 1, 2, 1, 3, 1, 2, 1, 3, \ldots$, and half with colors $1, 3, 1, 2, 1, 3, 1, 2, 1, 3, \ldots$. If G_2 has a path component, then the path component is a K_2, and there are odd number of, say $2p + 1$, cycles of length 2 (mod 4). Color p of these cycles with colors $1, 2, 1, 2, 1, 3, 1, 2, 1, 3, \ldots$, and the rest with colors $1, 3, 1, 2, 1, 3, 1, 2, 1, 3, \ldots$. Color the path component K_2 by the color 1. It is easy to see that this edge coloring of G_2 has the property that half of the vertices of G_2 have type 2 while the other half have type 3. Let a, b be two nonadjacent vertices of types 2 and 3 respectively, and color $a, ax_{n-1}, x_{n-1}, x_{n-1}b, b$ with colors $4, 2, 1, 3, 4$ respectively. Use the $n - 2$ vertices in $V(G_2) \setminus \{a, b\}$ and the standard coloring method to color the remaining vertices and edges of G_1 and $n - 2$ further edges of B. To complete the proof of the claim in this case, we take a matching of $n - 2$ uncolored edges between $V(G_1) \setminus \{x_{n-1}\}$ and $V(G_2) \setminus \{a, b\}$, and give these edges color 4.

Case 2: $n = 4k + 2$.

We first assume that G_2 has a cycle C of length 2 (mod 4). Color the edges of C by the colors $1, 2, 3, 1, 2, 3$ or $1, 2, 3, 1, 2, 3, 1, 2, 1, 3, 1, 2, 1, 3, \ldots$ if $|C| > 6$. $G_2 - C$ has order divisible by 4, so, as in Case 1, the edges of $G_2 - C$ can be colored by colors $1, 2, 3$ so that half of the vertices in $V(G_2 - C)$ have type 2 and half have type 3. In C, there are two vertices of type 1, denoted by a and b, respectively. Note that a and b are separated by at least two edges in C. Also note that half of the vertices in $C - \{a, b\}$ have type 2 and the other half have type 3. Therefore, the edge coloring of G_2 has the property that half of the vertices of $G_2 - \{a, b\}$ have type 2 and half have type 3.

Next, if G does not have a cycle C of length $2 (\bmod\ 4)$, then G_2 has a path component which is a K_2. Color the edge of the K_2 by the color 1. $G_2 - K_2$ has order divisible by 4, so that, as in Case 1, the edges of $G_2 - K_2$ can be colored by colors $1, 2, 3$ in such a way that half of the vertices in $V(G_2 - C)$ have type 2 and half have type 3. Therefore, the edge coloring of G_2 has the property that half of the vertices have type 2 and half have type 3.

Let a and b be two vertices of different types which are not adjacent. We now use the $n-2$ vertices from $G_2 - \{a, b\}$ and the standard coloring method to color $n-2$ edges of B and all vertices and edges of G_1 except for x_{n-1}. Give color 2 to x_{n-1} and color 1 to edge $x_{n-1}b$. As in Case 1, choose a matching of $n-1$ uncolored edges between $V(G_1)$ and $V(G_2) \setminus \{b\}$, and give these edges the color 4. We note that ax_{n-1} is colored with color 4. Color the vertex a with 1 and the vertex b with 4.

Case 3: n is odd.

In this case, G_2 has a path component P. If $n - |P| \equiv 0 (\bmod\ 4)$, then as in Case 1, we can color the edges in $G_2 - P$ with colors $1, 2, 3$ such that half of the vertices in $V(G_2) - V(P)$ have type 2 and half have type 3. Let a be an end vertex of P. The edges of $P - \{a\}$ can also be colored by $1, 2, 3$ such that a has type 1 and half of the vertices of $P - \{a\}$ have type 2 and half have type 3. Let b be any vertex in $G_2 - \{a\}$. Use the vertices of $V(G_2) \setminus \{a\}$ and the standard coloring method to color all vertices and edges of G_1 and $n-1$ edges of B; note that no vertex of G_1 gets color 1. Choose a matching of $n-1$ uncolored edges between $V(G_1)$ and $V(G_2) \setminus \{b\}$, and give these edges color 4. Color a with 1 and b with 4.

If $n - |P| \equiv 2 \ (\bmod\ 4)$, then G_2 has a cycle C of length $4t + 2$ for some t. Let c be an end vertex of P. Then by Case 2, we can color $G_2 - \{c\}$ by colors $1, 2, 3$ such that there are non-adjacent vertices, say a and b in C, half of the vertices in $G_2 - \{a, b, c\}$ have type 2, the other half have type 3, and the vertices a and b have type 1. Now apply the standard coloring to $G_2 - \{a, b, c\}$ to color $n-3$ edges of B and color the edges $x_1x_2, x_3x_4, \ldots, x_{n-4}x_{n-3}$ with color 1. Next, color a, b both with 1 (their type), both edges ax_{n-2}, bx_{n-1} with 4, and the edges cx_{n-2}, cx_{n-1} with 2 and 3 respectively, $x_{n-2}x_{n-1}$ with 1, and x_{n-2} and x_{n-1} with 3 and 2, respectively. Finally, color a matching of $\{x_1, \ldots, x_{n-3}\}$ and $Y - \{a, b, c\}$ consisting of edges that have not been colored with the color 4. Then we have produced a coloring required in the claim.

This completes the proof of the claim, and also the proof of Lemma 3.4. □

Lemma 3.5 *Let $G = G_1 + G_2$ where G_1 and G_2 are bipartite graphs with $n-2$ and n vertices respectively, and $\Delta(G_1) = 0$ and $\Delta(G_2) = 2$. Then G is Type 1.*

Proof. Let $V(G_1) = \{x_1, \ldots, x_{n-2}\}$ and $V(G_2) = \{y_1, \ldots, y_n\}$. Without loss of generality, we may assume that G_2 is the disjoint union of even cycles and, possibly, one path. Note that $\Delta(G) = n$, and so we must totally color G with $n+1$ colors. Let M be a maximum matching (with $\lfloor \frac{n}{2} \rfloor$ edges) of G_2 chosen so that $G_2 - M$ has no vertex of degree 2. Use color $n+1$ to color the vertices of G_1 and the edges of M.

Color the remaining edges of G_2 with the colors $1, \ldots, \lfloor \frac{n}{2} \rfloor$. Assign the remaining colors to vertices of G_2 so that each color is assigned to two non-adjacent vertices, and each vertex receives exactly one color if n is even, and the vertex which is isolated in $G_2 - M$ receives two colors (one of which may be discarded later) if n is odd; this is easy to do if one starts by giving one color to each pair of diametrically opposite vertices in each even cycle of G_2.

To color the edges of G between G_1 and G_2, form a bipartite graph J with vertex sets $\{c_1, \ldots, c_n\}$ (corresponding to the colors $1, \ldots, n$) and $\{y'_1, \ldots, y'_n\}$, and join c_i to y'_j if the i-th color is not present so far at y_j in G. Then J is a regular graph of degree $n-2$, and so, by Lemma 2.2, can be properly edge-colored with the $n-2$ colors $1, 2, \ldots, n-2$. If an edge $c_i y'_j$ is colored k, then color the edge of G joining the vertices y_j and x_k with color i. The result is the required total $(n+1)$-coloring of G. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1.

Proof. If G_1 and G_2 both have maximum degree at most 1, then the result follows from Lemma 3.2; so we may assume without loss of generality that $\Delta(G_2) = 2$. We may assume that we cannot add any vertices or edges to G without increasing its maximum degree or violating the hypotheses of the theorem; thus, $|V(G_1)| + 2 = |V(G_2)| + \Delta(G_1)$. If $\Delta(G_1) = 2$ then the result follows from Lemma 3.3, if $\Delta(G_1) = 1$ then it follows from Lemma 3.4, and if $\Delta(G_1) = 0$ then it follows from Lemma 3.5. This completes the proof of Theorem 3.1. □

4 Type 2 join graphs

We have determined the total chromatic number of $G_1 + G_2$ where G_1 and G_2 are bipartite graphs of maximum degree at most 2. One natural question now is, what is the total chromatic number of $G_1 + G_2$ when G_1 and G_2 are not bipartite graphs. We have the following general result.

Lemma 4.1 *Let $G = G_1 + G_2$ be regular with $|V(G_1)|$ and $|V(G_2)|$ both odd. Then G is Type 2.*

Proof. Let $V(G)| = 2n$. Then $|E(\bar{G})| = \frac{2n(2n-1)}{2} - n\Delta(G)$ and $\alpha'(\bar{G}) \leq n-1$. Therefore, $|E(\bar{G})| + \alpha'(\bar{G}) \leq \frac{2n(2n-1)}{2} - n\Delta(G)) + n - 1 = n(2n - \Delta(G)) - 1$. By Lemma 2.3, G is a Type 2 graph. □

Corollary 4.2 *Suppose $G = G_1 + G_2$ where both $|V(G_1)| = |V(G_2)| = n$, and both G_1 and G_2 are unions of cycles. Then G is Type 2 if n is odd.*

Now we have our second main result.

Theorem 4.3 *The graph $G = C_m + C_n$ is Type 2 if and only if $m = n$ and n is odd.*

Proof. If $m = n$ and n is odd, then G is Type 2 by Corollary 4.2.

To show the necessity, we need only to show that G is Type 1 if $m \neq n$ by Theorem 3.1.

Without loss of generality, let $m < n$, $C_m = x_1x_2\cdots x_mx_1$ and $C_n = y_1y_2\cdots y_ny_1$. Then $\Delta(G) + 1 = n + 3$. Let $B = G - (E(C_m) \cup E(C_n))$. We give a total coloring of G using $n + 3$ colors as follows.

1. Let $M = \{x_iy_i : i = 1, \ldots, m\}$, which is a matching of B, and let $K = M \cup E(C_m) \cup E(C_n)$. We totally color the edges in K and the vertices in $V(C_m) \cup \{y_n\}$ with colors $1, 2, 3, 4$, as follows.

   Color the vertices $x_1, x_2, \ldots, x_m$ alternately with colors 1 and 3 (starting with 1) except that, if m is odd, x_m gets color 2 instead of color 1.

   Color the edges $x_1x_2, x_2x_3, \ldots, x_mx_1$ alternately with colors 2 and 4 (starting with 2) except that, if m is odd, x_mx_1 gets color 3 instead of color 2.

   Give each edge x_iy_i of M the unique color that is available to it, which for $2 \leq i \leq m-1$ will be whichever of colors 1 and 3 has not been given to x_i; note that x_my_m always has color 1. Color the edge x_1y_1 with color 3 or 4 whichever is missing at x_1.

   Color the edges $y_1y_2, y_2y_3, \ldots, y_{m-1}y_m$ alternately with colors 2 and 4 (starting with 2), and give y_ny_1 the same color as x_mx_1.

   If m is odd, $y_{m-1}y_m$ has color 4, x_mx_1 and y_ny_1 have color 3, and the vertices of C_m have colors 1, 2 and 3. Color the edges $y_my_{m+1}, \ldots, y_{n-1}y_n$ alternately with colors 2 and 3, *ending* with color 2, and give y_n color 4.

   If m is even, $y_{m-1}y_m$ has color 2, x_mx_1 and y_ny_1 have color 4, and all the vertices of C_m have colors 1 and 3. Color the edges $y_my_{m+1}, \ldots, y_{n-1}y_n$ alternately with colors 3 and 4, *ending* with color 3, and give y_n color 2.

2. Let $G' = G - K = B - M$; note that G' is bipartite and $\Delta(G') = n - 1$. Form a new graph G^* from G' by adding a new vertex x^* that is adjacent to all vertices of C_n except for y_n. By considering separately the cases when $m = n - 1$ and $m \leq n - 2$, one sees easily that $\Delta(G^*) = \Delta(G') = n - 1$. Moreover G^* is bipartite, and so its edges can be properly colored with the $n - 1$ colors $5, \ldots, n + 3$. This causes colors to be assigned to all vertices and edges of G except for the vertices in $V(C_n) \setminus \{y_n\}$; give each such vertex y_i the color of the edge x^*y_i.

It is easy to see that this gives a proper total coloring of G with $n+3 = \Delta(G)+1$ colors. Therefore this shows that G is Type 1.

This completes the proof of Theorem 4.2. □

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