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Graph designs, packings and coverings of λK_v with a graph of six vertices and containing a triangle

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Abstract

Let λK_v be the complete multigraph with v vertices, where any two distinct vertices x and y are joined by λ edges (x, y). Let G be a finite simple graph. A G-design (G-packing, G-covering) of λK_v, denoted by (v, G, λ)-GD ((v, G, λ)-PD, (v, G, λ)-CD), is a pair $(X, \mathcal{B})$ where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly (at most, at least) λ blocks of $\mathcal{B}$. In this paper, we determine the existence spectrum for the G-designs of λK_v, $\lambda > 1$, and construct the maximum packings and the minimum coverings of λK_v with G for any positive integer λ, where the graph G has six vertices and contains a triangle.

1 Introduction

A complete multigraph of order v and index λ, denoted by λK_v, is a graph with v vertices, where any two distinct vertices x and y are joined by λ edges (x, y). Let G be a finite simple graph. A *G-design* (*G-packing, G-covering*) of λK_v, denoted by (v, G, λ)-GD ((v, G, λ)-PD, (v, G, λ)-CD), is a pair $(X, \mathcal{B})$ where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly (at most, at least) λ blocks of $\mathcal{B}$. A G-packing (G-covering) is said to be *maximum* (*minimum*), denoted by (v, G, λ)-MPD (MCD), if no other such G-packing (G-covering) has more (fewer) blocks. The number of blocks in a maximum G-packing (minimum G-covering), denoted by $p(v, G, \lambda)(c(v, G, \lambda))$, is called the *packing* (*covering*) *number*. It is well known that

$$p(v, G, \lambda) \le \lfloor \frac{\lambda v(v-1)}{2e(G)} \rfloor \le \lceil \frac{\lambda v(v-1)}{2e(G)} \rceil \le c(v, G, \lambda),$$

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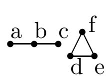
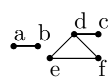
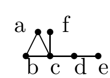
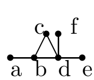
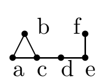
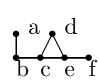
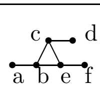
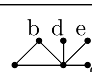
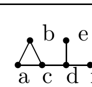
where $e(G)$ denotes the number of edges in G, $\lfloor x \rfloor$ denotes the greatest integer y such that $y \le x$ and $\lceil x \rceil$ denotes the least integer y such that $y \ge x$. A (v, G, λ)-PD ((v, G, λ)-CD) is said to be *optimal* and denoted by (v, G, λ)-OPD ((v, G, λ)-OCD) if the left (right) equality holds. Obviously, there exists a (v, G, λ)-GD if and only if $p(v, G, \lambda) = c(v, G, \lambda)$ and a (v, G, λ)-GD can be regarded as (v, G, λ)-OPD or (v, G, λ)-OCD.

By a $L_\lambda(\mathcal{D})$ of a packing $\mathcal{D}$, called *the leave edge graph*, we mean that it is a subgraph of λK_v and its edges are the supplement of $\mathcal{D}$ in λK_v. The number of edges in $L_\lambda(\mathcal{D})$ is denoted by $|L_\lambda(\mathcal{D})|$. Especially, when $\mathcal{D}$ is maximum, $|L_\lambda(\mathcal{D})|$ is called *leave edge number* and is denoted by $l_\lambda(v)$. Similarly, the *repeat edge graph* $R_\lambda(\mathcal{D})$ of a covering $\mathcal{D}$ is a subgraph of λK_v and its edges are the supplement of λK_v in $\mathcal{D}$. When $\mathcal{D}$ is minimum, $|R_\lambda(\mathcal{D})|$ is called the *repeat edge number* and is denoted by $r_\lambda(v)$. Generally, the symbols $L_\lambda(\mathcal{D})$, $l_\lambda(v)$, $R_\lambda(\mathcal{D})$ and $r_\lambda(v)$ can be denoted by L_λ, l_λ, R_λ and r_λ, briefly.

Many researchers have been involved in graph design, graph packing and graph covering of λK_v with five vertices or less (see [1–10]).

Yin [11] listed the spectrum of graph designs of K_v with six vertices and $e(G) \le 6$. (See Table A.)

Table A

note	G_1	G_2	G_3
graph			
spectrum	$v \equiv 0, 1\ (mod\ 5)$, $v > 6$	$v \equiv 0, 1\ (mod\ 5)$, $v \ge 6$	$v \equiv 0, 1, 4, 9\ (mod\ 12)$, $v \ge 6$
note	G_4	G_5	G_6
graph			
spectrum	$v \equiv 0, 1, 4, 9\ (mod\ 12)$, $v \ge 6$	$v \equiv 0, 1, 4, 9\ (mod\ 12)$, $v \ge 6$	$v \equiv 0, 1, 4, 9\ (mod\ 5)$, $v \ge 6$
note	G_7	G_8	G_9
graph			
spectrum	$v \equiv 0, 1, 4, 9\ (mod\ 12)$, $v \ge 6$	$v \equiv 0, 1, 4, 9\ (mod\ 12)$, $v \ge 6$	$v \equiv 0, 1, 4, 9\ (mod\ 12)$, $v \ge 6$

Throughout this paper, the graph G is denoted by $[a, b, c, d, e, f]$. In what follows, the notations ($a, b \in Z$): $[a, b] = \{x \in Z \mid a \le x \le b\}$, $[a, b]_k = \{x \in Z \mid a \le x \le$

$b, x \equiv a (\mathrm{mod}\ k)\}$ for $a, b \in Z$, $[a, b, \cdots, c] + i = [a+i, b+i, \cdots, c+i]$ and $(Z_n)_m = \{i_m \mid i \in Z_n\}$ are used frequently. The edge set $\{(a_1, a_2), (a_2, a_3), \cdots, (a_{n-1}, a_n)\}$ is denoted by $(a_1, a_2, \cdots, a_n)$.

In this paper, we prove the following theorem:

Theorem *For $i \in [1, 9]$, the $p(v, G_i, \lambda)$ and $c(v, G_i, \lambda)$ are determined.*

Example We construct $(14, G_i, 1)$-OPD ($i \in [3, 9]$) on the set $Z_{13} \cup \{a\}$ as follows:

i=3 $[5, 9, 3, 8, 11, 4] + i$, $i \in [0, 9]$, $[7, 10, a, 0, 6, 12]$, $[6, 9, a, 1, 0, 11]$, $[5, 8, a, 2, 6, 4]$, $[3, 7, 1, 2, 0, 6]$, $[4, 8, 2, 3, a, 7]$. Leave edge: $(0, 5)$.

i=4 $[4, 5, 9, 3, 6, 8] + i$, $i \in [0, 9]$, $[4, 1, 7, 3, 2, a]$, $[0, 2, 8, 4, 3, a]$, $[12, a, 5, 0, 6, 3]$, $[2, 1, 6, a, 10, 11]$, $[5, 2, 7, a, 8, 9]$. Leave edge: $(2, 6)$.

i=5 $[5, 9, 3, 4, 7, 12] + i$, $i \in [0, 9] - \{1\}$, $[6, 10, 4, 5, 2, 1]$, $[8, 2, 4, 1, a, 12]$, $[1, 3, 7, a, 8, 5]$, $[4, 9, a, 0, 2, 6]$, $[5, 10, a, 11, 6, 3]$, $[2, 3, a, 6, 0, 1]$. Leave edge: $(0, 8)$.

i=6 $[11, 8, 3, 9, 5, 4] + i$, $i \in [0, 9]$, $[12, a, 1, 7, 3, 2]$, $[11, a, 4, 8, 2, 0]$, $[6, 0, a, 10, 7, 2]$, $[4, 3, a, 9, 6, 1]$, $[6, 2, a, 8, 5, 0]$. Leave edge: $(1, 2)$.

i=7 $[9, 3, 4, 8, 6, 11] + i$, $i \in [0, 9] - \{1\}$, $[a, 0, 1, 5, 3, 8]$, $[a, 3, 5, 10, 2, 8]$, $[a, 5, 7, 12, 4, 10]$, $[10, a, 1, 2, 7, 3]$, $[11, a, 2, 4, 6, 0]$, $[12, a, 4, 1, 9, 5]$. Leave edge: $(a, 8)$.

i=8 $[4, 8, 2, 3, 5, 7] + i$, $i \in [0, 9]$, $[1, 5, 12, a, 2, 4]$, $[a, 7, 1, 2, 3, 4]$, $[2, 6, a, 4, 5, 8]$, $[0, 3, a, 9, 10, 11]$, $[1, 6, 0, 2, 5, 12]$. Leave edge: $(3, 7)$.

i=9 $[5, 10, 4, 8, 6, 11] + i$, $i \in [0, 9]$, $[2, 7, 1, 5, 8, a]$, $[3, 8, 2, a, 0, 1]$, $[3, 4, 9, a, 11, 12]$, $[7, 10, a, 6, 2, 4]$, $[7, 5, 3, a, 4, 8]$. Leave edge: $(6, 9)$.

Let the bipartite graph G have six vertices and let its edge number be not greater than 6. The G-design, maximum G-packing and minimum G-covering of λK_v was solved by Z. Liang [13]. When six vertex graph G contains a triangle and $e(G) \leq 6$, we give the G-design, maximum G-packing and minimum G-covering of λK_v in this paper.

2 Recursion

By $K_{n_1, n_2, \cdots, n_h}$ we mean the complete multipartite graph with h parts of sizes $n_1, n_2, \cdots, n_h$. Let $X = \bigcup_{1 \leq i \leq h} X_i$ be the vertex set of $K_{n_1, n_2, \cdots, n_h}$ where X_i $(1 \leq i \leq h)$ are disjoint sets with $|X_i| = n_i$ and $v = \sum_{1 \leq i \leq h} n_i$. For any fixed graph G, if $K_{n_1, n_2, \cdots, n_h}$ can be decomposed into edge-disjoint subgraphs isomorphic to G, then we call $(X, \mathcal{G}, \mathcal{A})$ a *holey G-design*, where $\mathcal{G} = \{X_1, X_2, \cdots, X_h\}$, and $\mathcal{A}$ is the collection of all subgraphs called *G-blocks* (or simply *blocks*). Each set $X_i (1 \leq i \leq h)$ is said to be a *hole* and the multiset $\{n_1, n_2, \cdots, n_h\}$ is called the type of the holey G-design. We denote the design by G-HGD$(n_1^1 n_2^1 \cdots n_h^1)$ (or $K_{n_1, n_2, \cdots, n_h}/G$) and use an "exponential" notation to describe its type in general: a type $1^i 2^j 3^k \cdots$, denotes i occurrences of 1, j occurrences of 2, etc. A G-HGD$(1^{v-w} w^1)$ is called an *incomplete G-design*, denoted by (v, w, G)-IGD. Obviously, a $(v, G, 1)$-GD is a G-HGD(1^v), which can be thought of as a (v, w, G)-IGD with $w = 0$ or 1.

Let S be a finite set and $H = \{S_1, S_2, \cdots, S_n\}$ be a partition of S. A *holey Latin square* having partition H is an $|S| \times |S|$ array L indexed by $S \times S$, satisfying the following condition:

1) every cell of L either contains an element of S or is empty;

2) every element of S occurs at most once in any row or any column of L;

3) the subarrays(called holes) indexed by $S_i \times S_i$ are empty for $1 \le i \le n$;

4) element $s \in S$ occurs in row s or column t if and only if $(s,t) \in (S \times S)\backslash(\bigcup_{i\in[1,n]} S_i \times S_i)$.

The order of L is $|S|$, and the type of L is the multiset $T = \{|S_i| : i \in [1,n]\}$. A holey Latin square is called *symmetric* if the element in cell (i,j) is the element in cell (j,i) for all i and j. We simply write HSL(T) for a holey symmetric Latin square of type T.

Theorem 2.1 [12] *There exist HSL(2^n) for all $n \ge 3$.*

Theorem 2.2 *Let $v = 2ne(G_i)$. There exist G_i-HGD($(2e(G_i))^n$) for $n \ge 3$ and $i \in [1,9]$.*

Proof By Theorem 2.1, let $A = (a_{ij})$ be a HSL(2^n), $S = [1,2n]$ and $H = \{S_t : S_t = \{2t-1, 2t\}, t \in [1,n]\}$. Vertex set $X = Z_{e(G_i)} \times S$, hole set $\mathcal{G} = \{Z_{e(G_i)} \times S_t : t \in [1,n]\}$. We construct $\mathcal{A}$ as follows:

<u>for G_1</u> $[(1,i),(3,a_{ij}),(1,j),(0,i),(0,j),(1,a_{ij})]$ (mod 5,–);

<u>for G_2</u> $[(0,i),(1,j),(3,j),(2,i),(2,j),(0,a_{ij})]$ (mod 5,–);

<u>for G_3</u> $[(0,j),(1,a_{ij}),(0,i),(3,j),(5,i),(2,j)]$ (mod 6,–);

<u>for G_4</u> $[(5,j),(2,i),(2,j),(0,a_{ij}),(1,i),(1,j)]$ (mod 6,–);

<u>for G_5</u> $[(1,a_{ij}),(0,i),(0,j),(2,i),(4,j),(1,i)]$ (mod 6,–);

<u>for G_6</u> $[(4,j),(2,i),(0,a_{ij}),(1,i),(1,j),(4,i)]$ (mod 6,–);

<u>for G_7</u> $[(4,j),(1,i),(0,a_{ij}),(2,i),(1,j),(3,i)]$ (mod 6,–);

<u>for G_8</u> $[(1,i),(1,j),(0,a_{ij}),(2,i),(2,j),(3,i)]$ (mod 6,–);

<u>for G_9</u> $[(1,i),(1,j),(0,a_{ij}),(2,i),(4,j),(5,j)]$ (mod 6,–).

Then $(X,\mathcal{G},\mathcal{A})$ is a G_i-HGD($(2e(G_i))^n$), $i \in [1,9]$. □

Theorem 2.3 *If both $(2e(G_i)+w, w, G_i)$-IGD and $(2e(G_i)+w, G_i, 1)$-MPD(MCD) exist, then a $(2ne(G_i)+w, G_i, 1)$-MPD(MCD) exists for $n \ge 3$ and $i \in [1,9]$.*

Proof By Theorem 2.2, there exists G_i-$HGD((2e(G_i))^n)$=$(X,\mathcal{G},\mathcal{A})$ for $i \in [1,9]$. Let Y=$(Z_n \times Z_{2e(G_i)}) \bigcup \{\infty_1, \infty_2, \cdots, \infty_w\}$, Y_j=$(\{j\} \times Z_{2e(G_i)}) \bigcup \{\infty_1, \infty_2, \cdots, \infty_w\}$, for $j \in Z_n$. On Y_j ($j \in Z_n^*$), let $(2e(G_i)+w, w, G_i)$-IGD=$(Y_j, \mathcal{A}_j)$. On Y_0, let $(2e(G_i)+w, G_i, 1)$-MPD=$(Y_0, \mathcal{A}_0)$. Since $|\mathcal{A}| = 2n(n-1)e(G_i)$,

$$|\bigcup_{1\le j\le n-1} \mathcal{A}_j| = (n-1)(2e(G_i)+2w-1)$$

and $|\mathcal{A}_0|$=$(2e(G_i)+2w-1) + \lfloor \frac{w(w-1)}{2e(G_i)} \rfloor$,

$$|\mathcal{A}| + |\bigcup_{1\le j\le n-1} \mathcal{A}_j| + |\mathcal{A}_0| \quad = \quad 2n^2e(G_i) + 2nw - n + \lfloor \frac{w(w-1)}{2e(G_i)} \rfloor$$

$$= \lfloor\frac{(2ne(G_i)+w)(2ne(G_i)+w-1)}{2e(G_i)}\rfloor.$$

Therefore $(Y, \mathcal{A}\bigcup(\bigcup_{0\le j\le n-1}\mathcal{A}_j))$ is a $(2ne(G_i)+w, G_i, 1)$-MPD.

In the same way we can prove an MCD exists. □

Theorem 2.4 *Let l be the leave edge number of the $(n, G, 1)$-OPD and $\bar{\lambda} = e(G)/gcd(e(G), l)$. If there exist (n, G, λ)-OPD and (n, G, λ)-OCD for $1 \le \lambda \le \bar{\lambda}$, then there exist (n, G, λ)-OPD and (n, G, λ)-OCD for any positive integer λ.*

The following theorem is a modified version of Theorem 4 in Section 3 of [14].

Theorem 2.5 *Given positive integers v, λ and μ, let X be a v-set.*
(1) Suppose that there exists a (v, G, λ)-MPD $= (X, \mathcal{D})$ with leave edge graph $L_\lambda(\mathcal{D})$, and a (v, G, μ)-MPD $= (X, \mathcal{E})$ with leave edge graph $L_\mu(\mathcal{E})$. If $|L_\lambda(\mathcal{D})| + |L_\mu(\mathcal{E})| = l_{\lambda+\mu}(v) < e(G)$, then there exists a $(v, G, \lambda+\mu)$-MPD with leave edge graph $L_\lambda(\mathcal{D}) \cup L_\mu(\mathcal{E})$.
(2) Suppose that there exists a (v, G, λ)-MCD $= (X, \mathcal{D})$ with repeat edge graph $R_\lambda(\mathcal{D})$ and a (v, G, μ)-MCD $= (X, \mathcal{E})$ with repeat edge graph $R_\mu(\mathcal{E})$. If $|R_\lambda(\mathcal{D})| + |R_\mu(\mathcal{E})| = r_{\lambda+\mu}(v) < e(G)$, then there exists a $(v, G, \lambda+\mu)$-MCD with repeat edge graph $R_\lambda(\mathcal{D}) \cup R_\mu(\mathcal{E})$.
(3) Suppose that there exists a (v, G, λ)-MPD $= (X, \mathcal{D})$ with leave edge graph $L_\lambda(\mathcal{D})$ and a (v, G, μ)-MCD $= (X, \mathcal{E})$ with repeat edge graph $R_\mu(\mathcal{E})$. If $R_\mu(\mathcal{E}) \subset L_\lambda(\mathcal{D})$ and $|L_\lambda(\mathcal{D})| - |R_\mu(\mathcal{E})| = l_{\lambda+\mu}(v) < e(G)$, then there exists a $(v, G, \lambda+\mu)$-MPD with leave edge graph $L_\lambda(\mathcal{D}) \backslash R_\mu(\mathcal{E})$.
(4) Suppose that there exists a (v, G, λ)-MCD $= (X, \mathcal{D})$ with repeat edge graph $R_\lambda(\mathcal{D})$ and a (v, G, μ)-MPD $= (X, \mathcal{E})$ with leave edge graph $L_\mu(\mathcal{E})$. If $L_\mu(\mathcal{E}) \subset R_\lambda(\mathcal{D})$ and $|R_\lambda(\mathcal{D})| - |L_\mu(\mathcal{E})| = r_{\lambda+\mu}(v) < e(G)$, then there exists a $(v, G, \lambda+\mu)$-MCD with repeat edge graph $R_\lambda(\mathcal{D}) \backslash L_\mu(\mathcal{E})$.

If we replace MPD and MCD by OPD and OCD respectively, then the theorem is also true.

Corollary 2.6 *If there exist (v, G, λ_1)-GD and (v, G, λ_2)-GD, then there exists a $(v, G, \lambda_1 + \lambda_2)$-GD.*

3 Incomplete graph designs

Theorem 3.1 *Let G be a graph and n a positive integer satisfying $n(n-1) < 2e(G)$. A $(v, G, 1)$-OPD exists and its leave edge graph is K_n if and only if there exists a (v, n, G)-IGD.*

Theorem 3.2 *For $w \in \{2, 3, 4, 7, 8, 9\}$ and $G \in \{G_1, G_2\}$, there exists a $(10 + w, w, G)$-IGD.*

Proof When $w = 2, 3$, see the proof of Theorem 4.4. When $w \in \{4, 7, 8, 9\}$, we can construct a $(10 + w, w, G)$-IGD (see the Appendix). □

Theorem 3.3 *When $w \in \{2, 3, 5, 6, 7, 8, 10, 11\}$, there exists a $(12 + w, w, G_i)$-IGD for $i \in [3, 9]$.*

Proof When $w = 2$, it follows from Example and Theorem 3.1 that the theorem is true. When $w \in \{3,5,6,7,8,10,11\}$, a $(12+w, w, G_i)$-IGD for $i \in [3,9]$ can be directly constructed (see the Appendix). □

Theorem 3.4 *When $i \in [1,9]$, if a $(v,G,1)$-MPD(MCD) exists for $6 \le v < 6e(G_i)$, then a $(v,G,1)$-MPD(MCD) exists for any $v \ge 6$.*

4 Packing and covering

Let P be the spectrum for the existence $(v,G,1)$-GD. In this section, we discuss (v,G,λ)-PD and (v,G,λ)-CD when v does not satisfy P.

Theorem 4.1 *If there exists a $(v,G,1)$-OPD(OCD) and $l_1 = 1$ $(r_1 = 1)$, then there exists a (v,G,λ)-OCD(OPD).*

Theorem 4.2 *(1). If there exists a $(v,G,1)$-GD, then a (v,G,λ)-OCD(OPD) exists for $\lambda \ge 1$.*
(2). Let G be a graph. If a $(v,G,1)$-OPD $= (X,A)$ exists, and $L_1 \subset G$, then a $(v,G,1)$-OCD exists.

Proof The following proves case (2). We take $R_1 = G \setminus L_1$; then $R_1 \cup L_1 = G$. The block of the graph G is denoted by $[a,b,c,d,e,f]$. Then $(X, A \cup \{[a,b,c,d,e,f]\})$ is a $(v,G,1)$-OCD, and its repeat edge graph is R_1. □

Lemma 4.3 *There does not exist a $(v,G_1,1)$-OPD(OCD) for $v = 6,7$, that is, $p(6,G_1,1) = 2$, $c(6,G_1,1) = 4$, $p(7,G_1,1) = 3$ and $c(7,G_1,1) = 6$.*

Proof Let Z_6 be the vertex set of K_6. Since $(Z_6, \{[2,4,5,0,1,3],[0,4,1,2,3,5]\})$ is a $(6,G_1,1)$-PD and $(Z_6,\{[2,4,5,0,1,3],[5,0,2,1,3,4],[4,0,1,2,3,5],[3,4,0,5,1,2]\})$ is a $(6,G_1,1)$-CD, $p(6,G_1,1) = 2$ and $c(6,G_1,1) = 4$. The leave edges are $(3,4),(2,0,5,1,2)$, and repeated edges are $(2,5),(1,3,4,0,1)$.

Let X be the vertex set of K_7. Suppose that there exists a $(7,G_1,1)$-OPD. Then the number of blocks is four, with leave an edge. Without loss of generality, let the leave edge be ab. The types of vertices a and b are $2^2 1^1$ and $2^1 1^3$. The types of other vertices are $2^2 1^2$ and 2^3. Vertex numbers of these types can be $\{0,2,1,4\}$, $\{1,1,2,3\}$ or $\{2,0,3,2\}$. No type can give rise to a $(7,G_1,1)$-*OPD*. Since $(Z_7,\{[2,6,5,0,1,3],[0,6,3,1,2,4],[1,6,4,2,3,5]\})$ is a $(7,G_1,1)$-PD, we have $p(7,G_1,1) = 3$ and the leave edges are $(0,4,5,0),(3,4),(0,2),(1,5)$. No leave edge graph of $(7,G_1,1)$-*MPD* can be covered by two blocks, therefore there is no $(7,G_1,1)$-OCD. Since $(Z_7,\{[6,2,5,0,1,4],[6,4,5,1,2,3],[3,4,2,1,5,6],[4,5,1,0,2,6],[2,1,3,0,3,5],[2,4,1,0,3,6]\})$ is a $(7,G_1,1)$-CD, we have $c(7,G_1,1) = 6$ and repeat edges are $(1,2,4,1),(1,3),(6,2),(4,5),(1,5)$ and $(6,0,3)$. □

Theorem 4.4 *There exists a $(v,G_i,1)$-OPD(OCD) for $i = 1,2$, except for $(v,i) = (7,1),(6,1)$.*

Proof <u>$v = 7$</u> On the set $X = Z_5 \cup \{a,b\}$, $(7,G_2,1)$-OPD $= (X,A)$,
A: $[a,2,b,4,0,1]$, $[a,4,b,3,1,2]$, $[4,2,1,a,0,3]$, $[3,4,1,b,0,2]$, leave edge is ab.

By Theorem 4.1, there exists a $(7,G_2,1)$-OCD.

<u>$v = 8$</u> On the set $X = Z_5 \cup \{a,b,c\}$, $(8,G_1,1)$-OPD $= (X,A)$, A: $[2,a,3,0,1,4]$, $[b,a,4,1,2,3]$, $[1,c,3,b,0,2]$, $[c,0,a,b,3,4]$, $[a,1,b,c,2,4]$, leave edges: $03, bc, ca$.

$(8, G_1, 1)$-OCD $= (X, A \cup \{[0, 3, 2, b, c, a]\})$, repeat edges: $32, ab$.

$(8, G_2, 1)$-OPD $= (X, A)$, A: $[a, 3, c, 0, 1, 4]$, $[a, 4, 0, 3, 1, 2]$, $[c, 1, a, b, 0, 2]$, $[a, 0, c, 3, 4, b]$, $[b, 1, a, c, 2, 4]$, leave edges: $2a, a1, bc$.

$(8, G_2, 1)$-OCD $= (X, A \cup \{[b, c, 4, a, 2, 1]\})$, repeat edges: $12, c4$.

$\underline{v = 9}$ On the set $X = Z_7 \cup \{a, b\}$, $(9, G_1, 1)$-OPD $= (X, A)$, A: $[a, 2, b, 0, 1, 3]$ (mod 7). $(9, G_2, 1)$-OPD $= (X, A)$, A: $[a, 2, b, 0, 1, 3]$ (mod 7). Their leave edge is ab. By Theorem 4.1, there exists a $(9, G_i, 1)$-OCD for i=1,2.

$\underline{v = 12}$ On the set $X = Z_{10} \cup \{a, b\}$, $(12, G_1, 1)$-OPD $= (X, A)$,
A: $[a, 9, 1, 0, 4, 3] + i$, $i \in Z_{10} \backslash \{0, 1\}$, $[a, 9, 1, 0, 5, b]$, $[a, 10, 2, 4, 9, b]$, $[4, 3, 0, 1, 6, b]$, $[0, 4, 1, 2, 7, b]$, $[1, 5, 4, 3, 8, b]$.

$(12, G_2, 1)$-OPD $= (X, A)$, A: $[a, 8, 2, 0, 4, 3] + i$, $i \in Z_{10} \backslash \{0, 1\}$, $[a, 8, 4, 0, 5, b]$, $[a, 9, 5, 1, 6, b]$, $[4, 1, 0, 2, 7, b]$, $[5, 4, 0, 3, 8, b]$, $[1, 2, 3, 4, 9, b]$.

Their leave edge is ab. It follows from Theorem 4.1 that there exists a $(12, G_i, 1)$-OCD for i=1,2.

$\underline{v = 13}$ On the set $X = Z_{10} \cup \{a, b, c\}$, $(13, G_1, 1)$-OPD $= (X, A)$,
A: $[a, 5, 7, 0, 4, 3] + i$, $i \in Z_{10}$, $[1, c, 6, 0, 5, b] + i$, $i \in [0, 4]$, leave edges: ab, ac, bc.

$(13, G_1, 1)$-OCD $= (X, A \cup \{[0, 1, 2, a, b, c]\})$, repeat edges: $(0, 1, 2)$.

$(13, G_2, 1)$-OPD $= (X, A)$, A: $[a, 5, c, 0, 4, 3] + i$, $i \in Z_{10}$, $[5, 7, 3, 1, 6, b] + i$, $i \in [0, 3]$, $[9, 1, 2, 0, 5, b]$, leave edges: ab, ac, bc.

$(13, G_2, 1)$-OCD $= (X, A \cup \{[2, 3, 0, b, a, c], [9, 1, b, 5, 0, 2]\} \backslash \{[9, 1, 2, 0, 5, b]\})$, repeat edges: $(5, 2, 3)$.

$\underline{v = 14}$ On the set $X = Z_{12} \cup \{a, b\}$, $(14, G_1, 1)$-OPD $= (X, A)$, A: $[a, 5, 7, 0, 4, 3]$ (mod 12), $[5, 10, 3, 0, 6, b]$, $[11, 4, 9, 1, 7, b]$, $[1, 6, 11, 2, 8, b]$, $[7, 0, 5, 3, 9, b]$, $[3, 8, 1, 4, 10, b]$, $[9, 2, 7, 5, 11, b]$.

$(14, G_2, 1)$-OPD $= (X, A)$, A: $[a, 5, 2, 0, 4, 3]$ (mod 12), $[6, 11, 7, 2, 8, b] + i$, $i \in [0, 3]$, $[10, 3, 5, 0, 6, b]$, $[11, 4, 6, 1, 7, b]$. Their leave edge is ab. It follows from Theorem 4.1 that there exist $(14, G_i, 1)$-OCD for $i = 1, 2$.

$\underline{v = 17}$ On the set $X = Z_{15} \cup \{a, b\}$, $(17, G_1, 1)$-OPD $= (X, A)$, A: $[a, 0, 5, 1, 8, 14]$ (mod 15), $[1, b, 8, 0, 4, 3] + i$, $i \in [0, 6]$, $[b, 0, 12, 7, 11, 10]$, $[13, 2, 14, 8, 12, 11]$, $[14, 3, 2, 9, 13, 12]$, $[12, 1, 0, 10, 14, 13]$, $[2, 1, 13, 11, 0, 14]$.

$(17, G_2, 1)$-OPD $= (X, A)$, A: $[a, 2, b, 0, 4, 3]$ (mod 15), $[12, 7, 5, 0, 6, 13] + i$, $i \in [0, 6]$, $[1, 8, 9, 7, 13, 5]$, $[0, 7, 10, 8, 14, 6]$, $[13, 4, 0, 9, 11, 2]$, $[4, 11, 1, 10, 12, 3]$, $[11, 13, 4, 14, 5, 12]$.

Their leave edge is ab. It follows from Theorem 4.1 that there exist $(17, G_i, 1)$-OCD for $i = 1, 2$.

$\underline{v = 18}$ On the set $X = Z_{15} \cup \{a, b, c\}$, $(18, G_1, 1)$-OPD $= (X, A)$, A: $[a, 1, 6, 0, 4, 3]$ (mod 15), $[b, 1, c, 0, 7, 13]$ (mod 15), leave edges: ab, ac, bc.

$(18, G_1, 1)$-OCD $= (X, A \cup \{[0, 1, 2, a, b, c]\})$, repeat edges: $(0, 1, 2)$.

$(18, G_2, 1)$-OPD $= (X, A)$, A: $[a, 1, 5, 0, 4, 3]$ (mod 15), $[b, 1, c, 0, 7, 13]$ (mod 15), leave edges: ab, ac, bc.

$(18, G_2, 1)$-OCD $= (X, A \cup \{[1, 2, c, 0, 7, 13], [2, 3, 1, b, a, c]\} \backslash \{[b, 1, c, 0, 7, 13]\})$, repeat edges: $(1,2,3)$.

<u>$v=19$</u> On the set $X=Z_{17}\cup\{a,b\}$, $(19,G_1,1)$-OPD $=(X,A)$, A: $[b,1,7,0,4,3]$ (mod 17), $[a,1,6,0,7,15]$ (mod 17), leave edge: ab.

$(19,G_2,1)$-OPD $=(X,A)$, A: $[b,1,6,0,4,3]$ (mod 17), $[a,1,5,0,7,15]$ (mod 17), leave edge: ab.

It follows from Theorem 4.1 that there exist $(19,G_i,1)$-OCD for $i=1,2$.

<u>$v=22$</u> On the set $X=Z_{22}$, $(22,G_1,1)$-OPD $=(X,A)$, A: $[12,4,13,0,1,6]$ (mod 22), $[4,6,17,0,3,10]+i$, $i\in[0,10]$, $[2,6,10,11,14,21]+i$, $i\in[0,3]$, $[10,14,18,19,0,7]+i$, $i\in[0,2]$, $[21,19,1,15,18,3]+i$, $i\in[0,2]$, $[15,17,21,0,2,4]$, $[16,18,20,1,3,5]$, $[13,17,19,18,21,6]$, leave edge: $(18,0)$.

$(22,G_2,1)$-OPD $=(X,A)$, A: $[12,4,9,0,1,6]$ (mod 22), $[4,6,11,0,3,10]+i$, $i\in[0,10]$, $[18,20,7,11,14,21]+i$, $i\in[0,3]$, $[4,8,11,15,18,3]+i$, $i\in[0,2]$, $[15,17,0,18,21,6]+i$, $i\in[0,2]$, $[15,19,6,2,4,0]+i$, $i\in[0,1]$, $[18,14,3,21,2,9]$, leave edge: $(17,21)$.

It follows from Theorem 4.1 that there exist $(22,G_i,1)$-OCD for $i=1,2$.

<u>$v=23$</u> On the set $X=Z_{23}$, $(23,G_1,1)$-OPD $=(X,A)$, A: $[12,4,13,0,1,6]$ (mod 23), $[19,8,10,1,4,11]+i$, $i\in[0,14]$, $[2,6,10,16,19,3]+i$, $i\in[0,1]$, $[1,12,8,18,21,5]+i$, $i\in[0,5]$, $[15,19,0,4,6,8]+i$, $i\in[0,1]$, $[18,22,3,0,2,4]$, $[17,21,2,1,3,5]$, leave edges: $(11,0),(7,18,14)$.

$(23,G_1,1)$-OCD $=(X,A\cup\{[11,0,1,7,18,14]\})$, repeat edges: $(0,1),(7,14)$.

$(23,G_2,1)$-OPD $=(X,A)$, A: $[12,4,9,0,1,6]$ (mod 23), $[8,12,11,0,3,10]+i$, $i\in[0,14]$, $[8,10,3,15,18,2]+i$, $i\in[0,7]$, $[16,18,6,2,4,0]+i$, $i\in[0,1]$, $[18,20,10,6,8,4]+i$, $i\in[0,1]$, leave edges: $(21,0),(20,22,1)$.

$(23,G_2,1)$-OCD $=(X,A\cup\{[21,0,2,22,20,1]\})$, repeat edges: $(1,20),(2,22)$.

<u>$v=24$</u> On the set $X=(Z_{11}\times Z_2)\cup\{a,b\}$, $(24,G_1,1)$-OPD $=(X,A)$, $A=\{[4_0,0_1,2_1,0_0,1_1,2_0],[a,0_1,4_1,0_0,2_1,3_1],[a,2_0,6_0,0_0,5_1,1_0],[b,1_0,6_0,0_1,5_0,2_0],[b,5_1,0_1,6_0,3_1,6_1]$ (mod $(11,-)$)$\}$, leave edge: ab.

$(24,G_2,1)$-OPD $=(X,A)$, $A=[4_0,0_1,3_1,1_1,2_0,0_0],[a,0_1,7_1,3_1,0_0,2_1],[a,2_0,4_0,0_0,5_1,1_0],[b,1_0,0_0,5_0,2_0,0_1],[b,5_1,1_1,6_1,6_0,3_1]$, (mod $(11,-)$)$\}$, leave edge: ab. It follows from Theorem 4.1 that there exist $(24,G_i,1)$-OCD for $i=1,2$.

<u>$v=27$</u> On the set $X=Z_{27}$, $(27,G_1,1)$-OPD $=(X,A)$, A: $[1,12,24,0,6,13]$ and $[2,10,20,0,4,9]$ (mod 27), $[16,17,19,0,1,3]+i$, $i\in[0,10]$, $[16,19,22,11,12,14]+i$, $i\in[0,2]$, $[1,25,22,14,15,17]+i$, $i\in[0,1]$, leave edge: $(0,24)$.

$(27,G_2,1)$-OPD $=(X,A)$, A: $[1,7,10,0,4,9]$ (mod 27), $[25,26,2,0,7,19]+i$, $i\in[11,26]$, $[15,1,2,0,7,19]+i$, $i\in[0,10]\backslash\{0,4\}$, $[12,15,11,0,13,24]+i$, $i\in[0,11]$, $[11,14,23,12,25,9]$, $[14,0,12,26,10,13]$, $[15,1,26,0,7,19]$, $[25,26,3,2,0,1]$, $[19,5,3,4,11,23]$, $[8,9,7,6,4,5]$, leave edge: $(7,8)$. It follows from Theorem 4.1 that there exist $(24,G_i,1)$-OCD for $i=1,2$.

<u>$v=28$</u> On the set $X=Z_{25}\cup\{a,b,c\}$, $(28,G_1,1)$-OPD $=(X,A)$ and $(28,G_2,1)$-OPD $=(X,A')$, $A=A'$: $[6,12,a,0,8,18]$, $[2,13,b,0,1,3]$ and $[1,13,c,0,4,9]$ (mod 25), leave edge: (a,b,c,a).

$\underline{v = 29}$ On the set $X = Z_{27} \cup \{a, b\}$, $(29, G_1, 1)$-OPD $= (X, A)$, A: $[1, 9, 19, 0, 6, 13]$, $[2, 13, 25, 0, 1, 3]$ and $[a, 2, b, 0, 4, 9]$ (mod 27), leave edge: ab.

$(29, G_2, 1)$-OPD $= (X, A)$, A: $[1, 9, 10, 0, 6, 13]$, $[2, 13, 12, 0, 1, 3]$ and $[a, 2, b, 0, 4, 9]$ (mod 27), leave edge: ab. It follows from Theorem 4.1 that there exist $(29, G_i, 1)$-OCD for $i = 1, 2$.

By Theorem 2.3, Theorem 3.2 and Lemma 4.3, we find that the theorem is true. □

Theorem 4.5 *There exist (v, G_i, λ)-OPD(OCD) for $i = 1, 2$, except for $(v, i) = (7, 1)$.*

Proof Set $A = \{[5, 3, 6, 0, 1, 4], [3, 4, 5, 6, 0, 2], [0, 5, 4, 1, 2, 3], [2, 4, 0, 1, 5, 6], [2, 6, 5, 0, 1, 3], [1, 6, 4, 2, 3, 5]\}$, $B = \{[6, 4, 3, 2, 5, 0], [0, 6, 3, 1, 2, 4]\}$, $C = \{[3, 6, 4, 2, 5, 0], [6, 3, 0, 1, 2, 4], [2, 5, 3, 0, 1, 4], [0, 6, 4, 1, 2, 3], [4, 5, 1, 6, 0, 2], [0, 3, 4, 1, 5, 6]\}$. It is easy to verify that $(Z_7, A \cup B)$ is a $(7, G_1, 2)$-OPD (leave edges: $(1, 5), (0, 3)$), and $(Z_7, A \cup C)$ is a $(7, G_1, 3)$-OPD (leave edges: $(0, 5), (3, 4, 2)$), and $(Z_7, A \cup C \cup \{[5, 0, 1, 3, 4, 2]\})$ is a $(7, G_1, 3)$-OCD (repeat edges: $(0, 1), (2, 3)$).

By Theorems 2.4–2.5 and following table, we find that $(7, G_1, \lambda)$-OPD(OCD) exists for $\lambda > 1$.

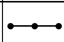
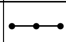
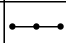
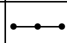
Table B

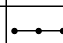
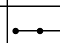
λ	1	2	3	4
L_λ	G_7	$P_2 \cup P_2$	$P_3 \cup P_2$	$P_3 \cup P_3$
R_λ	$G_7 \cup P_2 \cup P_3$	$P_2 \cup P_3$	$P_2 \cup P_2$	P_2

When $v \equiv 2, 4 \pmod 5$, the leave edge number is 1, and by Theorem 4.1 we know the theorem is true. When $v \equiv 3 \pmod 5$, we have $l_1 = 1$, $bar\lambda = 5$. By Theorems 2.4–2.5, we can list the following table to obtain (v, G_i, λ)-OPD(OCD) for $i = 1, 2$ and $\lambda > 1$.

For G_1:

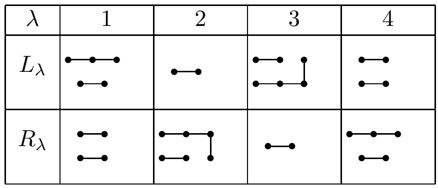
Table C

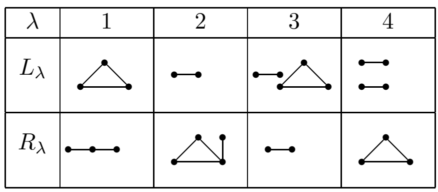
λ	1	2	3	4
L_λ				
R_λ				

λ	1	2	3	4
L_λ				
R_λ				

For G_2:

Table D





Theorem 4.6 *Let $l_1 = e(G)/2$ be an integer.*
(1) If there exist $(v, G, 1)$-OPD $= (X, \mathcal{A})$ and $(v, G, 1)$-OCD $= (X, \mathcal{B})$, and $L_1(\mathcal{A}) \cong R_1(\mathcal{B})$, then there exist (v, G, λ)-OPD(OCD) for any positive integer λ.
(2) If there exist two $(v, G, 1)$-OPD and their leave edge graphs are L_1 and L_1', then when $L_1 \cup L_1' = G$, there is a (v, G, λ)-OPD(OCD) for any positive integer λ.
(3) If (v, G, λ)-OPD exists for $\lambda = 1, 2$, and $L_1 \subset G$, then (v, G, λ)-OPD(OCD) exists for any positive integer λ.

Proof (1) When $\lambda = 1$, this is well-known. When $\lambda = 2$, we can construct an isomorphic mapping, which transforms $\mathcal{B}$ to $\mathcal{B}'$, and $R_1(\mathcal{B}) \cong R_1(\mathcal{B}')$ and $L_1(\mathcal{A}) = R_1(\mathcal{B}')$ are satisfied. We take $(X, \mathcal{A})$ and $(X, \mathcal{B}')$; then $(X, \mathcal{A} \cup \mathcal{B}')$ is a $(v, G, 2)$-GD. It follows from Theorem 2.4 that there exist (v, G, λ)-OPD(OCD) for any positive integer λ.

(2) Let a $(v, G, 1)$-OPD be $(X, \mathcal{B})$ and another be $(X, \mathcal{B}')$. We can construct an isomorphic mapping, which transforms $\mathcal{B}'$ to $\mathcal{B}''$, and $L_1 \cup L_1'' = G$ and $V(L_1 \cup L_1'') = V(G)$ are satisfied. If a block of the graph G is denoted by $[a, b, c, d, e, f]$, then $(X, \mathcal{B} \cup \mathcal{B}'' \cup \{[a, b, c, d, e, f]\})$ is a $(v, G, 2)$-GD. Since $L_1 \cup L_1' = G$, $L_1 \subset G$. It follows from Theorem 4.2 that a $(v, G, 1)$-OCD exists. By Theorem 2.4 we find that there exists a (v, G, λ)-OPD(OCD) for any positive integer λ.

(3) This part of the theorem is also true. □

Example On the set $X = (Z_3 \times Z_2) \cup \{a\}$, $(7, G_4, 1)$-OPD $= (X, A)$,
A: $[a, 0_1, 1_1, 0_0, 1_0, 2_1]$ mod $(3, -)$. Leave edges: $a0_0, a1_0, a2_0$.

$(X, A \cup \{[0_0, 1_1, 0_0, a, 1_0, 2_0]\})$ is a $(7, G_4, 1)$-OCD. We construct an isomorphic mapping f satisfying $1_0 \longmapsto 0_0$, $2_0 \longmapsto a$, $a \longmapsto 1_1$, $0_0 \longmapsto 0_1$. It is easy to see that $L_1 \cong R_1$. By the above theorem we find that a $(7, G_4, \lambda)$-OPD(OCD) exists for any positive integer λ.

We construct a $(7, G_8, 1)$-OPD $= (X, B)$ as follows: $[0_1, 1_1, 0_0, 1_0, 2_1, a]$ mod $(3, -)$. Leave edges: $a0_1, a1_1, a2_1$; and again construct $(7, G_8, 1)$-OPD $= (X, B')$ as follows: Replace the first block in B by $[a, 0_1, 0_0, 1_0, 1_1, 2_1]$; leave edges: $(2_1, a, 1_1, 0_1)$. We construct an isomorphic mapping f satisfying $2_1 \longmapsto 2_1$, $a \longmapsto 0_0$, $1_1 \longmapsto a$, $0_1 \longmapsto 1_0$. Thus $(X, B \cup f(B') \cup \{[0_0, 2_1, a, 0_1, 1_1, 1_0\})$ is a $(7, G_8, 2)$-GD. By the above theorem we find that a $(7, G_8, \lambda)$-OPD(OCD) exists for any positive integer λ.

On the set Z_7, let $A = \{[6,3,0,1,4,2],[6,4,1,2,5,0],[6,5,2,0,3,1]\}$; then (Z_7, A) is a $(7, G_7, 1)$-OPD, with leave edges: $(0,6,1),(6,2)$. $(Z_7, A \cup \{[5,0,6,2,1,4]\})$ is a $(7, G_7, 1)$-OCD, with repeated edges: $(5,0,1,4)$. B: $[2,0,6,5,3,1]$ (mod 7); then (Z_7, B) is a $(7, G_7, 2)$-GD. Therefore a $(7, G_7, \lambda)$-OPD(OCD) exists for any positive integer λ.

In the same way, we can obtain the following theorem:

Theorem 4.7 *There exists a (v, G_i, λ)-OPD(OCD) for $i \in [3, 9]$.*

5 Graph designs for $\lambda \geq 1$

Lemma 5.1 *The necessary conditions for a (v, G, λ)-GD to exist are*

(1) $\lambda v(v-1) \equiv 0 \pmod{2e(G)}$;

(2) $\lambda(v-1) \equiv 0 \pmod{n}$, where $n = \gcd(\{d(u) | u \in V(G)\})$.

By Corollary 2.6, Section 4 and Table A, we easily get following theorem:

Theorem 5.2 *If v satisfies the conditions in Lemma 5.1 and $v > 6$, then there exists a (v, G_i, λ)-GD for $i \in [1, 9]$ and $\lambda \geq 1$.*

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Appendix

Construction of $(10+w, w, G)$-IGD for $w \in \{4, 7, 8, 9\}$:

Let $X = Z_{10} \cup \{a_1, a_2, \cdots, a_w\}$ and $(10+w, w, G)$-IGD $= (X, \mathcal{A})$. We construct $\mathcal{A}$ as follows:

$\boxed{w=4}$ For G_1: $[a, 2, b, 0, 1, 4] + i$, $i \in Z_{10}$, $[4, d, 7, 0, 5, c] + i$, $i \in [0, 2]$, $[7, 9, 1, 3, 8, c]$, $[6, 8, 0, 4, 9, c]$, $[3, 5, 7, 0, 2, d]$, $[2, 4, 6, 1, 3, d]$.

For G_2: $[2, a_1, 3, 0, 1, y] + i$, $i \in Z_{10}$, when i is even $y = a_2$; when i is odd $y = a_3$; $[6, 8, 3, 7, a_4, 2] + i$, $i \in [0, 4]$, $[0, 4, 9, 3, 5, 1]$, $[5, 7, 8, 2, 4, 6]$.

$\boxed{w=7}$ For G_1: $[a_1, 3, a_2, 0, 4, a_5] + i$, $i \in [0, 3]$, $[a_1, 7, a_2, 4, 8, a_6] + i$, $i \in [0, 3]$, $[a_3, 2, a_4, 0,$
$5, a_7]+i$, $i \in [0, 4]$, $[a_3, 7, a_4, 3, 4, 6]+i$, $i \in [0, 4]$, $[9, 1, 8, 2, 3, a_6]$, $[a_1, 2, a_2, 8, 9, a_5]$, $[a_1, 1, a_2, 9, 0, 2]$, $[9, 3, 5, 1, 2, 4]$, $[8, 2, 5, 0, 1, 3]$.

For G_2: $[2, a_1, a_2, 0, 1, a_3]+2i$, $i \in [0, 4]$, $[3, a_1, a_2, 1, 2, a_4]+2i$, $i \in [0, 4]$, $[1, 6, a_5, 0, 3, a_6]$ $+2i$, $i \in [0, 4]$, $[6, 8, a_5, 1, 4, a_7]+2i$, $i \in [0, 2]$, $[9, 3, a_5, 7, 0, a_7]$, $[4, 8, a_5, 9, 2, a_7]$, $[0, 4, 9, 1, 3, 5]$, $[3, 7, 8, 2, 4, 6]$, $[6, 0, 1, 7, 9, 5]$.

$\boxed{w=8}$ For G_1: $[2,x,7,0,1,y]+i,\ i\in Z_{10}$, $[1,5,7,0,3,z]+i,\ i\in Z_{10}$, when i is even $x=a_1, y=a_2, z=a_3$; when i is odd $x=a_4, y=a_5, z=a_6$; $[1,a_7,6,0,5,a_8]+i,\ i\in[0,4]$.

For G_2: $[2,a_1,a_2,0,1,y]+i,\ i\in Z_{10}$, $[1,5,a_5,0,3,z]+i,\ i\in Z_{10}$, when i is even $y=a_3, z=a_6$; when i is odd $y=a_4, z=a_7$; $[0,2,8,6,a_8,1]+i,\ i\in[0,3]$, $[4,6,7,5,a_8,0]$.

$\boxed{w=9}$ For G_1: $[2,x,7,0,1,y]+i,\ i\in Z_{10}$, $[1,t,6,0,3,z]+i,\ i\in Z_{10}$, when i is even $x=a_1, y=a_2, z=a_3, t=a_4$; when i is odd $x=a_5, y=a_6, z=a_7, t=a_8$; $[6,4,8,0,5,a_9]+i,\ i\in[0,2]$, $[7,1,9,3,8,a_9]$, $[0,8,2,4,9,a_9]$, $[4,2,6,1,3,5]$, $[2,0,4,3,7,9]$.

For G_2: $[2,a_1,a_2,0,1,y]+i,\ i\in Z_{10}$, $[1,a_5,a_6,0,3,z]+i,\ i\in Z_{10}$, when i is even $y=a_3, z=a_7$; when i is odd $y=a_4, z=a_8$; $[4,8,7,5,a_9,0]+i,\ i\in[0,4]$, $[0,4,7,3,5,1]$, $[9,3,0,2,4,6]$.

Construction of $(12+w,w,G)$-IGD for $w\in\{3,5,6,7,8,10,11\}$:

Let $X=Z_{12}\cup\{a_1,a_2,\cdots,a_w\}$ and $(12+w,w,G)$-IGD $=(X,\mathcal{A})$. We construct $\mathcal{A}$ as follows:

$\boxed{w=3}$ For G_3: $[5,x,2,6,a_3,7]+i,\ i\in Z_{12}\backslash[0,1]$, when i is even $x=a_1$; when i is odd $x=a_2$; $[5,a_1,2,7,9,6]$, $[6,a_2,3,8,2,7]$, $[5,4,6,0,10,a_3]$, $[8,6,7,5,3,a_3]$, $[2,0,1,11,5,7]$, $[4,2,3,9,11,1]$, $[9,8,10,11,0,4]$.

For G_4: $[a_3,11,x,2,7,10]+i,\ i\in Z_{12}\backslash[0,1]$, when i is even $x=a_1$; when i is odd $x=a_2$; $[7,2,a_1,11,10,a_3]$, $[8,3,a_2,0,11,a_3]$, $[8,2,4,3,9,11]$, $[2,10,8,9,7,11]$, $[0,6,8,7,1,5]$, $[10,0,2,1,3,11]$, $[10,4,6,5,3,11]$.

For G_5: $[6,x,3,11,4,2]+i,\ i\in Z_{12}^*$, when i is even $x=a_1$; when i is odd $x=a_2$; $[6,a_1,3,2,1,0]$, $[0,a_3,6,7,8,9]$, $[1,7,a_3,11,10,9]$, $[9,a_3,3,4,5,6]$, $[4,10,a_3,5,11,0]$, $[8,a_3,2,4,11,3]$.

For G_6: $[1,5,10,x,7,9]+i,\ i\in Z_{12}^*$, when i is even $x=a_1$; when i is odd $x=a_2$; $[0,11,10,a_1,7,9]$, $[2,1,0,a_3,6,7]$, $[10,5,1,a_3,7,8]$, $[4,3,2,a_3,8,9]$, $[6,5,4,10,a_3,11]$, $[11,5,a_3,3,9,10]$.

For G_7: $[a_1,1,6,a_2,3,a_3]+i,\ i\in Z_{12}\backslash[0,2]$, $[3,a_1,1,0,2,4]$, $[8,a_2,6,3,7,2]$, $[5,a_3,3,1,4,10]$, $[2,3,8,7,5,11]$, $[6,0,8,9,4,5]+i,\ i=0,1$, $[8,2,10,11,6,1]$, $[9,3,11,0,7,4]$.

For G_8: $[3,x,6,4,11,a_3]+i,\ i\in Z_{12}^*$, when i is even $x=a_1$; when i is odd $x=a_2$; $[6,a_1,3,2,4,9]$, $[0,4,8,2,7,9]$, $[3,7,11,0,5,6]$, $[2,6,10,4,9,11]$, $[4,5,6,0,7,a_3]$, $[9,5,1,0,2,7]$.

For G_9: $[1,x,4,6,11,a_3]+i,\ i\in Z_{12}^*$, when i is even $x=a_1$; when i is odd $x=a_2$; $[1,a_1,4,6,7,a_3]$, $[0,6,11,10,4,9]$, $[8,4,0,1,2,7]$, $[1,5,9,3,2,4]$, $[2,10,6,5,4,11]$, $[3,11,7,8,9,2]$.

$\boxed{w=5}$ For G_3: $[x,0,1,y,2,a_1]+i,\ i\in Z_{12}$, when i is even $x=a_2,\ y=a_3$; when i is odd $x=a_4,\ y=a_5$; $[4,6,0,5,10,3]+i,\ i\in[0,4]$, $[3,11,5,9,6,8]$, $[1,3,10,0,9,7]$, $[11,4,2,10,6,0]$, $[1,9,11,8,0,7]$.

For G_4: $[a_1,0,x,1,a_2,a_3]+i,\ i\in Z_{12}$, when i is even $x=a_4$; when i is odd $x=a_5$; $[11,6,4,0,3,5]+i,\ i\in[0,3]$, $[7,4,8,10,3,6]$, $[7,10,2,0,8,9]$, $[5,3,11,1,9,10]$, $[6,9,11,5,8,10]$, $[9,4,2,11,7,8]$.

For G_5: $[x,0,1,y,2,a_1]+i,\ i\in Z_{12}$, when i is even $x=a_2, y=a_3$; when i is odd $x=a_4, y=a_5$; $[0,2,5,1,7,11]+i,\ i\in[0,4]$, $[5,7,10,6,8,11]$, $[2,9,11,6,0,4]$, $[9,7,0,3,10,1]$, $[11,4,1,8,10,0]$.

For G_6: $[2,x,1,y,0,a_1]+i,\ i\in Z_{12}$, when i is even $x=a_2, y=a_3$; when i is odd $x=a_4, y=a_5$; $[8,5,0,4,6,10]+i,\ i\in[0,4]$, $[1,10,5,11,9,2]$, $[4,7,0,10,3,1]$, $[5,2,4,11,1,8]$, $[0,2,11,6,3,5]$.

For G_7: $[a_1, 0, 4, a_2, x, 8] + i,\ i \in Z_{12}$, when $i \in [0, 3], x = a_3$, when $i \in [4, 7], x = a_4$, when $i \in [8, 11], x = a_5$; $[11, 6, 5, 7, 0, 9] + i,\ i = 0, 1, 3$, $[11, 8, 7, 9, 2, 4]$, $[0, 10, 9, 6, 4, 7]$, $[4, 1, 0, 2, 11, 9]$, $[8, 1, 3, 6, 2, 11]$, $[10, 3, 5, 8, 4, 6]$, $[4, 11, 10, 7, 5, 2]$.

For G_8: $[x, 0, 1, a_1, a_2, a_3] + i,\ i \in Z_{12}$, when i is even $x = a_4$; when i is odd $x = a_5$; $[1, 5, 7, 10, 11, 0] + i,\ i \in [0, 3]$, $[6, 4, 0, 2, 3, 10]$, $[5, 9, 11, 1, 2, 3]$, $[3, 8, 5, 0, 2, 10]$, $[2, 7, 4, 1, 9, 11]$, $[1, 3, 6, 9, 10, 11]$.

For G_9: $[1, x, 0, 4, a_1, a_2] + i,\ i \in Z_{12}$, when i is even $x = a_3$; when i is odd $x = a_4$; $[a_5, 6, 0, 5, 2, 3] + i,\ i \in [0, 5]\backslash[3, 4]$, $[3, a_5, 9, 0, 2, 7] + i,\ i \in [0, 1]$, $[1, 4, 11, 9, 6, 7]$, $[10, 0, 3, 8, 5, 6]$, $[9, 4, 2, 11, 6, 8]$.

$\boxed{w = 6}$ For G_3: $[x, 0, 1, y, 2, 5] + i,\ i \in Z_{12}$, when i is even $x = a_1,\ y = a_3$; when i is odd $x = a_2,\ y = a_4$; $[a_5, 6, 0, 3, 8, 2] + i,\ i \in [0, 5]$, $[8, 6, a_6, 7, 10, 1]$, $[a_6, 5, 0, 9, 6, 10]$, $[a_6, 2, 9, 11, 8, 7]$, $[a_6, 3, 10, 1, 6, 8]$, $[a_6, 4, 11, 2, 7, 1]$.

For G_4: $[a_1, 0, x, 1, 5, a_2] + i,\ i \in Z_{12}$, when i is even $x = a_3$; when i is odd $x = a_4$; $[11, 6, a_5, 0, 2, 5] + i,\ i \in [0, 5]$, $[1, 4, a_6, 7, 9, 10]$, $[2, 5, a_6, 8, 6, 11]$, $[0, 9, a_6, 11, 1, 2]$, $[3, 0, a_6, 10, 1, 8]$, $[9, 6, 3, a_6, 1, 2]$.

For G_5: $[1, x, 0, 4, y, 5] + i,\ i \in Z_{12}$, when i is even $x = a_1,\ y = a_3$; when i is odd $x = a_2,\ y = a_4$; $[6, a_5, 0, 2, 7, 4] + i,\ i \in [0, 5]$, $[2, a_6, 5, 0, 3, 10]$, $[3, a_6, 6, 8, 1, 4]$, $[10, 0, a_6, 11, 1, 6]$, $[7, 9, a_6, 8, 10, 1]$, $[2, 9, 11, 4, a_6, 1]$.

For G_6: $[2, x, 1, y, 0, 4] + i,\ i \in Z_{12}$, when i is even $x = a_1,\ y = a_3$; when i is odd $x = a_2,\ y = a_4$; $[5, 3, 0, a_5, 6, 1] + i,\ i \in [0, 5]$, $[1, 11, 9, 0, 2, a_6]$, $[1, 10, 7, 0, a_6, 9]$, $[2, 11, 8, a_6, 1, 3]$, $[0, 5, a_6, 11, 4, 2]$, $[9, 6, a_6, 3, 10, 0]$.

For G_7: $[a_1, 1, 0, a_2, 3, a_3] + i,\ i \in Z_{12}$, $[a_6, 4, a_5, 0, 8, 2] + i,\ i \in [0, 1]$, $[7, 0, a_4, 8, 4, 11] + i,\ i \in [0, 2]$, $[a_6, 6, a_5, 3, 10, 4]$, $[3, 7, a_5, 2, 11, 5]$, $[7, a_4, 3, 10, 11, 6]$, $[6, 0, a_6, 7, 8, 3]$, $[4, 9, a_6, 3, 1, 7]$, $[5, 10, a_6, 11, 2, 7]$.

For G_8: $[x, 0, 1, a_1, a_2, a_3] + i,\ i \in Z_{12}$, when i is even $x = a_4$; when i is odd $x = a_5$; $[6, a_6, 0, 3, 4, 5] + i,\ i = 2, 4, 5$, $[9, a_6, 3, 5, 7, 8]$, $[7, a_6, 1, 3, 4, 5]$, $[11, 6, 9, 0, 1, 2] + i, i = 0, 2$, $[10, 0, 7, 5, 9, 11]$, $[6, 8, 10, 1, 2, 3]$, $[4, 2, 0, 3, 5, 8]$, $[0, a_6, 6, 1, 3, 4]$.

For G_9: $[1, x, 0, 4, a_1, a_2] + i,\ i \in Z_{12}$, when i is even $x = a_3$; when i is odd $x = a_4$; $[a_5, 7, 1, 6, 3, 4] + i,\ i \in [0, 4]$, $[a_5, 0, 6, a_6, 5, 7]$, $[2, a_6, 9, 11, 1, 6]$, $[3, a_6, 10, 0, 7, 9]$, $[11, a_6, 8, 1, 3, 10]$, $[1, 4, a_6, 0, 2, 3]$, $[4, 11, 2, 5, 0, 3]$.

$\boxed{w = 7}$ For G_3: $[x, 0, 1, y, 2, 5] + i,\ i \in Z_{12}$, $[z, 5, 0, 2, a_7, t] + i,\ (i, t) \in \{(1, 7), (3, 9), (4, 10), (5, 11)\} \cup \{(6, 9) + j |\ j \in [0, 5]\}$, when i is even $x = a_1,\ y = a_3,\ z = a_5$; when i is odd $x = a_2,\ y = a_4,\ z = a_6$; $[2, a_7, 4, 7, a_5, 1]$, $[3, 6, 0, 5, a_5, 2]$, $[5, 8, 2, a_5, 0, 7]$.

For G_4: $[a_1, 0, x, 5, a_2, a_3] + i,\ i \in Z_{12}$, when i is even $x = a_4$; when i is odd $x = a_5$; $[a_7, 3, 6, 2, 8, a_6] + i,\ i \in [0, 3]$, $[a_7, 7, 10, 6, 8, a_6] + i,\ i \in [0, 5]$, $[6, 4, 0, 2, 5, a_7]$, $[7, 5, 3, 1, 2, 4]$, $[6, 0, a_6, 1, 7, a_7]$.

For G_5: $[0, x, 1, y, 2, a_5] + i,\ i \in Z_{12}$, when i is even $x = a_1,\ y = a_3$; when i is odd $x = a_2,\ y = a_4$; $[8, a_6, 2, 7, 5, 9] + i,\ i \in [0, 2]$, $[6, 11, 9, 1, a_7, 7] + i,\ i \in [0, 4]$, $[0, 6, a_6, 1, 7, 4]$, $[1, 4, 6, 3, 7, a_6]$, $[3, 5, 0, a_7, 6, 2]$, $[11, a_6, 5, 10, 8, 4]$, $[11, 4, 2, 5, 8, 0]$.

For G_6: $[1, x, 2, y, 9, 5] + i,\ i \in Z_{12}^*$, $[7, 1, 3, z, 0, a_7] + i,\ i \in [0, 5]\backslash\{2\}$, $[8, 7, 9, z, 6, a_7] + i,\ i \in [0, 5]$, when i is even $x = a_1,\ y = a_2,\ z = a_5$; when i is odd $x = a_3,\ y = a_4,\ z = a_6$; $[1, a_1, 2, a_2, 9, 3]$, $[7, 6, 5, a_5, 2, a_7]$, $[1, 2, 3, 4, 5, 9]$.

For G_7: $[a_1, 0, x, 8, 4, a_2] + i$, when $i \in [0, 3],\ x = a_3$; when $i \in [4, 7],\ x = a_4$; when $i \in [8, 11],\ x = a_5$; $[a_7, 0, 2, a_6, 5, 11] + i,\ i \in [0, 5]$, $[11, 0, 9, 8, a_6, 1]$, $[7, 8, a_6, 10, 11, 6]$, $[5, 6, a_7, 8, 7, 9]$; $[6, 8, 10, 9, 1, 2]$; $[a_7, 9, 11, 10, 2, 3]$, $[a_7, 10, 0, 7, 3, 4]$, $[a_7, 11, 1, 0, 4, 5]$.

For G_8: $[0,x,5,a_1,a_2,a_3]+i,\ i\in Z_{12}$, when i is even $x=a_4$; when i is odd $x=a_5$; $[5,a_7,6,8,9,10]+i,\ i\in[0,5]$, $[9,a_6,3,5,6,7]+i,\ i\in[0,2]$, $[2,3,0,1,4,11]$, $[4,5,2,1,8,6]$, $[3,4,1,a_6,7,5]$, $[0,6,a_6,2,7,8]$.

For G_9: $[x,7,2,6,a_1,a_2]+i,\ i\in Z_{12}^*$, $[3,y,0,2,a_7,8]+i,\ i\in[0,5]\backslash\{3\}$, $[9,y,6,8,a_7,7]+i,\ i\in[0,5]$, when i is even $x=a_3$, $y=a_5$; when i is odd $x=a_4$, $y=a_6$; $[a_3,2,7,6,a_2,a_1]$, $[3,a_6,6,5,11,a_7]$, $[5,4,3,2,1,6]$.

$\boxed{w=8}$ For G_3: $[x,1,0,y,11,2]+i,\ i\in Z_{12}$, $[z,8,3,t,0,7]+i,\ i\in Z_{12}\backslash\{0,1,2,4,5,6\}$, $[z,8,3,9,6,7]+i,\ i\in[0,2]$, $[z,0,7,t,3,11]+i,\ i\in[0,2]$, $[6,3,0,t,4,9]+i,\ i\in[0,2]$, when i is even $x=a_1,\ y=a_2,\ z=a_5,\ t=a_6$; when i is odd $x=a_3,\ y=a_4,\ z=a_7,\ t=a_8$.

For G_4: $[a_1,3,x,4,6,a_2]+i,\ i\in Z_{12}\backslash[0,2]$, $[8,0,y,5,a_5,a_6]+i,\ i\in Z_{12}\backslash[0,2]$, $[8,0,3,6,4,9]+i,\ i\in[0,2]$, $[a_2,4,x,3,a_1,9]+i,\ i\in[0,2]$, $[9,0,y,5,a_5,a_6]+i,\ i\in[0,2]$, when i is even $x=a_3,\ y=a_7$; when i is odd $x=a_4,\ y=a_8$.

For G_5: $[x,1,0,2,y,3]+i,\ i\in Z_{12}$, $[z,11,4,0,t,9]+i,\ i\in Z_{12}\backslash\{0,1,2,4,5,6\}$, $[0,6,3,9,t,1]+i,\ i\in[0,2]$, $[11,z,4,0,9,6]+i,\ i\in[0,2]$, $[11,z,4,0,t,8]+i,\ i\in[4,6]$, when i is even $x=a_1,\ y=a_2,\ z=a_5,\ t=a_6$; when i is odd $x=a_3,\ y=a_4,\ z=a_7,\ t=a_8$.

For G_6: $[1,x,0,y,5,9]+i,\ i\in Z_{12}$, $[11,z,0,t,1,3]+i,\ i\in Z_{12}\backslash\{0,1,2\}$, $[11,z,0,6,3,9]+i,\ i\in[0,2]$, $[6,9,0,t,1,3]+i,\ i\in[0,2]$, when i is even $x=a_1,\ y=a_2,\ z=a_5,\ t=a_6$; when i is odd $x=a_3,\ y=a_4,\ z=a_7,\ t=a_8$.

For G_7: $[a_1,0,4,a_2,x,8]+i$, when $i\in[0,3]$, $x=a_3$; when $i\in[4,7]$, $x=a_4$; when $i\in[8,11]$, $x=a_5$; $[a_7,6,11,a_6,8,a_8]+i,\ i\in[0,5]\backslash\{1\}$, $[0,1,a_7,4,2,3]$, $[4,5,a_6,8,6,7]$, $[3,9,a_6,0,10,11]$; $[a_7,0,5,11,2,a_8]$, $[7,1,6,0,3,a_8]$, $[8,2,7,a_6,4,a_8]$, $[a_7,3,8,7,5,a_8]$, $[3,4,9,8,6,a_8]$, $[a_7,5,10,4,7,a_8]$, $[a_7,7,0,11,9,a_8]$.

For G_8: $[2,x,0,a_1,a_2,a_3]+i$, when $i=4j,4j+1,\ x=a_4$, when $i=4j+2,4j+3,\ x=a_5,\ j\in[0,2]$, $[1,9,2,a_6,a_7,a_8]+i,\ i\in Z_{12}\backslash\{0,1,2\}$, $[1,2,9,0,3,6]+i,\ i\in[0,2]$, $[6,0,3,a_6,a_7,a_8]+i,\ i\in[0,1]$, $[5,8,2,a_6,a_7,a_8]$.

For G_9: $[1,x,0,2,a_1,a_2]+i,\ i\in Z_{12}$, $[5,y,0,4,a_5,a_6]+i,\ i\in Z_{12}\backslash\{0,1,2\}$, $[5,y,0,9,3,6]+i,\ i\in[0,2]$, $[6,3,0,4,a_5,a_6]+i,\ i\in[0,2]$, when i is even $x=a_3,y=a_7$, when i is odd $x=a_4,y=a_8$.

$\boxed{w=10}$ For G_3: $[0,3,x,9,a_1,6]+i$, when $i\in[0,2]$, $x=a_2$, when $i\in[3,5]$, $x=a_3$, when $i\in[6,8]$, $x=a_4$, when $i\in[9,11]$, $x=a_5$; $[5,y,0,z,1,2]+i,\ i\in Z_{12}$, when i is even $y=a_6,\ z=a_7$, when i is odd $y=a_8,\ z=a_9$; $[3,7,11,10,6,0]$, $[11,a_{10},5,9,10,1]$, $[6,a_{10},0,8,7,4]$, $[7,a_{10},1,9,8,0]$, $[8,a_{10},2,6,7,1]$, $[9,a_{10},3,2,10,4]$, $[10,a_{10},4,5,6,8]$.

For G_4: $[a_1,0,3,x,9,6]+i$, when $i\in[0,2]$, $x=a_2$, when $i\in[3,5]$, $x=a_3$, when $i\in[6,8]$, $x=a_4$, when $i\in[9,11]$, $x=a_5$; $[10,6,y,1,a_6,a_7]+i,\ i\in Z_{12}^*$, when i is even $y=a_8$, when i is odd $y=a_9$; $[8,6,a_{10},0,1,2]+i,\ i=1,3,4$, $[7,8,9,10,11,6]$, $[5,6,a_8,1,a_6,a_7]$, $[7,6,a_{10},0,1,2]$, $[6,8,a_{10},2,3,4]$, $[7,5,a_{10},11,0,1]$.

For G_5: $[0,4,x,8,y,9]+i$, when $i\in[0,3]$, $x=a_1$, when $i\in[4,7]$, $x=a_2$, when $i\in[8,11]$, $x=a_3$; $[9,z,6,t,5,7]+i,\ i\in Z_{12}^*$, when i is even $y=a_4,\ z=a_6,\ t=a_7$, when i is odd $y=a_5,\ z=a_8,\ t=a_9$; $[6,a_{10},0,11,4,9]+i,\ i=0,1,2,4$, $[9,a_6,6,a_7,5,4]$, $[5,a_{10},11,10,3,2]$, $[5,6,7,8,9,10]$, $[3,a_{10},9,2,7,0]$.

For G_6: $[1,x,0,4,y,8]+i$, when $i\in[0,3]$, $y=a_3$, when $i\in[4,7]$, $y=a_4$, when $i\in[8,11]$, $y=a_5$; $[10,z,11,t,2,9]+i,\ i\in Z_{12}^*$, when i is even $x=a_1,\ z=a_6,\ t=a_7$, when i is odd $x=a_2,\ z=a_8,\ t=a_9$; $[3,2,0,a_{10},6,8]+i,\ i=0,1,2,4,5$, $[6,5,3,a_{10},9,8]$, $[10,a_6,11,a_7,2,1]$, $[1,0,11,10,9,2]$.

For G_7: $[a_1,0,y,8,4,a_2]+i$, when $i\in[0,3]$, $y=a_3$, when $i\in[4,7]$, $y=a_4$, when $i\in[8,11]$, $y=a_5$; $[a_6,0,2,a_7,11,a_8]+i,\ i\in Z_{12}\backslash[0,4]$, $[1,4,6,a_7,3,a_8]$, $[a_8,1,a_6,4,2,0]$,

$[a_8, 2, a_7, 4, 3, 5]$, $[a_8, 0, 3, a_6, 1, 7]$, $[a_6, 0, 5, a_7, 11, a_8]$, $[2, 5, a_9, 3, 10, a_{10}]$, $[9, 4, a_9, 6, 11, a_{10}]$, $[11, 2, a_9, 0, 7, a_{10}]$, $[7, 0, a_{10}, 1, 6, 11]$, $[10, 3, a_{10}, 8, 9, 2]$, $[8, 2, a_{10}, 5, 4, 10]$, $[3, 8, a_9, 9, 1, 6]$.

For G_8: $[0, x, 5, a_1, a_2, a_3] + i$, $i \in Z_{12}$, $[0, y, 3, a_8, a_9, a_{10}] + i$, $i \in Z_{12}$, when i is even, $x = a_4, y = a_6$, when i is odd, $x = a_5, y = a_7$, $[6, 4, 0, 8, 10, 11] + i$, $i = 0, 1$, $[8, 9, 10, 11, 4, 6]$, $[11, 9, 5, 3, 4, 6]$, $[1, 2, 3, 4, 9, 11]$, $[4, 8, 2, 0, 6, 10]$, $[6, 8, 7, 3, 9, 11]$.

For G_9: $[0, 4, x, 8, a_1, a_2] + i$, when $i \in [0, 3]$, $x = a_3$, when $i \in [4, 7]$, $x = a_4$, when $i \in [8, 11]$, $x = a_5$; $[3, y, 0, 5, a_7, a_6] + i$, $i \in Z_{12}$, when i is even, $y = a_8$, when i is odd, $y = a_9$, $[6, a_{10}, 0, 1, 2, 3] + 2i$, $i \in [1, 2]$, $[3, a_{10}, 9, 8, 7, 10]$, $[5, a_{10}, 11, 9, 7, 10]$, $[1, a_{10}, 7, 6, 4, 8]$, $[6, a_{10}, 0, 1, 11, 3]$, $[10, 11, 0, 2, 1, 4]$.

$\boxed{w = 11}$ For G_3: $[5, x, 0, y, 1, a_1] + i, i \in Z_{12}$, $[3, z, 0, t, 1, a_2] + i$, $i \in Z_{12}^*$, when i is even $x = a_3$, $y = a_4$, $z = a_5$, $t = a_6$, when i is odd $x = a_7$, $y = a_8$, $z = a_9$, $t = a_{10}$; $[6, a_{11}, 0, 4, 5, 2] + i$, $i \in [0, 5]$, $[3, a_5, 0, a_6, 1, 8]$, $[6, 8, 10, 0, a_2, 2]$, $[11, 0, 1, 2, 3, 9]$, $[7, 9, 11, 3, 4, 10]$.

For G_4: $[a_1, 0, x, 3, a_2, a_3] + i$, $i \in Z_{12}$, $[a_4, 0, y, 5, a_6, a_5] + i$, $i \in Z_{12}$, when i is even $x = a_7, y = a_8$, when i is odd $x = a_9, y = a_{10}$; $[5, 6, a_{11}, 0, 2, 4] + i$, $i = 0, 1, 2$, $[3, 11, 0, 1, 2, 9]$, $[0, 8, 6, 10, 9, 2]$, $[8, 9, 11, 7, 3, 5]$, $[5, 9, a_{11}, 3, 2, 4]$, $[10, 11, a_{11}, 5, 3, 4]$, $[0, 10, a_{11}, 4, 6, 8]$.

For G_5: $[5, x, 0, y, 1, a_1] + i$, $i \in Z_{12}$, $[3, z, 0, t, 1, a_2] + i$, $i \in Z_{12}$, when i is even $x = a_3, y = a_4, z = a_5, t = a_6$, when i is odd $x = a_7, y = a_8, z = a_9, t = a_{10}$; $[0, 4, 6, 5, a_{11}, 11] + i$, $i = 0, 2, 5$, $[7, 5, 1, 9, 8, 0]$, $[2, 10, 0, 11, 1, 3]$, $[3, 11, 7, 9, 10, 6]$, $[10, 4, 8, a_{11}, 6, 7]$, $[5, 3, 4, 2, a_{11}, 0]$, $[9, a_{11}, 3, 2, 1, 0]$.

For G_6: $[1, x, 0, y, 3, a_1] + i$, $i \in Z_{12}$, $[1, z, 0, t, 5, a_2] + i$, $i \in Z_{12}$, when i is even $x = a_3, y = a_4, z = a_5, t = a_6$, when i is odd $x = a_7, y = a_8, z = a_9, t = a_{10}$; $[2, 4, 0, a_{11}, 6, 10] + i$, $i = 1, 2, 3, 4$, $[6, 10, 0, 1, 2, 3]$, $[5, 4, 3, 1, 11, 0]$, $[6, 7, 8, 10, 9, 11]$, $[2, 4, 0, a_{11}, 6, 5]$, $[7, 9, 5, a_{11}, 11, 10]$.

For G_7: $[a_1, 0, x, 8, 4, a_2] + i$, when $i \in [0, 3]$, $x = a_3$, when $i \in [4, 7]$, $x = a_4$, when $i \in [8, 11]$, $x = a_5$; $[a_6, 0, 5, a_7, 2, a_8] + i$, $i \in Z_{12}$, $[a_{10}, 0, a_9, 5, 6, 7]$, $[a_{10}, 7, a_9, 9, 1, a_{11}]$, $[1, 2, a_9, 11, 8, 7]$, $[a_{11}, 2, a_{10}, 1, 3, a_9]$, $[a_9, 4, a_{10}, 8, 10, 11]$, $[a_{11}, 6, a_{10}, 9, 5, 11]$, $[9, 3, a_{11}, 8, 4, 5]$, $[a_9, 10, a_{11}, 7, 9, 8]$, $[a_{10}, 11, a_{11}, 5, 0, 1]$.

For G_8: $[0, x, 5, a_1, a_2, a_3] + i$, $i \in Z_{12}$, $[0, y, 3, a_8, a_9, a_{10}] + i$, $i \in Z_{12}$, when i is even, $x = a_4, y = a_6$, when i is odd, $x = a_5, y = a_7$, $[6, a_{11}, 0, 1, 2, 4] + i$, $i = 0, 3, 4, 5$, $[7, a_{11}, 1, 9, 3, 5]$, $[8, a_{11}, 2, 3, 4, 1]$, $[6, 7, 8, 9, 10, 0]$, $[7, 9, 11, 0, 1, 3]$, $[2, 6, 10, 0, 9, 11]$.

For G_9: $[8, x, 5, y, 1, 9] + i$, $i \in Z_{12}^*$, $[5, z, 0, t, 4, 8] + i$, $i \in Z_{12}$, when i is even, $x = a_1, z = a_2$, when i is odd, $x = a_3, z = a_4$, when $i \in [0, 3]$, $y = a_5, t = a_6$, when $i \in [4, 7]$, $y = a_7, t = a_8$, when $i \in [8, 11]$, $y = a_9, t = a_{10}$; $[6, a_{11}, 0, 4, 2, 3] + i$, $i \in [1, 3]$, $[10, a_{11}, 4, 8, a_1, 6]$, $[7, 8, 9, 5, a_{11}, 11]$, $[1, 9, a_5, 5, a_1, 8]$, $[0, 6, a_{11}, 11, 3, 7]$, $[1, 11, 0, 4, 2, 3]$, $[11, 9, 10, 0, 2, 8]$, $[1, 3, 2, 10, 8, 6]$.

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