A construction of Hadamard matrices from BIBD$(2k^2 - 2k + 1, k, 1)$

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Abstract

It is shown that the existence of a BIB design with parameters $v = 2k^2 - 2k + 1$, $b = 2v$, $r = 2k$, k, $\lambda = 1$ implies the existence of Hadamard matrices of orders $4v$ and $8vt$, where t is an integer for which an orthogonal design of order $4t$ and type (t, t, t, t) exists.

1 Introduction

We first recall the following definitions.

A balanced incomplete block (BIB) design with parameters v, b, r, k, λ is a block design, denoted by BIBD(v, k, λ), with v points and b blocks of size k each such that every point occurs in r blocks and any two distinct points occur together in exactly λ blocks.

A partially balanced incomplete block (PBIB) design, based on a 2-associate association scheme, with parameters $v, b, r, k, \lambda_1, \lambda_2$ is a block design with v points and b blocks of size k each such that every point occurs in r blocks and any two distinct points being the ith associates occur together in exactly λ_i blocks for $i = 1, 2$ (see Raghavarao [5]).

A PBIB design with parameters $v, b, r, k, \lambda_1, \lambda_2$ is called a partial geometry (r, k, t) if (i) $\lambda_1 = 1,\ \lambda_2 = 0$, and (ii) when a point α is not incident with a block E, there are t blocks through α that intersect E (see Bose [2]). Now let C_1 be the set of pairs $\alpha\beta$

such that points α, β ($\alpha \neq \beta$) are incident with a block of the design, and let C_2 be the set of the remaining pairs of distinct points. Then C_1 and C_2 are the classes of the underlying 2-associate association scheme, yielding two association matrices S_1 and S_2 of order v (see the proof of Theorem 2.1 later). The parameters are expressed by

$$\begin{gathered} v = k[(r-1)(k-1)+t]/t,\ b = r[(r-1)(k-1)+t]/t, n_1 = r(k-1), \\ n_2 = (r-1)(k-1)(k-t)/t,\ p_{11}^1 = (t-1)(r-1)+k-2, \\ p_{12}^1 = (r-1)(k-t),\ p_{11}^2 = rt,\ p_{12}^2 = r(k-t-1), \end{gathered} \tag{1.1}$$

where $1 \leq t \leq \min\{r, k\}$ (see Raghavarao [5; p.192]).

An Hadamard matrix H of order v is a $v \times v$ matrix with entries ± 1 such that $HH' = vI_v$.

The Williamson matrices are four circulant symmetric $(1, -1)$-matrices A, B, C, D of order v satisfying

(i) $MN = NM$ for $M, N \in \{A, B, C, D\}$,

(ii) $A^2 + B^2 + C^2 + D^2 = 4vI_v$.

An orthogonal design, denoted by $\mathrm{OD}(4t; t, t, t, t)$, of order $4t$ and type $(t,\ t,\ t,\ t)$ is a matrix P of order $4t$ with entries 0, $\pm x_1$, $\pm x_2$, $\pm x_3$, $\pm x_4$ (x_i being commutative indeterminates) such that $PP' = t(x_1^2 + x_2^2 + x_3^2 + x_4^2)I_{4t}$. Such designs are known to exist for infinitely many values of t (see Colbourn and Dinitz [3; p.404], Seberry and Yamada [6]).

The following lemma (see Baumert and Hall [1]) extends the Williamson method.

Lemma 1.1. *The existence of an* $\mathrm{OD}(4t; t, t, t, t)$ *and four Williamson matrices of order* v *implies the existence of an Hadamard matrix of order* $4vt$.

The Hadamard matrices given by Lemma 1.1 are said to be of Williamson type. Methods of their construction have been found in abundance in the literature (see Colbourn and Dinitz [3; IV, Chapters 24 and 31], Seberry and Yamada [6]).

The purpose of this note is to provide a new method of constructing Hadamard matrices from a series of BIB designs with $\lambda = 1$.

2 Statement

Theorem 2.1. *The existence of a BIB design with parameters*

$$v = 2k^2 - 2k + 1,\ b = 2v,\ r = 2k,\ k\,(\geq 2),\ \lambda = 1$$

implies the existence of two Hadamard matrices:

(i) of order $4(2k^2 - 2k + 1)$*,*

(ii) of Williamson type with order $8t(2k^2 - 2k + 1)$*, where* t *is an integer for which an* $OD(4t; t, t, t, t)$ *exists.*

Proof. It is clear that any two blocks of the BIB design are incident with at most one point. Note that if a block E is not incident with a point α, then there are k points $x_1, x_2, ..., x_k$ incident with E such that each pair (x_i, α) is incident with one and only one block. Thus the dual of this design is a partial geometry with $v' = 4k^2 - 4k + 2, b' = 2k^2 - 2k + 1, r' = k, k' = 2k,\ \lambda_1' = 1, \lambda_2' = 0, t' = k$. Hence by (1.1) it follows that

$$\begin{gathered} n_1 = k(2k-1), n_2 = (k-1)(2k-1), p_{11}^1 = k^2 - 1, \\ p_{12}^1 = k(k-1), p_{11}^2 = k^2, p_{12}^2 = k(k-1), \\ p_{ij}^\ell = p_{ji}^\ell,\ i, j, \ell = 1, 2. \end{gathered} \tag{2.1}$$

Let S_1 and S_2 be the association matrices, i.e., satisfy

$$S_i S_j = n_i \delta_{ij} I_{v'} + \sum_{\ell=1}^{2} p_{ij}^\ell S_\ell,\ i, j = 1, 2, \tag{2.2}$$

where δ_{ij} denotes the Kronecker delta (see also Raghavarao [5]). Consider the symmetric matrices A, B, C, D given by

$$\begin{aligned} A &= I_{v'} + a_1 S_1 + a_2 S_2,\ B = I_{v'} + b_1 S_1 + b_2 S_2, \\ C &= I_{v'} + c_1 S_1 + c_2 S_2,\ D = I_{v'} + d_1 S_1 + d_2 S_2, \end{aligned} \tag{2.3}$$

where $a_i, b_i, c_i, d_i = \pm 1$ are to be chosen such that A, B, C, D are the Williamson matrices. A sufficient condition for this is, by the definition,

$$A^2 + B^2 + C^2 + D^2 = 4v' I_{v'}, \tag{2.4}$$

where $I_{v'}$ denotes the identity matrix of order v'. Let $\sum f(x_1, x_2)$ stand for $f(a_1, a_2)$ $+ f(b_1, b_2) + f(c_1, c_2) + f(d_1, d_2)$. Then (2.4) becomes $\sum (I_{v'} + x_1 S_1 + x_2 S_2)^2 = 4v' I_{v'}$. Using (2.2), this can be expanded as

$$\begin{aligned} &4I_{v'} + 4(n_1 I_{v'} + p_{11}^1 S_1 + p_{11}^2 S_2) + 4(n_2 I_{v'} + p_{22}^1 S_1 + p_{22}^2 S_2) \\ &\quad + 2\sum x_1 S_1 + 2\sum x_2 S_2 + 2\sum x_1 x_2 (p_{12}^1 S_1 + p_{12}^2 S_2) = 4v' I_{v'}. \end{aligned} \tag{2.5}$$

It follows from (2.1) that (2.5) is satisfied if

$$\sum x_1 = 0 = \sum x_2,\quad \sum x_1 x_2 = -4,$$

i.e.,

$$a_1 + b_1 + c_1 + d_1 = a_2 + b_2 + c_2 + d_2 = 0,\ a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2 = -4. \tag{2.6}$$

It is easy to choose a_i, b_i, c_i, d_i satisfying (2.6), which make A, B, C, D the Williamson matrices of order $2v$. Now the existence of Hadamard matrices of order $8vt$ follows from Lemma 1.1. Using the preceding method it can also be seen that $A^2 + B^2 =$

$2v'I_{v'}$ provided $a_1 + b_1 = 0 = a_2 + b_2$ and $a_1a_2 + b_1b_2 = -2$. Thus taking $a_1 = b_2 = 1$ and $b_1 = a_2 = -1$ in (2.3), we finally have an Hadamard matrix

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

of order $2v' = 4(2k^2 - 2k + 1)$. Hence the proof is completed. □

Remark 2.1. The idea behind the construction is to observe that the dual of a BIB design with parameters $v = 2k^2 - 2k + 1$, $b = 2v, r = 2k$, $k(> 1)$, $\lambda = 1$ is a partial geometry and to use the association matrix on the partial geometry to obtain (i) two commuting symmetric $(1, -1)$-matrices and (ii) four Williamson matrices, of appropriate orders. These are just linear combinations (with coefficient ± 1) of the identity matrix, the association matrix and its complement.

In the following table, we list known BIB designs having the parameters of Theorem 2.1 and yielding Hadamard matrices for infinitely many values of t.

Table: BIB designs and the corresponding Hadamard matrices

BIBD No. of Mathon and Rosa [4]	Parameters (v, b, r, k, λ)	Order of Hadamard matrices
existent	$(5, 10, 4, 2, 1)$	$20, 40\text{t}$
8	$(13, 26, 6, 3, 1)$	$52, 104\text{t}$
22	$(25, 50, 8, 4, 1)$	$100, 200\text{t}$
45	$(41, 82, 10, 5, 1)$	$164, 328\text{t}$

The existence of the BIB designs are unknown for $k \geq 6$ (see Mathon and Rosa [4; Table 1.28]). New Hadamard matrices are likely to be discovered when such BIB designs are known, as seen from Theorem 2.1 and Remark 2.2 below.

Remark 2.2. Though it is generally conjectured that an OD$(4t; t, t, t, t)$ exists for all t, the existence is known for $t \leq 72$ (see, e.g., Colbourn and Dinitz [3; p.404]). In fact, within the range of $t \leq 130$, the unknown cases are $t =$ 73, 79, 83, 89, 97, 103, 107, 109, 113, 127.

References

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