Extending Skolem sequences, how can you begin?

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Abstract

For each $n \geq 1$ and $1 \leq j \leq n$ we show the existence of an extended Skolem sequence of order n starting with the symbol j.

1 Introduction

In this paper $[x, y] = \{n \mid x \leq n \leq y,\ n \text{ an integer}\}$. A **Skolem sequence** of **order** n is a partition of the set $[1, 2n]$ into pairs $\{(a_i, b_i)\}_{i=1}^n$ such that $b_i - a_i = i$ for $1 \leq i \leq n$. The partition into pairs $\{(7,8), (2,4), (3,6), (1,5)\}$ is then a Skolem sequence of order 4, and it may be equivalently stored in the sequence 42324311, where $i \in [1, 4]$ occurs in positions a_i, b_i. The existence of Skolem sequences was settled in 1958 by Skolem [8]: a Skolem sequence exists if and only if $n \equiv 0, 1 \pmod 4$. Since then a number of variations have been considered and a growing list of names have cropped up in the literature. To describe these to the unfamiliar reader we employ an explicit notation. For sets of integers $D = \{d_1, \ldots, d_n\}$, $S = \{s_1, \ldots, s_{2n}\}$ a (D, S)-sequence is a partition of S into pairs $\{(a_i, b_i)\}_{i=1}^n$ such that $b_i - a_i = d_i$ for $1 \leq i \leq n$. A Skolem sequence is then more compactly described as a $([1, n], [1, 2n])$-sequence. A summary of some variations on Skolem's theme is given in Section 2, and we also refer the reader to the survey [6] in this regard. Our immediate goal however is to develop terminology that is not weighed down with historical references, but that is also descriptive without being too terse. Viewing Skolem and Skolem-like sequences as packings or tilings seems to satisfy these requirements, and fits in nicely with the terminology already in the literature (e.g. see [7]).

The notion of packing arises from the sequence representation of a general (D, S)-sequence, where underscores (some authors prefer the star, $*$, or a zero) indicate unused positions or **holes** in the sequence. For instance 242_34_311 is a $([1, 4], [1, 10]\backslash\{4, 7\})$-sequence with two holes, suggesting that we view the sequence 4232411 without holes as a perfect packing or tiling of the "difference tiles" 4___4, 3__3, 2_2, 11 into a set of eight contiguous positions. The underscores in a difference tile hold positions not used by the tile, and these may be occupied in the course of forming a packing by filling them with symbols from other tiles. In general we say that a set of differences D **packs perfectly** if a $(D, [1, 2|D|])$-sequence exists. A packing such as 11232_3 that leaves the next to last position unused is called **hooked**, and we say that the differences $[1, 3]$ (really the difference tiles 11, 2_2, 3__3) **can be hooked** if they can be packed into a hooked sequence with no other holes. Joining hooks is a useful way to reduce the number of holes in a sequence, e.g. 64758463573_8 joins to 2_211 to give a perfect packing of differences $[1, 8]$. A packing is **extended** if it has exactly one hole, suggesting that the positions are no longer contiguous due to an extension of the length of the sequence. For example, 11232_3, 232_311 and 3_23211 are all extended, and all possible (up to a parity constraint) positions of the hole are achieved. We say that differences $[1, 3]$ can be **extended in all possible ways**. Sequences _11232_3, 11_232_3 and 3113_2_2 show that $[1, 3]$ can be **hooked-extended** in all possible ways, i.e. a hook is left at the end, after which all possible positions of the remaining hole can be achieved. In what follows we will use ϵ as a convenient shorthand to represent "_".

We now describe the main result of this paper. As noted in [1] it is trivial to construct for each order $n \geq 1$ an extended Skolem sequence, (i.e. a $([1, n], [1, 2n] \backslash \{k\})$-sequence) for example 8642_2468 75311357 is one of order 8. However, it is considerably more difficult to construct such a sequence if the difference at the start of the sequence is specified. Such sequences are a natural first step towards solving the problem of constructing Skolem and Skolem-like sequences with a specified difference in a specified position. Small examples are easily constructed: 11 for $n = 1$ and 112_2 begins and ends with 1 and 2. For $n = 3$ we assemble 11 and 3_232 as needed. For $n = 4$ we exhibit a single sequence, $11\vec{3}4\vec{2}324$, where any symbol with an arrow over it can be rewritten at the end of the sequence. We call this operation **pivoting**, since one occurrence of the symbol remains fixed, but the other occurrence moves to the opposite side of the fixed occurrence. Continuing, $52\vec{4}\tilde{2}\vec{3}54311$, solves our problem for $n = 5$, where a tilde, ˜, indicates that we pivot to the left instead of to the right. We shall prove that, without exception, an extended Skolem sequence of order n can have its starting symbol specified, thereby answering the query posed in the title.

2 Some Useful Results

A short inventory of sequences we require later is given below.

Lemma 2.1 *We have*

1. (a) [2] *A* $([1, n], [1, 2n+1] \backslash \{k\})$*-sequence exists if and only if* $n \equiv 0, 1 \pmod 4$ *for* k *odd and* $n \equiv 2, 3 \pmod 4$ *for* k *even.*

   (b) [4] *A* $([2, n], [1, 2n-1] \backslash \{k\})$*-sequence exists if and only if* $n \equiv 0, 1 \pmod 4$ *for* k *odd and* $n \equiv 2, 3 \pmod 4$ *for* k *even.*

   (c) [4] *A* $([3, n], [1, 2n-3] \backslash \{k\})$*-sequence exists if and only if* $n \equiv 2, 3 \pmod 4$ *for* k *odd and* $n \equiv 0, 1 \pmod 4$ *for* k *even, with the exception of* $n \leq 4$ *and* $(n, k) = (5, 2), (5, 6), (6, 1), (6, 5), (6, 10)$*.*

2. (a) [4] *A* $([1,n],[1,2n+2]\backslash\{k,2n+1\})$*-sequence exists if and only if* $n \equiv 2,3 \pmod 4$ *for* k *odd and* $n \equiv 0,1 \pmod 4$ *for* k *even, with the exception of* $(n,k)=(1,2)$*.*

   (b) [5] *A* $([2,n],[1,2n]\backslash\{k,2n-1\})$*-sequence exists if and only if* $n \equiv 2,3 \pmod 4$ *for* k *odd and* $n \equiv 0,1 \pmod 4$ *for* k *even, with the exception of* $(n,k)=(3,3)$*.*

3. [3], [7]

   (a) *A* $([d,d+m-1],[1,2m])$*-sequence exists if and only if* $m \geq 2d-1$ *and* $m \equiv 0,1 \pmod 4$ *for* d *odd and* $m \equiv 0,3 \pmod 4$ *for* d *even.*

   (b) *A* $([d,d+m-1],[1,2m+1]\backslash\{2m\})$*-sequence exists if and only if* $m(m-2d+1)+2 \geq 0$ *and* $m \equiv 2,3 \pmod 4$ *for* d *odd and* $m \equiv 1,2 \pmod 4$ *for* d *even.*

The $([d,d+m-1],[1,2m])$-sequences of Lemma 2.1(3) are called **Langford sequences** of **defect** d and **length** m. Similarly a $([2,n],[1,2n]\backslash\{k,2n-1\})$-sequence is called a hooked-extended Langford sequence of defect 2 and length $n-1$, whereas a $([1,n],[1,2n+2]\backslash\{k,2n+1\})$-sequence is called a hooked-extended Skolem sequence and position k is the **floating hole**. Note that difference set $[1,n]$ can hooked, extended with a hole at position k, or hooked-extended with hole at position k under precisely the same general conditions that difference set $[2,n]$ can packed into these shapes. This is because appending the difference tile "11" to any one of these three shapes that $[2,n]$ is packed into will give a packing of $[1,n]$ into the same general shape, i.e., the parity of the hole does not change. Note that hooking an extended sequence causes the parity of the floating hole to change, so that extended and hooked-extended sequences with the same differences fall into complementary congruence classes modulo 4.

3 The Main Theorem

Let $\mathcal{S}(j;n)$, $h\mathcal{S}(j;n)$, $E\mathcal{S}(j;n)$ respectively denote a Skolem, hooked Skolem or extended Skolem sequence that begins with symbol j. A few special cases need to be handled separately.

Lemma 3.1 *The following sequences exist:*

1. *A* $\mathcal{S}(n;n)$ *for all* $n \equiv 0,1 \pmod 4$*.*

2. *A* $h\mathcal{S}(n;n)$ *for all* $n \equiv 2,3 \pmod 4$*,* $n \neq 2$*.*

Proof: If $n \equiv 0,1 \pmod 4$, $n \geq 4$ then put n at the start of an extended Skolem sequence of order $n-1$ with hole in position n (possible by Lemma 2.1(1)). If $n \equiv 2,3 \pmod 4$, $n \geq 3$ then put n at the start of a hooked, extended Skolem sequence of order $n-1$ with hole in position n (possible by Lemma 2.1(1)). Since a

$\mathcal{S}(1;1)$ is trivial to construct, this completes the the proof of the lemma. ■

For each $n \geq 1$ let $d(n)$ be the largest integer such that the differences $[d(n), n]$ can either be packed perfectly or can be hooked; then $d(n) = \lfloor \frac{n+2}{3} \rfloor$ by Lemma 2.1(3). The following lemma uses Langford sequences to construct $E\mathcal{S}(j;n)$ with j in approximately the first third of the full set of differences.

Lemma 3.2 *An $E\mathcal{S}(k;n)$ exists for all k, $1 \leq k \leq d(n) - 1$.*

Proof: By Lemma 2.1(3) for any $1 \leq k \leq d(n)$ the set $[k, n]$ is either perfect or hooked, so let $\mathcal{L}$ be the resulting sequence. If $k - 1 = 2$ then place 11 at the front of $\mathcal{L}$ and hook or append 2_2 at the end of $\mathcal{L}$ depending on whether $\mathcal{L}$ is hooked or not respectively. This gives an $E\mathcal{S}(2;n)$. If $k - 1 \neq 2$ then by Lemma 3.1 $[1, k-1]$ packs into a sequence, $\mathcal{S}$, which is either a $\mathcal{S}(k-1;k-1)$ or a $E\mathcal{S}(k-1;k-1)$. Hooking or appending $\mathcal{S}$ at the end of $\mathcal{L}$ as we did with the tile 2_2 and reversing gives a $E\mathcal{S}(k-1;n)$. This gives constructions of $E\mathcal{S}(k-1;n)$ for each $1 \leq k \leq d(n)$, and completes the proof. ■

The next lemma gives direct constructions for extended Skolem sequences that begin with differences in the middle third of the full set of differences.

Lemma 3.3 *An $E\mathcal{S}(k;m)$ exists for all k, $d(m) \leq k \leq 2d(m)$.*

Proof: We make cases on m modulo 3.

Let $m \equiv 1 \pmod 3$. If $m = 1$ the solution is trivial, so write $m = 3n - 2$, $n \geq 2$, and form the sequence

$$\begin{aligned}&\overline{2n}, \overline{2n+1}, \ldots, \overline{3n-3}, \overline{3n-2}, n, n+1, \ldots, 2n-2,\\&2n-1, n, \overline{2n}, n+1, \overline{2n+1}, \ldots, \overline{3n-3}, 2n-2, \overline{3n-2}, 2n-1.\end{aligned} \tag{1}$$

If $n = 4$ the sequence is $\overline{8}, \overline{9}, \overline{10}, 4, 5, 6, 7, 4, \overline{8}, 5, \overline{9}, 6, \overline{10}, 7$. The diacritical marks identify differences that fall into a pattern and/or they may indicate a small number of differences that don't fall into a pattern. The key property of this sequence is that any one of the second occurrences of $4, 5, 6$ and 7 can be pivoted to the front. The sequences we will construct subsequently will all be variations of this basic sequence.

By Lemma 2.1(1) the differences $[1, n-1]$ can be arranged into a sequence, $\mathcal{S}$, which is either a Skolem sequence or a hooked Skolem sequence. Appending $\mathcal{S}$ to (1) or its reverse gives $E\mathcal{S}(j;3n-2)$ for $j = 2n, 2n-1$.

Pivoting any one of the differences $j \in [n, 2n-2]$ to the left in (1) puts j at the start of the sequence, and leaves a hole at position $2j+1$. If $n - 1 \equiv 0, 1 \pmod 4$ append any Skolem sequence of order $n-1$ at the end of the pivoted sequence to get a $E\mathcal{S}(j;3n-2)$, $n \leq j \leq 2n-2$. On the other hand if $n - 1 \equiv 2, 3 \pmod 4$ then replace the difference $2n-1$ into positions $2j+1$, $2n+2j$, and (by Lemma 2.1(1)) append an extended Skolem sequence of order $n-1$ with the hole aligned at position

$2n+2j$, thereby leaving one hole in the finished sequence at position $2n$. This gives sequences $E\mathcal{S}(j;3n-2)$, $n \le j \le 2n-2$, when $n-1 \equiv 2,3 \pmod 4$. Altogether we have sequences $E\mathcal{S}(j;3n-2)$, $n \le j \le 2n$ for $n \ge 2$, as required.

Now let $m \equiv 2 \pmod 3$. If $m=2$ the solution is easy, so write $m = 3n-1$, $n \ge 2$. Consider again the pivoted version of sequence (1) with difference $j \in [n, 2n-1]$ at the start of the sequence. It occupies $4n-1$ positions and has a hole at position $2j+1$. Put (new) difference $3n-1$ into positions $2j+1, 3n+2j$ and note that $3n+2j \le 6n-1$ if and only if $j \in [n, \frac{1}{2}(3n-1)]$ and $6n-1$ is the number of positions required for a $E\mathcal{S}(j;3n-1)$. By Lemma 2.1, parts (1), (2), the differences $[1, n-1]$ can be extended/hooked extended in all possible ways, and the floating holes have opposite parities in the hooked and unhooked cases. Thus if $j \in [n, \frac{1}{2}(3n-1)]$ we may fill positions $4n+2, 4n+3, \ldots$ with a subsequence made of differences $[1, n-1]$ so that the second occurrence of $3n-1$ is aligned with the floating hole and the subsequence is hooked/unhooked as required by parity. This gives sequences $E\mathcal{S}(j;3n-1)$ for $j \in [n, \frac{1}{2}(3n-1)]$. To obtain sequences with $j \in (\frac{1}{2}(3n-1), 2n]$ we begin by forming the $(\{1,n\} \cup [n+2, 3n-1], [1, 4n])$-sequence

$$\begin{gathered}\widetilde{n}, \overline{2n+1}, \overline{2n+2}, \ldots, \overline{3n-1}, \widetilde{n}, n+2, n+3, \ldots, 2n \\ 1, 1, \overline{2n+1}, n+2, \overline{2n+2}, n+3, \ldots, \overline{3n-1}, 2n\end{gathered} \tag{2}$$

for $n \ge 3$.

Next we require a $([2, n-1] \cup \{n+1\}, [1, 2n-1] \backslash \{k\})$-sequence starting with $n+1$ for $n \ge 3$; denote this sequence $\mathcal{A}_n$. To show the existence of $\mathcal{A}_n$ we begin $\mathcal{A}_3 = 42\_24$, $\mathcal{A}_4 = 5\_23253$ and for $n \ge 5$ we use (by Lemma 2.1(1)) an extended or hooked extended Langford sequence with differences $[2, n-1]$ and a hole at position $n+1$, which is then filled when difference $n+1$ is placed at the start.

By placing $\mathcal{A}_n$ at the front or back of (2) we obtain sequences $E\mathcal{S}(j;3n-1)$, $j = n, n+1$. Pivoting a given $j \in [n+2, 2n]$ in (2) puts j at the start and leaves a hole at position $2j+1$. Placing difference $n+1$ into positions $2j+1, 2j+n+2$ will cause a collision if $2j+n+2 \le 4n+1$, since $4n+1$ is the length of the pivoted sequence. However, if $2j+n+2 > 4n+1$, or equivalently $j \in (\frac{1}{2}(3n-1), 2n]$, then depending on parity (by Lemma 2.1 parts (1), (2)) we fill positions $4n+2, 4n+3, \ldots$ with an extended or hooked-extended Skolem sequence with the floating hole aligned on the second occurrence of $n+1$. This gives sequences $E\mathcal{S}(j;3n-1)$, $j \in (\frac{1}{2}(3n-1), 2n]$, so altogether we have obtained sequences $E\mathcal{S}(j;3n-1)$, $n \le j \le 2n$, $n \ge 3$.

Finally let $m \equiv 0 \pmod 3$. If $m=3$ the solution is easy, so write $m = 3n$, $n \ge 2$. For $n \ge 2$ we construct

$$\begin{gathered}\overline{2n+2}, \overline{2n+3}, \ldots, \overline{3n}, n, n+1, n+2, \ldots, 2n-1, n, \epsilon, n+1 \\ \overline{2n+2}, n+2, \overline{2n+3}, \ldots, 2n-1, \overline{3n}, \widetilde{2n}, \epsilon^{2n-1}, \widetilde{2n};\end{gathered} \tag{3}$$

where all differences $[n, 3n]$ occur except $2n+1$ and there is a hole at position $2n+1$. If $n-1 \equiv 2,3 \pmod 4$ then put difference $2n+1$ into positions $2n+1, 4n+2$ and fill the resulting subsequence $\widetilde{2n}, \epsilon, 2n+1, \epsilon^{2n-3}, \widetilde{2n}$ with a hooked Skolem sequence of order $n-1$. The resulting sequence has no holes, ends with $2n$ and any difference

$j \in [n, 2n-1]$ can be pivoted to the front. On the other hand if $n-1 \equiv 0, 1 \pmod 4$ then first pivot $j \in [n, 2n-1]$ to the front of (3) leaving a hole at position $2j$, put difference $2n+1$ into positions $2j, 2n+2j+1$ and fill the resulting subsequence $\widetilde{2n}, \epsilon^{2j-2n}, 2n+1, \epsilon^{4n-2j-2}, \widetilde{2n}$ with an extended Skolem sequence of order $n-1$; the resulting sequence ends with $2n$, has one hole at position $2n+2$ and the pivoted difference j at the front. Putting together the cases on $n-1$ modulo 4, we obtain sequences $E\mathcal{S}(j; 3n)$, $n \le j \le 2n$, $n \ge 2$. ■

The next two lemmas construct sequences with the starting symbol in approximately the last third of the full set.

Lemma 3.4 *A* $E\mathcal{S}(j; 3n-1)$*,* $2n+1 \le j \le 3n-1$*, exists for* $n \ge 3$*, and* $n \equiv 2, 3 \pmod 4$.

Proof: For $n = 3$ the sequence $84\vec{7}3643587625211$ gives the required values $7 \le j \le 8$, so we take $n \ge 6$ and form the sequence:

$$\begin{aligned}&\widetilde{2n-1}, \epsilon^{n-1}, \widehat{3n-1}, \epsilon^{n-4}, \widehat{2n}, \epsilon, \widetilde{2n-1}, \overline{2n+1},\\ &\overline{2n+2}, \ldots, \overline{3n-2}, n, n+1, n+2, \ldots, 2n-2, \widehat{2n}, n,\\ &\widehat{3n-1}, n+1, \overline{2n+1}, n+2, \overline{2n+2}, \ldots, 2n-2, \overline{3n-2}, 1, 1.\end{aligned} \qquad (4)$$

If $n \equiv 2, 3 \pmod 4$ then we may pivot any $j \in [2n+1, 3n-2]$ to the front of this sequence and then (by Lemma 2.1(1)) fill the subsequence $\widetilde{2n-1}$, ϵ^{n-1}, $\widehat{3n-1}$, ϵ^{n-4}, $\widehat{2n}$, ϵ, $\widetilde{2n-1}$, with a hooked, extended Langford sequence with differences $[2, n-1]$, thereby obtaining sequences $E\mathcal{S}(j; 3n-1)$, $2n+1 \le j \le 3n-2$. An $E\mathcal{S}(3n-1; 3n-1)$ exists by Lemma 3.1. Altogether we have sequences $E\mathcal{S}(j; 3n-1)$ for $2n+1 \le j \le 3n-1$, $n \equiv 2, 3 \pmod 4$ and $n \ge 3$. ■

We now describe a recursive construction called **twinning**, which is useful for making extended Skolem sequences that begin with differences in the last third of the full set of differences. It is best illustrated by an example. Let $S = 423243\_11$ and consider the four occurrences of the symbol 3 in $S\epsilon S$. The first and fourth 3's are in positions with difference $9+1+3$, and the second and third 3's are in positions with difference $9+1-3$, so we replace the four 3's with two 13's and two 7's. Doing this for all the differences and filling in the underscores with 10's gives S' below

$$\begin{array}{lccccccccccccccccccc} S\epsilon S: & 4 & 2 & 3 & 2 & 4 & 3 & 0 & 1 & 1 & \_ & 4 & 2 & 3 & 2 & 4 & 3 & 0 & 1 & 1 \\ S': & 14 & 12 & 13 & 8 & 6 & 7 & 10 & 11 & 9 & \_ & 6 & 8 & 7 & 12 & 14 & 13 & 10 & 9 & 11. \end{array}$$

Clearly S' has differences $9+1 \pm [0,4] = [6,14]$, and S' begins with $9+1+4 = 14$. In general, taking a sequence S, forming $S\epsilon^k S$ and replacing differences as above yields a sequence S' with differences $\ell + k \pm d$ for all d a difference of S, where ℓ is the length of S. Note that S' begins with $\ell + k + j$, where j is the first symbol of S, and that positions $\ell+1, \ldots, \ell+k$ are unused. We say that S' is the result of **twinning** S **at distance** k.

Lemma 3.5 *The following recursions hold:*

1. $\forall j \in [1,n] \exists E\mathcal{S}(j;n) \Rightarrow \forall j \in [2n+2, 3n+1] \exists E\mathcal{S}(j; 3n+1)$.
2. $\forall j \in [1,n] \exists E\mathcal{S}(j;n) \Rightarrow \forall j \in [2n+3, 3n+2] \exists E\mathcal{S}(j; 3n+2)$, $n \equiv 0, 3 \pmod 4$.
3. $\forall j \in [1,n] \exists E\mathcal{S}(j;n) \Rightarrow \forall j \in [2n+3, 3n+3] \exists E\mathcal{S}(j; 3n+3)$.

Proof: Twinning a sequence $E\mathcal{S}(j;n)$ at distance 0 and then appending the same $E\mathcal{S}(j;n)$ to the twinned sequence gives a $E\mathcal{S}(2n+1+j; 3n+1)$. As j varies over $[1,n]$ the quantity $2n+1+j$ varies over $[2n+2, 3n+1]$, and (1) is proved.

Twin a sequence $E\mathcal{S}(j;n)$ at distance 1 to get $([n+2, 3n+2], [1, 4n+3]\backslash\{2n+2\})$-sequence begining with $2n+2+j$. If $n+1 \equiv 0, 1 \pmod 4$ then append a Skolem sequence of order $n+1$ to get a $E\mathcal{S}(2n+2+j; 3n+2)$. As j varies over $[1,n]$ the quantity $2n+2+j$ varies over $[2n+3, 3n+2]$, and (2) is proved.

By what we have just done we have at our disposal a $([n+2, 3n+2], [1, 4n+3]\backslash\{2n+2\})$-sequence starting with $2n+3$. Put difference $3n+3$ in positions $2n+2, 5n+5$, and (by Lemma 2.1(2)) depending on whether $[1, n+1]$ can be extended or hooked-extended in all possible ways, construct an extended or hooked extended packing of $[1, n+1]$ on positions $[4n+4, 6n+6]$ or $[4n+4, 6n+7]$ with the hole aligned at position $5n+5$. This gives an $E\mathcal{S}(2n+3; 3n+3)$. Twinning a $E\mathcal{S}(j;n)$ at distance 2 gives a $([n+3, 3n+3], [1, 4n+4]\backslash\{2n+2, 2n+3\})$-sequence starting with $2n+3+j$. Put difference 1 in positions $2n+2, 2n+3$, and (by Lemma 2.1(1)) append any $([2, n+2], [1, 2n+3]\backslash\{k\})$-sequence at the end to get a $E\mathcal{S}(2n+3+j; 3n+3)$. Altogether the $E\mathcal{S}(2n+3; 3n+3)$ obtained earlier together with $E\mathcal{S}(2n+3+j; 3n+3)$ for all $j \in [1,n]$ gives (3). ■

The proof of the main theorem is now straightforward.

Theorem 3.1 *For each $n \geq 1$ and each $1 \leq j \leq n$ there exists an extended Skolem sequence of order n starting with the symbol j.*

Proof: The proof is by induction on n. We have already verified the theorem for $1 \leq n \leq 5$ in the introduction. We now take $n > 5$ and make cases depending on n modulo 3.

Case: $n \equiv 0 \pmod 3$. Write $n = 3m$, $m \geq 2$. By Lemma 3.2 we have sequences $E\mathcal{S}(j; 3m)$, $1 \leq j \leq m-1$, which together with sequences $E\mathcal{S}(j; 3m)$, $m \leq j \leq 2m$, from Lemma 3.3 gives sequences $E\mathcal{S}(j; 3m)$, $1 \leq j \leq 2m$. By Lemma 3.5 we recursively obtain sequences $E\mathcal{S}(j; 3m)$, $2m+1 \leq j \leq 3m$, which completes the construction of sequences $E\mathcal{S}(j; 3m)$, $1 \leq j \leq 3m$, in this case.

Case: $n \equiv 1 \pmod 3$. Write $n = 3m+1$, $m \geq 2$. By Lemma 3.2 we have sequences $E\mathcal{S}(j; 3m+1)$, $1 \leq j \leq m$, which together with the sequences $E\mathcal{S}(j; 3m+1)$, $m+1 \leq j \leq 2m+2$, from Lemma 3.3 gives sequences $E\mathcal{S}(j; 3m+1)$, $1 \leq j \leq 2m+2$. By Lemma 3.5 we recursively obtain sequences $E\mathcal{S}(j; 3m+1)$, $2m+2 \leq j \leq 3m+1$, which completes the construction of sequences $E\mathcal{S}(j; 3m+1)$, $1 \leq j \leq 3m+1$, in this case.

Case: $n \equiv 2 \pmod 3$. Write $n = 3m+2$, $m \geq 2$. By Lemma 3.2 we have sequences $E\mathcal{S}(j; 3m+2)$, $1 \leq j \leq m$, which together with sequences $E\mathcal{S}(j; 3m+2)$, $m+1 \leq j \leq 2m+2$ from Lemma 3.3 gives sequences $E\mathcal{S}(j; 3m+2)$, $1 \leq j \leq 2m+2$.

By Lemma 3.4 we have sequences $E\mathcal{S}(j; 3m+2)$, $2m+3 \leq j \leq 3m+2$, for $m \geq 2$ and $m \equiv 1, 2 \pmod 4$. By Lemma 3.5 we recursively obtain sequences $E\mathcal{S}(j; 3m+2)$, $2m+3 \leq j \leq 3m+2$, for $m \equiv 0, 3 \pmod 4$, so altogether we have sequences $E\mathcal{S}(j; 3m+2)$, $2m+3 \leq j \leq 3m+2$. This completes the range of starting symbols in this case and gives all sequences $E\mathcal{S}(j; 3m+2)$, $1 \leq j \leq 3m+2$. The main theorem is now proved. ■

4 Conclusion

On the surface, the problem we have solved appears to be a two parameter problem: the order of the sequence and the starting symbol being the two parameters. However, the position of the hole in our constructions was not fixed and this compensates by adding a degree of freedom, making this problem more like a "$1\frac{1}{2}$" parameter problem. For some potential applications of our sequences this is not an issue, but of course one always is looking for sharper and more versatile tools to gain more control in constructing sequences. An obvious next step then is to find better techniques that will show the existence of $\mathcal{S}(j; n)$ and $h\mathcal{S}(j; n)$ for all j, and that will also have an impact on solving other problems of a similar nature.

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