Uniform coverings of 2-paths with 5-paths in K_{2n}

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Abstract

We give a construction of a uniform covering of 2-paths with 5-paths in K_n for all even $n \geq 6$, i.e., we construct a set S of 5-paths in K_n having the property that each 2-path in K_n lies in exactly one 5-path in S for all even $n \geq 6$.

1 Introduction

Let K_n be the complete graph on n vertices. A *path* of *length* l, or an l*-path*, is the graph induced by the edges in $\{\{v_i, v_{i+1}\} \mid 0 \leq i \leq l-1\}$; it is denoted by $[v_0, v_1, ..., v_l]$.

A uniform covering of the 2-paths in K_n with k-paths (k-cycles) is a set S of k-paths (k-cycles) having the property that each 2-path in K_n lies in exactly one k-path (k-cycle) in S. Only the following cases of the problem of constructing a uniform covering of the 2-paths in K_n with k-paths or k-cycles have been solved:

1. with 3-cycles,
2. with 3-paths [2],
3. with 4-cycles [3],
4. with 4-paths [5],
5. with n-cycles (Hamilton cycles) when n is even [4].

In this paper, we solve the problem in the case of 5-paths when n is even, that is, we prove:

Theorem *Let n be even and $n \geq 6$. Then there exists a set S of 5-paths in K_n having the property that each 2-path in K_n lies in exactly one path in S.*

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2 Preliminaries

We prove the theorem by induction on even $n \geq 6$.

Proposition 2.1 *There exists a uniform covering of* 2*-paths with* 5*-paths in* K_6*.*

Proof. The round table problem has a solution in the case of 7 people [1], that is, there exists a uniform covering S of 2-paths with Hamilton cycles in K_7. Remove one vertex and all edges incident to the vertex from the uniform covering S. Then we get a uniform covering of 2-paths with 5-paths (Hamilton paths) in K_6. More precisely, for the vertex set $V = \{0,1,2,3,4,5\}$ of K_6 and the permutation $\sigma = (012345)$, the set of 5-paths $S = S_1 \cup S_2$ is a uniform covering of 2-paths with 5-paths in K_6, where

$$S_1 = \{\sigma^j(P) \mid P = [0,5,4,1,3,2] \text{ or } [2,0,5,3,4,1]\ ; j = 0,1,2,3,4,5\},$$
$$S_2 = \{\sigma^j(P) \mid P = [1,3,5,2,0,4]\ ; j = 0,1,2\}.$$

□

Let n be even and $n \geq 6$. Suppose that there exists a uniform covering S_0 of 2-paths with 5-paths in K_n. Let V be the vertex set of K_n and let $V' = \{x,y\} \cup V$ be the vertex set of K_{n+2}. Put $V = \{0,1,2,...,n-1\}$. We denote by σ the permutation $(x)(y)(012...n-1)$. Put

$$\Sigma = \{\sigma^j \mid j = 0,1,2,...,n-1\} \text{ and } \Sigma^* = \{\sigma^j \mid j = 0,1,2,...,\tfrac{n-2}{2}\}.$$

We construct a set S of 5-paths in K_{n+2} such that the union $S \cup S_0$ is a uniform covering of 2-paths in K_{n+2}. Then each element of the set S is one of the following types:

type 1:	$[a,b,x,y,c,d]$,	type 2:	$[a,x,b,c,y,d]$,
type 3:	$[x,a,b,y,c,d]$,	type 4:	$[y,a,b,x,c,d]$,
type 5:	$[a,x,b,y,c,d]$,	type 6:	$[a,y,b,x,c,d]$,

where a,b,c,d are four different vertices in V.

Proposition 2.3 *There exists a uniform covering of* 2*-paths with* 5*-paths in* K_8*.*

Proof. Let S_0 be a uniform covering of 2-paths with 5-paths in K_6 obtained in Proposition 2.1. Define

$$S_1 = \{[1,3,y,x,0,4],\ [x,5,0,y,1,3],\ [y,5,0,x,1,3]\},$$
$$S_2 = \{[2,x,0,3,y,5],\ [3,0,y,2,x,5],\ [0,3,x,5,y,2]\}.$$

Put

$$S'_1 = \Sigma S_1 = \{\sigma^j(P) \mid P \in S_1;\ j = 0,1,2,3,4,5\},$$
$$S'_2 = \Sigma^* S_2 = \{\sigma^j(P) \mid P \in S_2;\ j = 0,1,2\}.$$

Put $S = S'_1 \cup S'_2$. Each 2-path including at least one of x, y lies in the set S exactly once. Hence the set $S_0 \cup S$ is a uniform covering of 2-paths with 5-paths in K_8. □

3 Proof of Theorem

Suppose there exists a uniform covering S_0 of 2-paths with 5-paths in K_n for even $n \geq 8$. We will prove that there exists a uniform covering S of 2-paths with 5-paths in K_{n+2}. Let V and V' be the vertex sets defined in Section 2.

Case 1 : $n = 4k$ for $k \geq 2$.

If $k \geq 3$, then let

$$S_1 = \{[0, l, y, k, k+l, x], [k+l, x, 2k, 2k+l, y, 3k], [l, 0, x, 3k+l, 3k, y] \mid l = 2, 3, ..., k-1\}.$$

Put

$$S_1' = \Sigma^* S_1 = \{\sigma^j(P) \mid P \in S_1;\ j = 0, 1, 2, .., \tfrac{n-2}{2}\}.$$

Then the set S_1' covers all 2-paths $[x, a, b]$ and $[y, a, b]$ with $|a - b| \equiv l \,(\mathrm{mod}\, n)$ for some $l = 2, .., k-1$. Also the set S_1' covers all 2-paths $[a, x, b]$ and $[a, y, b]$ with $|a - b| \equiv l \,(\mathrm{mod}\, n)$ for some $l = 1, 2, .., k-2$.

Let

$$S_2 = \{[x, k, 2k+l, y, 0, k+l], [k, x, 3k+l, 2k, y, l], [0, k+l, x, 3k, l, y] \mid l = 1, 2, .., k-1\},$$

Put

$$S_2' = \Sigma^* S_2 = \{\sigma^j(P) \mid P \in S_2;\ j = 0, 1, .., \tfrac{n-2}{2}\}.$$

Then the set S_2' covers all 2-paths $[x, a, b]$ and $[y, a, b]$ with $|a - b| \equiv l \,(\mathrm{mod}\, n)$ for some $l, k+1 \leq l \leq 2k-1$ and all 2-paths $[a, x, b]$ and $[a, y, b]$ with $|a - b| \equiv l \,(\mathrm{mod}\, n)$ for some $l, k+1 \leq l \leq 2k-1$.

Let

$$S_3 = \{[1, 0, x, y, 2k, 2k+1], \quad [0, 1, y, k, 2k, x], \quad [0, 1, x, k, 2k, y]\},$$
$$S_3' = \Sigma S_3 = \{\sigma^j(P) \mid P \in S_3;\ j = 0, 1, .., n-1\}.$$

Then S_3' covers all 2-paths $[x, a, b]$ and $[y, a, b]$ with $|a - b| \equiv 1$ or $k \,(\mathrm{mod}\, n)$, and all 2-paths $[a, x, b]$ and $[a, y, b]$ with $|a - b| \equiv k - 1 \,(\mathrm{mod}\, n)$. It also covers all 2-paths $[a, x, y]$ and $[a, y, x]$ in K_{n+2}.

Let

$$S_4 = \{[2k, 0, y, k, x, 3k], \quad [0, 2k, x, 3k, y, k], \quad [k, x, 0, 2k, y, 3k]\},$$
$$S_4' = \Sigma^* S_4 = \{\sigma^j(P) \mid P \in S_4;\ j = 0, 1, .., \tfrac{n-2}{2}\}.$$

Then S_4' covers all 2-paths $[x, a, b]$, $[y, a, b]$ and $[x, a, y]$ with $|a - b| \equiv 2k \,(\mathrm{mod}\, n)$. It also covers all 2-paths $[a, x, b]$ and $[a, y, b]$ with $|a - b| \equiv k$ or $2k \,(\mathrm{mod}\, n)$.

Put $S = S_1' \cup S_2' \cup S_3' \cup S_4'$. Then the set S covers all 2-paths including at least one of x, y and it covers them exactly once. Hence the set $S_0 \cup S$ is a uniform covering of 2-paths with 5-paths in K_{n+2}.

Case 2 : $n = 4k + 2$ for even $k \geq 2$.

Let

$$\begin{aligned}S_{11} &= \{[0, l+1, y, k+1, k+l+1, x], [k+l+1, x, 2k+1, 2k+l+2, y, 3k+2],\\ &\quad [y, 3k+2, 3k+l+2, x, 0, l+1] \mid l \text{ is odd, } 1 \le l \le k-1\},\\ S_{12} &= \{[l+1, 0, y, k+l+1, k+1, x], [k+1, x, 2k+l+2, 2k+1, y, 3k+l+2],\\ &\quad [0, l+1, x, 3k+2, 3k+l+2, y] \mid l \text{ is odd, } 1 \le l \le k-1\}.\end{aligned}$$

Put $S_1 = S_{11} \cup S_{12}$ and $S_1' = \Sigma^* S_1$. Then the set S_1' covers all 2-paths $[x, a, b]$ and $[y, a, b]$ with $|a-b| \equiv l \pmod{n}$ for some $l, 1 \le l \le k$, and it also covers all 2-paths $[a, x, b]$ and $[a, y, b]$ with $|a-b| \equiv l \pmod{n}$ for some odd $l, 1 \le l \le k-1$ or for some even $l, k+2 \le l \le 2k$. If $k \ge 4$, then let

$$\begin{aligned}S_{21} &= \{[0, k+l, y, k, 2k+l+1, x], [2k+l+1, x, 2k+1, 3k+l+1, y, 3k+1],\\ &\quad [k+l, 0, x, l, 3k+1, y] \mid l \text{ is even, } 2 \le l \le k-2\},\\ S_{22} &= \{[k+l, 0, y, 2k+l+1, k, x], [k, x, 3k+l+1, 2k+1, y, l],\\ &\quad [0, k+l, x, 3k+1, l, y] \mid l \text{ is even, } 2 \le l \le k-2\}.\end{aligned}$$

Put $S_2 = S_{21} \cup S_{22}$ and $S_2' = \Sigma^* S_2$. Then the set S_2' covers all 2-paths $[x, a, b]$ and $[y, a, b]$ with $|a-b| \equiv l \pmod{n}$ for some l, $k+2 \le l \le 2k-1$ and it also covers all 2-paths $[a, x, b]$ and $[a, y, b]$ with $|a-b| \equiv l \pmod{n}$ for some even $l, 2 \le l \le k-2$ or for some odd $l, k+3 \le l \le 2k-1$.

Let

$$\begin{aligned}S_3 &= \{[2k, 0, x, y, 2k+1, 4k+1], [0, 2k, y, 3k, 4k+1, x], [0, 2k, x, 3k, 4k+1, y]\},\\ S_4 &= \{[k, x, 3k+1, y, 0, 2k+1], [0, 2k+1, x, k, y, 3k+1], [k, y, 2k+1, 0, x, 3k+1]\}.\end{aligned}$$

Put $S_3' = \Sigma S_3$. Then the set S_3' covers all 2-paths $[x, a, b]$ and $[y, a, b]$ with $|a-b| \equiv k+1$ or $2k \pmod{n}$ and it also covers all 2-paths $[a, x, b]$ and $[a, y, b]$ with $|a-b| \equiv k \pmod{n}$. It also covers all 2-paths $[a, x, y]$ and $[a, y, x]$ in K_{n+2}. Put $S_4' = \Sigma^* S_4$. Then the set S_4' covers all 2-paths $[x, a, b]$, $[y, a, b]$ with $|a-b| \equiv 2k+1 \pmod{n}$, all 2-paths $[a, x, b]$ and $[a, y, b]$ with $|a-b| \equiv k+1$ or $2k+1 \pmod{n}$ and all 2-paths $[x, a, y]$.

Put $S = S_1' \cup S_2' \cup S_3' \cup S_4'$. Then the union $S \cup S_0$ is a uniform covering of 2-paths with 5-paths in K_{n+2}.

Case 3 : $n = 4k+2$ for odd $k \ge 3$.

Let

$$\begin{aligned}S_{11} &= \{[0, l+1, y, k+1, k+l+1, x], [k+l+1, x, 2k+1, 2k+l+2, y, 3k+2],\\ &\quad [y, 3k+2, 3k+l+2, x, 0, l+1] \mid l \text{ is odd, } 1 \le l \le k-2\},\\ S_{12} &= \{[x, k+1, k+l+1, y, 0, l+1], [k+1, x, 2k+l+2, 2k+1, y, 3k+l+2],\\ &\quad [0, l+1, x, 3k+2, 3k+l+2, y] \mid l \text{ is odd, } 1 \le l \le k-2\}.\end{aligned}$$

Put $S_1 = S_{11} \cup S_{12}$ and $S_1' = \Sigma^* S_1$. Then the set S_1' covers all 2-paths $[x, a, b]$ and $[y, a, b]$ with $|a-b| \equiv l \pmod{n}$ for $l, 1 \le l \le k-1$, and it also covers all 2-paths

$[a, x, b]$ and $[a, y, b]$ with $|a-b| \equiv l \pmod{n}$ for even $l, 2 \le l \le k-1$ or for odd $l, k+2 \le l \le 2k-1$.

Let

$$\begin{aligned} S_{21} &= \{[0, k+l, y, k, 2k+l+1, x], [2k+l+1, x, 2k+1, 3k+l+1, y, 3k+1], \\ &\quad [k+l, 0, x, l, 3k+1, y] \mid l \text{ is odd}, 1 \le l \le k-2\}, \\ S_{22} &= \{[k+l, 0, y, 2k+l+1, k, x], [l, y, 2k+1, 3k+l+1, x, k], \\ &\quad [0, k+l, x, 3k+1, l, y] \mid l \text{ is odd}, 1 \le l \le k-2\}. \end{aligned}$$

Put $S_2 = S_{21} \cup S_{22}$ and $S_2' = \Sigma^* S_2$. Then the set S_2' covers all 2-paths $[x, a, b]$ and $[y, a, b]$ with $|a-b| \equiv l \pmod{n}$ for $k+1 \le l \le 2k-1$ and it also covers all 2-paths $[a, x, b]$ and $[a, y, b]$ with $|a-b| \equiv l \pmod{n}$ for odd $l, 1 \le l \le k-2$ or for even $l, k+3 \le l \le 2k$.

Let

$$\begin{aligned} S_3 &= \{[2k, 0, x, y, 2k+1, 4k+1], [0, 2k, y, 3k+1, 4k+1, x], \\ &\qquad [0, 2k, x, 3k+1, 4k+1, y]\}, \\ S_4 &= \{[2k+1, 0, x, k, y, 3k+1], [k, x, 3k+1, y, 2k+1, 0], [k, y, 0, 2k+1, x, 3k+1]\}. \end{aligned}$$

Put $S_3' = \Sigma S_3$ and $S_4' = \Sigma^* S_4$. Then the set S_3' covers all 2-paths $[x, a, b]$, $[y, a, b]$ with $|a-b| \equiv k$ or $2k \pmod{n}$, and all 2-paths $[a, x, b]$ and $[a, y, b]$ with $|a-b| \equiv k+1 \pmod{n}$. The set S_3' also covers all 2-paths $[a, x, y]$ and $[a, y, x]$ in K_{n+2}. The set S_4' covers all 2-paths $[x, a, b]$ and $[y, a, b]$ with $|a-b| \equiv 2k+1 \pmod{n}$, all 2-paths $[a, x, b]$ and $[a, y, b]$ with $|a-b| \equiv k$ or $2k+1 \pmod{n}$ and all 2-paths $[x, a, y]$.

Put $S = S_1' \cup S_2' \cup S_3' \cup S_4'$. Then the union $S \cup S_0$ is a uniform covering of 2-paths with 5-paths in K_{n+2}.

Completion of the proof of Theorem

In the case of $n = 6$ and 8, it was proved in Proposition 2.1 and Proposition 2.3. So by Case 1, Case 2 and Case 3, we have completed the proof of the theorem by induction on $n \ge 6$. □

Remark

If there exists a uniform covering of 2-paths with 5-paths in K_n, then n should be even or $n = 8k+1$, because four 2-paths lie in each 5-path and hence ${}_nP_3/2 = n(n-1)(n-2)/2$ should be divisible by four. In the case of $n = 8k+1$, the construction of a uniform covering of 2-paths with 5-paths in K_n is a little complicated. The proof of this will be shown in a separate paper.

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