Semi-planar Steiner quasigroups of cardinality $3n$

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Abstract

It is well known that for each $n \equiv 1$ or $3 \pmod 6$ there is a planar Steiner quasigroup (briefly, squag) of cardinality n (Doyen (1969) and Quackenbush (1976)). A simple squag is semi-planar if every triangle either generates the whole squag or the 9-element subsquag (Quackenbush (1976)). In fact, Quakenbush has stated that there should be such semi-planar squags. In this paper, we construct a semi-planar squag of cardinality $3n$ for all $n > 9$ and $n \equiv 1$ or $3 \pmod 6$. For $n = 9$, we give a construction for a semi-planar squag of cardinality 27 which is not planar. Steiner triple systems are in 1–1 correspondence with the squags (see Quackenbush (1976)). In this article, the Steiner triple system associated with a semi-planar squag will be called semi-planar. Consequently, we may say that there is a semi-planar Steiner triple system of cardinality m which is not planar for all $m > 9$ and $m \equiv 3$ or $9 \pmod{18}$. Quackenbush has also proved that the variety generated by a finite simple planar squag covers the variety of all medial squags. Similarly, it is easy to show that the variety generated by a finite semi-planar squag also covers the variety of all medial squags.

1 Introduction

A *squag* (or Steiner quasigroup) is a groupoid $\boldsymbol{S} = (S; \cdot)$ satisfying the identities:

$$x \cdot x = x, \quad x \cdot y = y \cdot x, \quad x \cdot (x \cdot y) = y.$$

A squag is called *medial* if it satisfies the medial law:

$$(x \cdot y) \cdot (z \cdot w) = (x \cdot z) \cdot (y \cdot w).$$

A Steiner triple system (briefly, triple system) is a pair $(P; B)$, where P is a set of points and B is a set of 3-element subsets of P called blocks such that for distinct points $p_1, p_2 \in P$, there is a unique block $b \in B$ such that $\{p_1, p_2\} \subseteq b$. Triple systems are in 1–1 correspondence with the squags [6, 10].

The associated squag $(P; \cdot)$ with the triple system $(P; B)$ is defined as follows: $x \cdot x = x$ for all $x \in P$ and for each pair $\{x, y\} \subseteq P$, $x \cdot y = z$ if and only if $\{x, y, z\} \in B$ [6, 9].
If the cardinality of P is equal to n, then $(P; B)$ and $(P; \cdot)$ are said to be of order n (or of cardinality n), and written STS(n) and SQG(n), respectively.

It is well known that the necessary and sufficient condition for an STS(n) to exist is that $n \equiv 1$ or $3 \pmod 6$ [3, 6]. In fact, there is a 1–1 correspondence between the subsquags of the coordinatising squag $\boldsymbol{Q} = (P; \cdot)$ and the subspaces (or subSTSs) of the underlying triple system $(P; B)$ [6].

A subsquag $\boldsymbol{N} = (S; \cdot)$ of a squag $\boldsymbol{Q} = (P; \cdot)$ is called *normal* if and only if $\boldsymbol{N}$ is a congruence class of $\boldsymbol{Q}$. In the following theorem, Quackenbush [10] has given a necessary and sufficient condition for a large subsquag $\boldsymbol{S}_1$ of a finite squag $\boldsymbol{Q}$ to be normal.

Theorem 1 [10]. *If $\boldsymbol{S}_1 = (P_1; \cdot)$ and $\boldsymbol{S}_2 = (P_2; \cdot)$ are two subsquags of a finite squag $\boldsymbol{Q}$ such that $P_1 \cap P_2 = \emptyset$ and $|P| = 3|P_1| = 3|P_2|$, then $\boldsymbol{S}_i$, for $i = 1, 2, 3$, are normal subsquags, where $\boldsymbol{S}_3 = (P_3; \cdot)$ and $P_3 = P - (P_1 \cup P_2)$.*

Moreover, the author [2] has shown that there is a subsquag $\boldsymbol{S}_1 = (P_1; \cdot)$ of a finite squag $\boldsymbol{Q} = (P; \cdot)$ with $|P| = 3|P_1|$ and $\boldsymbol{S}_1$ not normal. This means that a subsquag $\boldsymbol{S}_1 = (P_1; \cdot)$ of a finite squag $\boldsymbol{Q} = (P; \cdot)$ with $|P| = 3|P_1|$ is normal if and only if the set $P - P_1$ can be divided into two subsquags of $\boldsymbol{Q}$ of cardinality $|P_1|$.

Quackenbush [10] also proved that squags have permutable, regular, and Lagrangian congruences. Moreover, he showed that the lattice of normal subsquags of a squag $\boldsymbol{Q}$ containing a fixed element is isomorphic to the lattice of congruences of $\boldsymbol{Q}$.

Basic concepts of universal algebra and properties of squags can be found in [7] and [4].

A squag is called *simple* if it has only the trivial congruences. Guelzow [8] and the author [1] have constructed examples of non-simple squags (and not medial, of course).

An STS is *planar* if it is generated by every triangle and contains a triangle. A planar STS(n) exists for each $n \geq 7$ and $n \equiv 1$ or $3 \pmod 6$ [5]. Quackenbush has also shown in the next theorem that the only nonsimple finite planar squag has 9 elements.

Theorem 2 [10]. *Let $(P; B)$ be a planar STS(n) and let $\boldsymbol{Q} = (P; \cdot)$ be the corresponding squag. Then either $\boldsymbol{Q}$ is simple or $n = 9$.*

Accordingly, we may say that there is always a simple SQG(n) for all $n > 9$ and $n \equiv 1$ or $3 \pmod 6$. In fact, the planar squag SQG(n) associated with the planar STS(n) given in [5] is the only known construction of a simple squag for each possible n.

In the comments and problems section of [10], Quackenbush has stated that there should be semi-planar squags that are simple squags and each of whose triangles either generates the whole squag or the 9-element subsquag. We observe that any planar squag is semi-planar and the converse is not true.

In the following section, we construct semi-planar squags of cardinality $3n$ which are not planar for all $n > 9$ and $n \equiv 1$ or $3 \pmod 6$.

2 Construction of semi-planar squags of cardinality $3n$

Let $\boldsymbol{P}_i = (P_i; B_i)$ be a triple system with $(P_i; \cdot)$ the corresponding squag for $i = 1, 2$. The direct product $\boldsymbol{P}_1 \times \boldsymbol{P}_2$ of the two triple systems can be obtained from the underlying triple system of the direct product $(P_1; \cdot) \times (P_2; \cdot)$ [6].

Let $\boldsymbol{P}_1 = (P_1; B_1)$ be a planar triple system of cardinality n, and let $P_1 = \{a_1, a_2, \ldots, a_n\}$. We consider the direct product $\boldsymbol{P}_1 \times \boldsymbol{C}_3$, where $\boldsymbol{C}_3$ is the STS(3) on the set $\{1, 2, 3\}$. The direct product $\boldsymbol{P}_1 \times \boldsymbol{C}_3 = (P; B)$ is formed by the usual tripling of $(P_1; B_1)$. Namely, $(P; B)$ is an STS($3n$), where the set of triples B is obtained by:

$$\begin{aligned} B &= \{\{(a_i, 1), (a_j, 2), (a_k, 3)\} \mid \{a_i, a_j, a_k\} \in B_1 \text{ or } a_i = a_j = a_k\} \\ &\quad \cup \{\{(a_i, i), (a_j, i), (a_k, i)\} \mid \{a_i, a_j, a_k\} \in B_1 \text{ and } i \in \{1, 2, 3\}\}. \end{aligned}$$

We denote the squag $(P_1; \cdot_1)$ associated with $\boldsymbol{P}_1$ by $\boldsymbol{Q}_1$ and the squag $(P; \cdot_2)$ associated with $\boldsymbol{P}_1 \times \boldsymbol{C}_3 := (P; B)$ by $\boldsymbol{Q}_2$.

Without loss of generality, we may assume that $A_1 = \{a_1, a_2, a_3\}$ is a block of B_1; then the triple system $(P; B)$ contains the subsystem $(A; R)$, where $A = A_1 \times C_3$ and the set of blocks R is given by:

$$\begin{aligned} R &= \{\{(a_1, i), (a_2, i), (a_3, i)\} : i \in \{1, 2, 3\}\} \\ &\quad \cup \{\{(x, 1), (x, 2), (x, 3)\} : x \in \{a_1, a_2, a_3\}\} \\ &\quad \cup \{\{(x, i), (y, j), (z, k)\} : \{x, y, z\} = \{a_1, a_2, a_3\} \text{ and } \{i, j, k\} = \{1, 2, 3\}\}. \end{aligned}$$

Define on the subset A the set of triples H as follows:

$$\begin{aligned} H = \{ & \{(a_3, 1), (a_3, 2), (a_1, 3)\}, \{(a_2, 2), (a_2, 3), (a_2, 1)\}, \{(a_1, 1), (a_1, 2), (a_3, 3)\}, \\ & \{(a_3, 1), (a_2, 2), (a_1, 1)\}, \{(a_3, 2), (a_2, 3), (a_1, 2)\}, \{(a_1, 3), (a_2, 1), (a_3, 3)\}, \\ & \{(a_3, 1), (a_2, 3), (a_3, 3)\}, \{(a_2, 2), (a_1, 2), (a_1, 3)\}, \{(a_1, 1), (a_2, 1), (a_3, 2)\}, \\ & \{(a_1, 3), (a_2, 3), (a_1, 1)\}, \{(a_2, 2), (a_3, 2), (a_3, 3)\}, \{(a_1, 2), (a_2, 1), (a_3, 1)\}\}. \end{aligned}$$

Each of $(A; R)$ and $(A; H)$ is isomorphic to the affine plane over GF(3).

Using the replacement property by interchanging the two sets of blocks R and H in $(P; B)$, we again get an STS$(3n) = (P; \underline{B})$, where $\underline{B} := (B - R) \cup H$ [6, 9]. In fact, the sub-STS formed by the direct product of $\{a_1, a_2, a_3\}$ and $\{1, 2, 3\}$ is replaced with an isomorphic copy on the same set of points. We denote the squag associated with the STS $(P; \underline{B})$ by $\boldsymbol{Q} = 3 \otimes_A \boldsymbol{Q}_1 = (P; \cdot)$. Observe that the difference between the binary operations "$\cdot_2$" and "$\cdot$" depends only on the elements of A.

Theorem 3 *If $\boldsymbol{Q}_1$ is a planar squag of cardinality n, then the constructed squag $\boldsymbol{Q} = 3 \otimes_A \boldsymbol{Q}_1$ is semi-planar of cardinality $3n$ for all $n > 9$ and $n \equiv 1$ or 3 (mod 6).*

Proof. Let $S = \{(x_1, i_1), (x_2, i_2), (x_3, i_3)\}$ be a triangle in $\boldsymbol{Q}$; i.e., $(x_1, i_1) \cdot (x_2, i_2) \neq (x_3, i_3)$. First we have to prove that the subsquag $\langle S \rangle_Q$ generated by S either generates the whole squag $\boldsymbol{Q}$ or a subsquag of cardinality 9.

In general, there are only four possible cases:

(i) $S \subseteq A$, (ii) $|\langle S \rangle_Q \cap A| = 1$ or 0,
(iii) $|\langle S \rangle_Q \cap A| = 3$, (iv) $S \not\subset A$ and $A \subset \langle S \rangle_Q$.

(i) If $S \subseteq A$, then $\langle S \rangle_Q \subseteq A$, and hence $\langle S \rangle_Q$ is the 9-element subsquag on A.

(ii) If $|\langle S \rangle_Q \cap A| = 1$ or 0, then $\langle S \rangle_Q$ is the same as that of $\langle S \rangle_{\boldsymbol{Q}_2}$, and hence the set of first components of $\langle S \rangle_Q$ is a subsquag of $\boldsymbol{Q}_1$.

Since $|\langle S \rangle_Q \cap A| = 1$ or 0, the set of first components of $\langle S \rangle_Q$ does not equal P_1. Hence the set of first components of $\langle S \rangle_Q$ forms a 3-element subsquag of $\boldsymbol{Q}_1$, which implies that $\langle S \rangle_Q$ is a 9-element subsquag of $\boldsymbol{Q}$.

(iii) If $|\langle S \rangle_Q \cap A| = 3$, then we have $\langle S \rangle_Q \cap A = \{(a_2, 2), (a_2, 3), (a_2, 1)\}$ or any other block in H.

If $\langle S \rangle_Q \cap A = \{(a_2, 2), (a_2, 3), (a_2, 1)\}$, then $\langle S \rangle_Q = \langle S \rangle_{Q_2}$. This means that the set of first components of $\langle S \rangle_Q$ forms a subsquag of $\boldsymbol{Q}_1$. Since $\langle S \rangle_Q \cap A$ is a 3-element subsquag, then the set of first components of $\langle S \rangle_Q$ does not equal P_1. Therefore, the set of first components of $\langle S \rangle_Q$ must be a 3-element subsquag of $\boldsymbol{Q}_1$, which implies that $\langle S \rangle_Q$ is a 9-element subsquag of $\boldsymbol{Q}$.

Let $\langle S \rangle_Q \cap A$ be a 3-element subsquag not equal to $\{(a_2, 2), (a_2, 3), (a_2, 1)\}$. We show that any other choice of $\langle S \rangle_Q \cap A$ leads to a contradiction by considering the following two cases.

First, let $\langle S \rangle_Q \cap A = \{(a_1, 1), (a_2, 2), (a_3, 1)\}$, $\{(a_1, 2), (a_2, 3), (a_3, 2)\}$, $\{(a_1, 3), (a_2, 1), (a_3, 3)\}$, $\{(a_1, 1), (a_2, 1), (a_3, 2)\}$ or $\{(a_1, 2), (a_2, 1), (a_3, 1)\}$.

For any choice in this case, the set of the second components of the elements of $\langle S \rangle_Q \cap A$ is a 2-element subset of $\{1, 2, 3\}$. We can easily see that the set of second components of the elements of $\langle S \rangle_Q$ consists of $\{1, 2, 3\}$ for any choice of $\langle S \rangle_Q \cap A$.

Let $\{i, j, k\} = \{1, 2, 3\}$ and let the maximum number of distinct elements of $\langle S \rangle_Q$, having second components i, be equal to r. Let the values of second components of the three elements of $\langle S\rangle_Q \cap A$ be i, i and j. If $(y,k) \in \langle S\rangle_Q$, then the product of any element (x,i) of $\langle S\rangle_Q$ by (y,k) gives an element of $\langle S\rangle_Q$ having a second component equal to j; that is, $(x,i)\cdot(y,k) = (z,j)$. This means that $\langle S\rangle_Q$ also contains r elements having second components equal to j. Also, let $(y,j) \in \langle S\rangle_Q$; then $(x,i)\cdot(y,j) = (z,k)$, which means that $\langle S\rangle_Q$ also contains r distinct elements having second components equal to k. Accordingly, we may deduce that $\langle S\rangle_Q$ consists exactly of an r-element subset of pairs with second components i, an r-element subset of pairs with second components j and an r-element subset of pairs with second components k.

Each of the r-element subsets of $\langle S\rangle_Q$ with second components j or k forms a subsquag of $\langle S\rangle_Q$. According to Theorem 1, the third r-element subset of $\langle S\rangle_Q$ with second component equal to i must be a subsquag of $\langle S\rangle_Q$, contradicting the choice that $\langle S\rangle_Q \cap A = \{(a_1,i),(a_2,i),(a_3,j)\}$. This is a contradiction.

$$\begin{aligned}\text{Next, let}\quad \langle S\rangle_Q \cap A \;=\;& \{(a_1,1),(a_1,2),(a_3,3)\}, \{(a_2,2),(a_1,2),(a_1,3)\},\\ & \{(a_3,1),(a_3,2),(a_1,3)\}, \{(a_3,1),(a_2,3),(a_3,3)\},\\ & \{(a_1,1),(a_2,3),(a_1,3)\} \text{ or } \{(a_2,2),(a_3,2),(a_3,3)\}.\end{aligned}$$

For any possible choice of $\langle S\rangle_Q \cap A$, the set of first components of the elements of $\langle S\rangle_Q \cap A$ is a 2-element subset of $\{a_1,a_2,a_3\}$. Hence we may say that the set of first components of the elements of $\langle S\rangle_Q$ does not contain all elements of $\{a_1,a_2,a_3\}$.

Let $\{i,j,k\} = \{1,2,3\}$ and let $\langle S\rangle_Q \cap A = \{(a_i,\ell),(a_i,n),(a_j,m)\}$ with $\ell \neq n$ and $\ell,m,n \in \{1,2,3\}$.

Let $(b,\ell) \in \langle S\rangle_Q - A$ and $(b,\ell)\cdot_2(a_i,\ell) = (b,\ell)\cdot(a_i,\ell) = (c,\ell)$; then $(b,\ell)\cdot_2(a_i,n) = (b,\ell)\cdot(a_i,n) = (c,k)$ with $\{\ell,n,k\} = \{1,2,3\}$. Hence $(c,\ell)\cdot_2(c,k) = (c,\ell)\cdot(c,k) = (c,n)$ and accordingly $(c,n)\cdot_2(b,\ell) = (c,n)\cdot(b,\ell) = (a_i,k) \in \langle S\rangle_Q$. On the other hand, we have $(a_i,k) \in A$, contradicting the choice that $\langle S\rangle_Q \cap A = \{(a_i,\ell),(a_i,n),(a_j,m)\}$ and $\{\ell,n,k\} = \{1,2,3\}$. We will get the same contradiction if we choose (b,n) or (b,k) in $\langle S\rangle_Q - A$ instead of (b,ℓ). Therefore the second possible case of $\langle S\rangle_Q \cap A$ is also ruled out.

(iv) Let $S \not\subset A$ and $A \subset \langle S\rangle_Q$, and let $(\langle S\rangle_Q; B_S)$ be the associated STS of the subsquag $\boldsymbol{S} = (\langle S\rangle_Q;\cdot)$; then $(\langle S\rangle_Q; (B_S - H) \cup R)$ is a sub-STS of $(P;B)$ (the associated STS of $\boldsymbol{Q}_2$). As a consequence, the associated squag of $(\langle S\rangle_Q; (B_S - H) \cup R)$ is equal to $\langle S\rangle_{Q_2}$. This means that $\langle S\rangle_Q$ and $\langle S\rangle_{Q_2}$ have the same set of points. Indeed, $\langle S\rangle_Q$ differs from $\langle S\rangle_{Q_2}$ in the binary operations.

On the other hand, since $S \not\subset A$, there is an element $(x,i) \in \langle S\rangle_Q - A$. Hence we may assume that $\langle S\rangle_{Q_2}$ contains three elements $(x,i),(y,i),(z,i)$ with x,y,z forming a triangle in $\boldsymbol{Q}_1$. Consequently, the elements (x,i), (y,i) and (z,i) form a triangle in $\boldsymbol{Q}_2$. Since $\boldsymbol{Q}_1$ is planar, $\boldsymbol{Q}_1 \times \{i\}$ is a subsquag of $\langle S\rangle_{Q_2}$. Also, $\langle S\rangle_{Q_2}$ contains another element (a,j) with $i \neq j$, so $(a,j)\cdot_2(\boldsymbol{Q}_1 \times \{i\}) = \boldsymbol{Q}_1 \times \{k\}$ forms a subsquag of $\langle S\rangle_{Q_2}$, where $\{i,j,k\} = \{1,2,3\}$. According to Theorem 1, $\boldsymbol{Q}_1 \times \{j\}$ also forms a subsquag of $\langle S\rangle_{Q_2}$. Hence $\langle S\rangle_{Q_2} = \boldsymbol{Q}_2$, which implies that $\langle S\rangle_Q = \boldsymbol{Q}$. This completes the proof of the first part of the theorem.

Now we need only show that $\boldsymbol{Q}$ is a simple squag. Assume that $\boldsymbol{Q}$ has a proper congruence θ. Since $[(x,i)]\theta$ is a subsquag of $\boldsymbol{Q}$, the cardinality of $[(x,i)]\theta$ must be equal to 9 or 3. On the other hand, for any 3-element subsquag X of $\boldsymbol{Q}$, the set $[X]\theta$ forms a subsquag of $\boldsymbol{Q}$. Hence, if $|[(x,i)]\theta| = 9$, then there is a 3-element subsquag X such that the subsquag $[X]\theta$ is of cardinality 27. Therefore, in light of the first part of the proof, we may say that the case $|[(x,i)]\theta| = 9$ is ruled out.

If $|[(x,i)]\theta| = 3$, then we will choose $(x,i) \in A$ and $x \neq a_2$; i.e., $x = a_1$ or a_3. If $[(x,i)]\theta \not\subset A$, then $[A]\theta$ is a subsquag of $\boldsymbol{Q}$ of cardinality 27. If $[(x,i)]\theta \subseteq A$, we may choose a 3-element subsquag $X_1 = \{x, x_2, x_3\}$ of $\boldsymbol{Q}_1$ satisfying $X_1 \cap A_1 = \{x\}$ where $x \neq a_2$. Then $X_2 = X_1 \times \{1,2,3\}$ forms a subsquag of each of $\boldsymbol{Q}_2$ and $\boldsymbol{Q}$ with $X_2 \cap A = \{(x,1),(x,2),(x,3)\}$, where $\{(x,1),(x,2),(x,3)\}$ is not a block of $\boldsymbol{Q}$. This means that $[X_2]\theta$ is also a subsquag of $\boldsymbol{Q}$ of cardinality 27. Both cases $[(x,i)]\theta \not\subset A$ and $[(x,i)]\theta \subseteq A$ contradict the fact that the maximal cardinality of a proper subsquag of $\boldsymbol{Q}$ is 9. This means that there is no proper congruence θ of $\boldsymbol{Q}$ with $|[(x,i)]\theta| = 3$. Therefore, $\boldsymbol{Q}$ is a simple squag. This completes the proof of the theorem.

In this article, the Steiner triple system STS(m) associated with a semi-planar squag SQG(m) will be called semi-planar (for much more precision, it may be called semi-9-planar). In other words, one may say that a triple system STS(m) is semi-planar if the STS(m) has no proper a-normal subsystems (see [11]) (equivalently, the corresponding SQG(m) is simple) and has subsystems only of cardinality 1, 3, 9 and m. According to the previous theorem, we may deduce that there is a semi-planar triple system of cardinality $m = 3n$ which is not planar, for all $n > 9$ and $n \equiv 1$ or 3 (mod 6).

We are faced with the question: is there a semi-planar squag of cardinality m which is not planar for the other possible values of m such as $m = 7, 9, 13, 15, 19, 25, 27, \ldots$? Indeed, for $m = 7, 9, 13, 15$ there are only planar squags. Also, we observe that if m is prime or if $m = r \times n$ with r or n not congruent to 1 or 3 (mod 6), then each SQG(m) must be simple. Hence for $m = 19$ each SQG(19) is simple. Indeed, one can easily show that there is a simple SQG(19) having a 9-element subsquag as follows:
For this purpose, we now briefly review the concept of derived triple systems.

A Steiner quadruple system (briefly SQS) is a pair $(P; B)$ where P is a finite set and B is a collection of 4-subsets (called blocks) of P such that every 3-subset of P is contained in exactly one block of B [9]. Let SQS$(m) = (P; B)$ denote a Steiner quadruple system of cardinality m. If one considers $P_x = P - \{x\}$ for any point $x \in P$ and deletes that point from all blocks which contain it, then the resulting system $(P_x; B(x))$ is a triple system where $B(x) = \{b' = b - \{x\} : b \in B \text{ and } x \in b\}$. Now $(P_x; B(x))$ is called a derived triple system [6, 9]. The direct product SQS(10)$\times$ SQS(2) is an SQS(20). Each derived triple system of the Steiner quadruple system SQS(10)$\times$ SQS(2) is an STS(19) having a subSTS(9). This implies that there is a simple SQG(19) which has a 9-element subsquag. Actually this SQS(19) is still not semi-planar.

In the following section we turn our attention to construct a semi-planar SQG(m) for $m = 27$.

3 Construction of a semi-planar SQG(27)

From the foregoing discussion, we may directly say that the number 21 is the smallest known cardinality of a semi-planar squag which is not planar. As a result of the previous construction, we see that given an SQG(7) $= \boldsymbol{Q}_1$ one can construct a semi-planar squag SQG(21) $= \boldsymbol{Q}$ which is not planar.

Theorem 3 is not valid for $n = 9$. In this section, we are only interested in constructing a semi-planar squag of cardinality 27 which is not planar. The techniques used in the above theorem (the replacement property) can be applied to the direct product STS(9) $\times$ STS(3) to get a semi-planar STS(27) $= (P; B)$, where the sets P and B are given by: $P = \{a_i, b_i, c_i \mid i = 1, 2, \ldots, 9\}$ and

$$
\begin{array}{lllllll}
B = & a_1\,a_2\,b_1 & b_1\,b_2\,b_4 & b_1\,c_1\,c_2 & a_2\,a_3\,b_3 & a_2\,b_2\,c_2 & b_3\,c_2\,c_3 \\
& a_4\,a_5\,a_6 & b_4\,b_5\,b_6 & c_4\,c_5\,c_6 & a_1\,b_2\,c_3 & a_1\,a_3\,c_2 & a_2\,c_1\,c_3 \\
& a_7\,a_8\,a_9 & b_7\,b_8\,b_4 & c_7\,c_8\,c_9 & a_1\,b_4\,c_7 & a_2\,b_5\,c_8 & a_3\,b_6\,c_9 \\
& a_1\,a_4\,a_7 & b_1\,b_6\,b_7 & c_1\,c_4\,c_7 & a_1\,b_5\,c_9 & a_2\,b_6\,c_7 & a_3\,b_5\,c_7 \\
& a_2\,a_5\,a_8 & b_2\,b_5\,b_8 & c_2\,c_5\,c_8 & a_1\,b_6\,c_8 & a_2\,b_4\,c_9 & a_3\,b_4\,c_8 \\
& a_3\,a_6\,a_9 & b_3\,b_6\,b_2 & c_3\,c_6\,c_9 & b_1\,a_3\,c_3 & a_1\,c_1\,b_3 & a_3\,b_2\,c_1 \\
& a_1\,a_5\,a_9 & b_1\,b_5\,b_9 & c_1\,c_5\,c_9 & a_1\,c_4\,b_7 & a_2\,b_8\,c_5 & a_3\,b_9\,c_6 \\
& a_3\,a_5\,a_7 & b_3\,b_5\,b_7 & c_3\,c_5\,c_7 & a_1\,c_5\,b_9 & a_2\,b_7\,c_6 & a_3\,b_7\,c_5 \\
& a_1\,a_6\,a_8 & b_9\,b_6\,b_8 & c_1\,c_6\,c_8 & a_1\,c_6\,b_8 & a_2\,b_9\,c_4 & a_3\,b_8\,c_4 \\
& a_2\,a_6\,a_7 & b_1\,b_3\,b_8 & c_2\,c_6\,c_7 & a_2\,a_4\,a_9 & b_2\,b_7\,b_9 & c_2\,c_4\,c_9 \\
& a_7\,b_7\,c_7 & a_8\,b_8\,c_8 & a_9\,b_9\,c_9 & a_3\,a_4\,a_8 & b_3\,b_4\,b_9 & c_3\,c_4\,c_8 \\
& a_7\,b_8\,c_9 & a_8\,b_7\,c_9 & a_9\,b_7\,c_8 & a_4\,b_4\,c_4 & a_5\,b_5\,c_5 & a_6\,b_6\,c_6 \\
& a_7\,b_1\,c_4 & a_8\,b_2\,c_5 & a_9\,b_3\,c_6 & a_4\,b_5\,c_6 & a_5\,b_4\,c_6 & a_6\,b_4\,c_5 \\
& a_7\,b_3\,c_5 & a_8\,b_1\,c_6 & a_9\,b_1\,c_5 & a_4\,b_1\,c_7 & a_5\,b_2\,c_8 & a_6\,b_3\,c_9 \\
& a_7\,b_2\,c_6 & a_8\,b_3\,c_4 & a_9\,b_2\,c_4 & a_4\,b_3\,c_8 & a_5\,b_3\,c_7 & a_6\,b_1\,c_8 \\
& a_7\,b_9\,c_8 & a_8\,b_9\,c_7 & a_9\,b_8\,c_7 & a_4\,b_2\,c_9 & a_5\,b_1\,c_9 & a_6\,b_2\,c_7 \\
& a_7\,b_4\,c_1 & a_8\,b_5\,c_2 & a_9\,b_6\,c_3 & a_4\,b_6\,c_5 & a_5\,b_6\,c_4 & a_6\,b_5\,c_4 \\
& a_7\,b_5\,c_3 & a_8\,b_6\,c_1 & a_9\,b_5\,c_1 & a_4\,b_7\,c_1 & a_5\,b_8\,c_2 & a_6\,b_9\,c_3 \\
& a_7\,b_6\,c_2 & a_8\,b_4\,c_3 & a_9\,b_4\,c_2 & a_4\,b_8\,c_3 & a_5\,b_7\,c_3 & a_6\,b_8\,c_1 \\
& a_4\,b_9\,c_2 & a_5\,b_9\,c_1 & a_6\,b_7\,c_2 & & &
\end{array}
$$

Let $\boldsymbol{Q} = (P; \cdot)$ be the squag associated with the STS(27) $= (P; B)$. In the following, we want to prove that $\boldsymbol{Q} = (P; \cdot)$ is a semi-planar SQG(27).

It is clear that the set $A = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9\}$ is a sub-SQG(9); if A is normal, then $a_1 \cdot A$ must also be a sub-SQG(9) [10, 2]. Indeed, we have:

$$
\begin{aligned}
a_1 \cdot A &= \{a_1 \cdot b_1,\ a_1 \cdot b_2,\ a_1 \cdot b_3,\ a_1 \cdot b_4,\ a_1 \cdot b_5,\ a_1 \cdot b_6,\ a_1 \cdot b_7,\ a_1 \cdot b_8,\ a_1 \cdot b_9\} \\
&= \{a_2, c_3, c_1, c_7, c_9, c_8, c_4, c_6, c_5\}.
\end{aligned}
$$

Moreover, $a_2 \cdot c_4 = b_9$, so $a_1 \cdot A$ is not a sub-SQG(9) and hence A is not normal in

$\boldsymbol{Q}$. Since A is not normal, any normal sub-SQG(9) $= N$ of $\boldsymbol{Q}$ must intersect A, so that $|N \cap A| = 3$.

To prove that $\boldsymbol{Q}$ has no congruence θ with $|[x]\theta| = 9$, we need only prove that there is no normal sub-SQG(9) containing a block of A and a fixed element $x \in P - A$.

In the following, we calculate $\langle b_i, b_j, b_k, x\rangle$ in which the fixed element $x \in P - A$ may be chosen for instance as the element a_1 and $\{b_i, b_j, b_k\}$ is any block on A. This means that we have to calculate 12 different cases.

Consider the sub-SQG $\langle b_3, b_5, b_7, a_1\rangle$ in $\boldsymbol{Q}$; then we have: $a_1 \cdot b_3 = c_1$, $a_1 \cdot b_5 = c_9$, $a_1 \cdot b_7 = c_4$, $c_1 \cdot b_5 = a_9$, $c_1 \cdot b_7 = a_4$, $c_1 \cdot c_9 = c_5$, $c_1 \cdot c_4 = c_7$, $c_4 \cdot c_9 = c_2$, $c_1 \cdot c_2 = b_1$, $b_1 \cdot b_3 = b_8$. Then $\langle b_3, b_5, b_7, a_1\rangle$ contains more than 13 elements. This means that $\langle b_3, b_5, b_7, a_1\rangle = \boldsymbol{Q}$.

Consider the sub-SQG $\langle b_2, b_5, b_8, a_1\rangle$ in $\boldsymbol{Q}$; we have:
$a_1 \cdot b_2 = c_3$, $a_1 \cdot b_5 = c_9$, $a_1 \cdot b_8 = c_6$, $c_3 \cdot c_9 = c_6$, $c_3 \cdot b_8 = a_4$, $c_3 \cdot b_5 = a_7$, $a_4 \cdot b_2 = c_9$, $a_4 \cdot b_5 = c_6$, $c_3 \cdot c_9 = a_8$. Then $\langle b_4, b_7, b_8, a_1\rangle = SQG(9) = \{a_1, b_2, b_5, b_8, c_6, c_3, c_9, a_4, a_7\}$. On the other hand, we have: $a_2 \cdot \langle b_4, b_7, b_8, a_1\rangle = a_2 \cdot \{a_1, b_2, b_5, b_8, c_6, c_3, c_9, a_4, a_7\} = \{b_1, c_2, c_8, c_5, b_7, c_1, b_4, a_9, a_6\}$ and $c_1 \cdot c_5 = c_9$. This means that the SQG(9) $= \langle b_2, b_5, b_8, a_1\rangle$ is not normal in $\boldsymbol{Q}$.

Similarly, by considering the other 10 cases one can prove that $\langle b_i, b_j, b_k, a_1\rangle_{\boldsymbol{Q}}$ is the whole squag $\boldsymbol{Q}$ or a sub-SQG(9) but not normal, where $\{b_i, b_j, b_k\}$ is always a block on A.

Therefore, we may say that $\boldsymbol{Q}$ has no congruence θ with $|[x]\theta| = 9$. Hence $\boldsymbol{Q}$ also has no congruence θ with $|[x]\theta| = 3$; thus $\boldsymbol{Q}$ is a simple squag. This implies that $\boldsymbol{Q}$ is a semi-planar squag of cardinality 27 which is not planar.

Finally, we may improve the result of Theorem 3 as follows:

Theorem 4 *There is a semi-planar squag of cardinality* $3n$ *which is not planar, for all* $n > 3$ *and* $n \equiv 1$ *or* 3 *(mod* 6*).*

In other words, one may say that there is a semi-planar triple system of cardinality m which is not planar, for all $m > 9$ and $m \equiv 3$ or 9 (mod 18).

Quackenbush [10] has proved that the variety $V(\boldsymbol{Q})$ generated by a simple planar squag $\boldsymbol{Q}$ has only two subdirectly irreducible squags $\boldsymbol{Q}$ and the 3-element squag SQG(3) and then $V(\boldsymbol{Q})$ covers the smallest nontrivial subvariety (the class of all medial squags).

Similarly, if $\boldsymbol{Q}$ is a semi-planar squag, then one can prove that the variety $V(\boldsymbol{Q})$ generated by $\boldsymbol{Q}$ has only two subdirectly irreducible squags $\boldsymbol{Q}$ and the 3-element squag SQG(3). And hence we deduce the same result that each semi-planar squag $\boldsymbol{Q}$ generates another variety $V(\boldsymbol{Q})$ which covers also the smallest nontrivial subvariety (the class of all medial squags).

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References

[1] M. H. Armanious, S. F. Tadros and N. M. Dhshan, Subdirectly irreducible squags of cardinality $3n$, Ars Combinatoria **64** (2002), 199–210.

[2] M. H. Armanious, Subsquags and normal subsquags, Ars Combinatoria **59** (2001), 241–243.

[3] R. H. Bruck, A survey of binary systems, Springer-Verlag, Berlin Heidelberg, New York (1971).

[4] O. Chein, H. O. Pflugfelder and J. D. H. Smith, Quasigroups and loops: theory and aplications, Sigma Series in Pure Math. **8**, Heldermann Verlag, Berlin, 1990.

[5] J. Doyen, Sur la structure de certains systems triples de Steiner, Math. Z. **111**(1969) 289–300.

[6] B. Ganter and H. Werner, Co-ordinatizing Steiner systems, Ann. Discrete Math. **7** (1980) 3–24.

[7] G. Gratzer, Universal Algebra, Springer Verlag New York, Heidelberg, and Berlin, 2nd edition, 1979.

[8] A. J. Guelzow, Representation of finite nilpotent squags, Discrete Mathematics **154** (1996), 63–76.

[9] C. C. Lindner and A. Rosa, Steiner quadruple systems: a survey, Discrete Mathematics **21** (1979), 147–181.

[10] R. W. Quackenbush, Varieties of Steiner loops and Steiner quasigroups, Canadian J. Math. **28** (1976), 1187–1198.

[11] R. W. Quackenbush, Nilpotent block design I: Basic concepts for Steiner triple and quadruple systems, J. Combin. Designs **7** (1999), 157–171.

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