The decycling number of graphs*

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Abstract

For a graph G and $S \subset V(G)$, if $G - S$ is acyclic, then S is said to be a decycling set of G. The size of a smallest decycling set of G is called the decycling number of G. The purpose of this paper is to provide a review of recent results and open problems on this parameter. Results to be reviewed include recent work on decycling numbers of cubes, grids and snakes and bounds on the decycling number of cubic graphs, and expected bounds on the decycling numbers of random regular graphs. A structural description of graphs with a fixed decycling number based on connectivity is also presented.

1 Decycling a Graph

The minimum number of edges whose removal eliminates all cycles in a given graph has been known as the *cycle rank* of the graph, and this parameter has a simple expression: $b(G) = \|G\| - |G| + \omega$ ([14], Chapter 4) where, as in [12], $|G|$ and $\|G\|$ are respectively the number of vertices and the number of edges of G and ω is the number of components of G. The corresponding problem of eliminating all cycles from a graph by means of deletion of vertices goes back at least to the work of Kirchhoff [16] on spanning trees. This problem does not have a simple solution.

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This latter question is difficult even for some simply defined graphs. For general graph theoretic notations, we follow Diestel [12].

Let G be a graph. If $S \subseteq V(G)$ and $G-S$ is acyclic, then S is said to be a *decycling set* of G. The smallest size of a decycling set of G is said to be the *decycling number* of G and is denoted by $\phi(G)$. A decycling set of this cardinality is said to be a *minimum decycling set*. Determining the decycling number of a graph is equivalent to finding the greatest order of an induced forest and the sum of the two numbers equals the order of the graph. It was shown in [15], that determining the decycling number of an arbitrary graph is NP-complete (see Problem 7 on the feedback node set in the main theorem of [15], which asks for a set $S \subseteq V$ of minimum cardinality in a digraph G such that every directed cycle of G contains a member of S). In fact, the computation of decycling numbers of the following families of graphs is shown to be NP-hard: planar graphs, bipartite graphs, perfect graphs, and comparability graphs (graphs with a transitive orientation). On the other hand, the problem is known to be polynomial for various other families, including cubic graphs (see [17, 25]), permutation graphs (see Liang [18]), and interval and comparability graphs (see Liang and Chang [19]). These results naturally suggest further investigations as to good bounds on the parameter and exact results when possible.

In [4], Bafna, Berman and Fujito have found a polynomial time algorithm for a decycling set of cardinality at most $2\phi(G)$ in an arbitrary graph G (this is referred to as a 2-approximation).

Clearly, $\phi(G) = 0$ if and only if G is a forest, and $\phi(G) = 1$ if and only if G has at least one cycle and a vertex is on all of its cycles. It is also easy to see that $\phi(K_p) = p-2$ and $\phi(K_{r,s}) = r-1$ if $r \leq s$. This is easily extendable to all complete multipartite graphs. For the Petersen graph P, $\phi(P) = 3$.

All results cited in this section are from [8].

Lemma 1.1 *Let G be a connected graph with degrees $d_1, d_2, \ldots, d_p$ in non-increasing order. If $\phi(G) = s$, then*

$$\sum_{i=1}^{s}(d_i - 1) \geq \|G\| - |G| + 1.$$

Corollary 1.1 *If G is a connected graph with maximum degree Δ, then*

$$\phi(G) \geq \frac{\|G\| - |G| + 1}{\Delta - 1}.$$

For graphs regular of degree r, one may wonder whether there is a constant c such that

$$\phi(G) \leq \frac{\|G\| - |G| + 1}{r - 1} + c?$$

This is not the case, even for cubic graphs (graphs that are regular of degree 3). Let G be any cubic graph of order $2n$. Replace each vertex of G with a triangle and denote the resulting graph by H. Then $|H| = 6n$ and $\phi(H) \geq 2n$. Thus

$$\phi(H) - \frac{\|H\| - |H| + 1}{2} \geq 2n - \frac{3n+1}{2} \geq 2n - \frac{3n}{2} = \frac{n}{2}.$$

Problem 1.1 *Which cubic graphs G with $|G| = 2n$ satisfy $\phi(G) = \left\lceil \frac{n+1}{2} \right\rceil$?*

Problem 1.2 *Which cubic planar graphs G with $|G| = 2n$ satisfy $\phi(G) = \left\lceil \frac{n+1}{2} \right\rceil$?*

While these problems are still open, a polynomial time algorithm has been obtained in [17] for finding a minimum decycling set of vertices in a cubic graph. Bounds on the decycling number of cubic graphs and expected bounds on the decycling numbers of random regular graphs will be reviewed in Sections 5 and 6.

Let $S \subseteq V(G)$ and let $G|_S$ denote subgraph induced by the set S in G. Define

$$\sigma(S) = \sum_{v \in S} d(v), \quad \epsilon(S) = \|G|_S\|$$

Define the *outlay* of S to be

$$\theta(S) = \sigma(S) - |S| - \epsilon(S) - \omega(G - S) + 1.$$

Lemma 1.2 *Let G be a connected graph. If S is a decycling set of G, then*

$$\theta(S) = \|G\| - |G| + 1.$$

Lemma 1.3 *If G and H are homeomorphic graphs then $\phi(G) = \phi(H)$.*

Denote by $\alpha(G)$ and $\beta(G)$ the independence and the covering numbers of G respectively. Then these two parameters are related by the equality $\alpha(G) + \beta(G) = |G|$.

Lemma 1.4 *For any nonnull graph G, $\phi(G) \leq \beta(G) - 1$.*

Let G and H be two graphs. Then the *cartesian product* $G \times H$ of G and H is defined by assigning

$$V(G \times H) = V(G) \times V(H),$$

$$E(G \times H) = \{\{(x, y), (x', y')\} : [x = x' \wedge yy' \in E(H)] \vee [y = y' \wedge xx' \in E(G)]\}.$$

Theorem 1.1 *For any graph G,*

$$2\phi(G) \leq \phi(K_2 \times G) \leq \phi(G) + \beta(G).$$

The equalities in Theorem 1.1 are satisfied by a graph of each possible order. For example, if $G = K_p^c$, then $\phi(G) = \phi(K_2 \times G) = 0$ and both equalities hold. Also, for the equality to the lower bound, if $p \geq 2$ then $\phi(K_2 \times K_p) = 2p - 4 = 2\phi(K_p)$. The path of order p gives equality to the upper bound.

2 Cubes

As we have remarked in the previous section, the determination of the decycling number of an arbitrary graph is NP-complete [14]. However, results on the decycling number of several classes of simply defined graphs have been obtained in [5, 6, 8].

In [8], upper and lower bounds for the decycling numbers of cubes and grids were obtained. The results in [5, 8] will be reviewed in this section.

The n-dimensional cube (or n-cube) Q_n can be defined recursively: $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$. An equivalent formulation, as the graph having the 2^n n-tuples of 0's and 1's as vertices with two vertices adjacent if they differ in exactly one position, gives a coordinatization of the cube. The following result of [8] gives a lower bound on $\phi(Q_n)$.

Lemma 2.1 *Let $n \geq 2$. Then*
(1) $\phi(Q_n) \geq 2\phi(Q_{n-1})$.
(2) $\phi(Q_n) \geq 2^{n-1} - \dfrac{2^{n-1}-1}{n-1}$.

For $n \leq 8$, $\phi(Q_n)$ was determined in [8].

n	1	2	3	4	5	6	7	8
$\phi(Q_n)$	0	1	3	6	14	28	56	112

Upper and lower bounds for the decycling nunbers of n-cubes for $9 \leq n \leq 13$ were also obtained in [8].

Cubes	Lower bounds for ϕ	Upper bounds for ϕ
Q_9	224	312
Q_{10}	448	606
Q_{11}	896	1184
Q_{12}	1792	2224
Q_{13}	3584	4680

These results were improved in [5], from which all results in the remainder of this section are cited.

Lemma 2.2 *For any bipartite graph G with partite sets of cardinality r and s with $r \leq s$, $\phi(G) \leq r - 1$.*

Since the cartesian product of two bipartite graphs is a bipartite graph, Q_n is bipartite. With this observation, the above upper bounds can be lowered a little. For example, they are $255, 511, 1023, 2047$ and 4095 respectively. Applying Lemma 2.1 (2), one can lift the lower bound a little as well. That is, these lower bounds can be lifted to $225, 456, 922, 1862$ and 3755 respectively. However, one can still go a little further.

Lemma 2.3 *If e and f are two adjacent edges of the n-cube Q_n, then there is a unique 4-cycle containing $\{e, f\}$.*

Corollary 2.1 (1) *Every edge of Q_n is contained in precisely $n-1$ 4-cycles;*
(2) *If $n \geq 3$, then Q_n has precisely $n(n-1)2^{n-3}$ 4-cycles.*

Denote by $\rho(u, v)$ the distance between points u and v. Let $x_0 \in Q_n$ and define

$$V_k(Q_n, x_0) = \{x \in Q_n : \rho(x, x_0) = k\}.$$

Then there is a nice connection between sizes of the sets $V_k(Q_n, x_0)$ and the binomial coefficients.

Theorem 2.1

$$|V_k(Q_n, x_0)| = \binom{n}{k}, \quad 0 \leq k \leq n.$$

Let the two partite sets of Q_n be denoted by X_n and Y_n. Then $|X_n| = |Y_n| = 2^{n-1}$. If $x \in X_n$, then the induced subgraph $G|_{\{x\} \cup N(x)}$ is a star $S(x)$ of order $n+1$ centered at x. Call a vertex of degree 1 of a tree a *leaf*.

Theorem 2.2 *For $n \geq 2$, let $x, x' \in X_n$. If $S(x)$ and $S(x')$ are stars then either $S(x) \cap S(x') = \emptyset$ or $S(x)$ and $S(x')$ have precisely two leaves in common.*

Theorem 2.3 (1) $225 \leq \phi(Q_9) \leq 237$;
(2) $456 \leq \phi(Q_{10}) \leq 493$;
(3) $922 \leq \phi(Q_{11}) \leq 1005$;
(4) $1862 \leq \phi(Q_{12}) \leq 2029$;
(5) $3755 \leq \phi(Q_{13}) \leq 4077$.

To obtain the upper bounds given in this theorem, decycling sets of the given cardinality were to be exhibited in each case and Lemma 2.1 (2) is to be applied. For a proof, the reader is referred to [5].

3 Grids

Another class of graphs for which the decycling number has been studied to some precision are the grid graphs $P_m \times P_n$, where P_m is the path with m vertices. A standard notation corresponding to matrix notation is to be adopted for convenience. Thus the ith vertex in the jth copy of P_m will be denoted $v_{i,j}$.

If S is a set of vertices in $P_m \times P_n$, then $S(j)$ will denote the vertices of S in the jth column, and put $S(j, k) = S(j) \cup S(j+1) \cup \cdots \cup S(k)$. Let $N(j) = |S(j)|$ and $N(j, k) = |S(j, k)|$.

The following results were obtained in [8]. First is a general lower bound. This theorem, together with Theorem 3.4, provides general asymptotic bounds for all grids.

Theorem 3.1 *If $m, n \geq 3$, then*

$$\phi(P_m \times P_n) \geq \left\lfloor \frac{mn - m - n + 2}{3} \right\rfloor.$$

Theorem 3.2 *For $n \geq 4$,*

(1) $\phi(P_2 \times P_n) = \left\lfloor \frac{n}{2} \right\rfloor$;

(2) $\phi(P_3 \times P_n) = \left\lfloor \frac{3n}{4} \right\rfloor$;

(3) $\phi(P_4 \times P_n) = n$;

(4) $\phi(P_5 \times P_n) = \left\lfloor \frac{3n}{2} \right\rfloor - \left\lfloor \frac{n}{8} \right\rfloor - 1$;

(5) $\phi(P_6 \times P_n) = \left\lfloor \frac{5n}{3} \right\rfloor$;

(6) $\phi(P_7 \times P_n) = 2n - 1$.

Theorem 3.3 *Let $m = 6q + r$ and $n = 6s + t$ with $1 \leq r, t \leq 6$. Then*

$$\phi(P_m \times P_n) \leq \min \{q(2n-1) + \phi(P_r \times P_n), s(2m-1) + \phi(P_t \times P_m)\}.$$

Theorem 3.4 *For $m, n > 2$,*

$$\phi(P_m \times P_n) \leq \frac{(m+4)(2n-1)}{6} = \frac{mn}{3} + \frac{8n-m-4}{3}.$$

Theorem 3.5 *Suppose that $n \equiv 0 \pmod 2$ and $m = 3r + 1$. Then*

$$\phi(P_m \times P_n) = rn - r + 1.$$

Theorem 3.6 *If S is a minimum decycling set of $P_m \times P_n$ with*

$$\phi(P_m \times P_n) = \left\lceil \frac{mn - m - n + 2}{3} \right\rceil$$

and

$$T = \{v_{ij} : i = 2, 4, \cdots, 3m-2; j = 2, 4, \cdots, 2n-2\}$$

then $S \cup T$ is a minimum decycling set of $P_{2m-1} \times P_{2n-1}$.

Theorem 3.7 *For any positive integers r and s*

$$\phi(P_{6r+1} \times P_{4r-1}) = 8rs - 4r + 1.$$

This theorem, obtained in [5], covers some cases other than that covered by Theorem 3.5.

The problem of determining the decycling numbers of the remaining cases of the grid graphs is open. For the cartesian products $C_m \times C_n$, the following problem is also open.

Problem 3.1 $\phi(C_m \times C_n) =?$

4 Snakes

In this section, a chordless cycle is referred to as a *cell*. A *snake* can be defined recursively as follows. A snake with two cells consists of two cycles with one common edge, one of the two cells will be designated the *head* and the other the *tail*. A snake with $n+1$ cells is obtained from a snake with n cells by identifying an edge of a new cell with an edge of the tail of the old snake that lies on no other cell. The tail of the new snake is the new cell, and the head remains the same. The *length* of a snake is the number of its cells.

Determining a minimum decycling set for a snake is algorithmically straightforward and in [6], an exact formula for the decycling numbers of snakes was given. Given a snake G, let v be a vertex on the head with largest possible degree. Put v in the decycling set, then delete it along with all vertices which lie only on cells that contain v. What remains is either a shorter snake or a single cell. Either repeating this process on the snake which remains (where the new head is the cell originally adjacent to the old head) or choosing any vertex from a single cell clearly results in a decycling set S of G. That S has the minimum cardinality follows from the fact that each vertex in S is on some cell that has none of its other vertices in S. Thus, G has a set of $|S|$ vertex disjoint cycles. Hence $\phi(G) = |S|$ and S is a minimum decycling set.

Let G be a snake. A *major pair* is a pair of vertices of degree 3 such that the edge joining them lies on two cells. A *minor pair* is a pair of vertices of degree 3 in a cell which contains exactly two vertices of degree 4. A minor pair will be said to lie *between* the two vertices of degree 4. A vertex of degree at least 4 is called a *major* vertex.

Note that adding a new tail cell to an existing snake increases the degree of two of its vertices of the old tail by 1 each. Since the new tail cell can be incident with at most one vertex of degree at least 3 in the old snake, its addition either creates a new major vertex, adds 1 to the degree of an old major vertex, or creates a major pair. This idea gives a natural order, from head to tail, $(u_1, u_2, \cdots, u_s)$ to the set of major vertices, major pairs and minor pairs. Define the *name* of a snake G to be the sequence $(n_1, n_2, \cdots, n_s)$ where

$$n_i = \begin{cases} 2 & \text{if } u_i \text{ represents a minor pair} \\ 3 & \text{if } u_i \text{ represents a major pair} \\ d(u_i) & \text{if } u_i \text{ is a major vertex.} \end{cases}$$

With this definition, there may be several snakes with the same name even if the cells are of uniform length. It is easily seen that every finite sequence of integers greater than or equal to 2 is the name of some snake.

Given a snake G and its name $N(G) = (n_1, n_2, \cdots, n_s)$, define the *nickname* $C(G)$ of G (as a subset of $\{1, 2, \cdots, s\}$) as follows.

(1) $1, s \in C(G)$.

(2) Assume that for $i < s-1$ it has been determined whether or not each of $1, 2, \cdots i$ is in $C(G)$. Then

(i) If $n_{i+1} \geq 6$, then $i+1 \in C(G)$;

(ii) If $n_{i+1} = 5$, then $i+1 \in C(G)$ if and only if $i \notin C(G)$;

(iii) If $n_{i+1} = 4$ then $i+1 \in C(G)$ if and only if either $n_i \geq 5$ and $i \notin C(G)$, or $n_i = 4$ and $i, i-1 \notin C(G)$.

With this definition, it can be shown that the decycling number of a snake whose cells are all 4-cycles is the cardinality of its nickname. From this result the following theorem of [6] follows.

Theorem 4.1 *Let G be a snake with nickname $C(G)$. Then $\phi(G) = |C(G)|$.*

A *subsnake* of a snake G is a subgraph of G that is itself a snake. A *straight segment* of a snake whose cells are all squares is a subsnake in which the vertices of each of the shared edges is a major pair. A *maximal* straight segment T of a square-celled snake G is a straight segment of G such that for each cell $s \notin T$, $T \cup s$ is not a straight segment of G. A square-celled snake G is said to be *nonsingular* if each its maximal straight segment has at least three cells; otherwise it is said to be *singular*. The *segment sequence* of a square-celled snake G is the sequence of lengths of maximal straight segments of G ordered from head to tail. The following theorem was also proved in [6].

Theorem 4.2 *If $(d_1, d_2, \cdots, d_k)$ is the segment sequence of a nonsingular snake G, then*

$$\phi(G) = \sum_{i=1}^{k} \left\lceil \frac{d_i}{2} \right\rceil - k + 1.$$

The decycling number of a singular snake is certainly related to that of a nonsingular one by means of a certain transformation (surgery) of the snake. The decycling numbers of snakes with cell size not equal to 4 are related to those of snakes with cell size 4 by means of simple transformations. It is therefore possible to consider the decycling problem with restriction to square-celled snakes only.

A snake with triangular cells is a special type of triangulation of a polygon, namely that in which every triangle contains at least one edge of the polygon. This raises the question of decycling triangulations of polygons, or equivalently, the maximal outerplanar graphs. In general, this seems to be considerably more complicated than decycling snakes. At present, we content ourselves with bounds (cited from [6]).

Theorem 4.3 *If G is a maximal outerplanar graph of order n, then*

$$1 \leq \phi(G) \leq \left\lfloor \frac{n}{3} \right\rfloor.$$

Even an algorithm similar to that described at the beginning of this section is not known for the computation of the decycling number of a triangulation of a polygon. An interesting open problem is to determine the decycling number of outerplanar graphs.

Problem 4.1 *Is there a fast algorithm for computing the decycling numbers of (maximal) outerplanar graphs?*

Problem 4.2 *Determine the decycling numbers of 2-dimensional trees.*

Albertson and Berman [1] conjectured that every planar graph has an induced acyclic subgraph with at least half the vertices.

Conjecture 4.1 If G is a planar graph, then $\phi(G) \leq \frac{|G|}{2}$.

This is related to a theorem of Borodin [11] on the *acyclic chromatic number* of a graph, defined to be the minimum number of colours in a proper colouring of the graph so that no cycle has only two colours. From [11], we have

Theorem 4.4 *If G is a planar graph, then $\phi(G) \leq \frac{3|G|}{5}$.*

5 Cubic Graphs

While Questions 1.1 and 1.2 remain open, bounds on the decycling number of cubic graphs have been investigated for some time.

For a connected cubic graph G with $g(G) = g$, Speckenmeyer obtained in [24] that

$$\phi(G) \leq \frac{g+1}{4g-2}|G| + \frac{g-1}{2g-1}.$$

This improved upon his earlier result in [23]. Zheng and Lu showed in [27] that if G is a connected cubic graph without triangles and $|G| \neq 8$, then

$$\phi(G) \leq \left\lceil \frac{|G|}{3} \right\rceil$$

thus settling a conjecture by Bondy, Hopkins and Staton in [10] in the affirmative.

A sharp upper bound for the decycling number of cubic graphs has been obtained in [21] by Liu and Zhao and that for connected graphs with maximum degree 3 has been obtained in [3]. Let $\mathcal{G}$ denote the family of cubic graphs obtained by taking cubic trees and replacing each vertex of degree 3 by a triangle and attaching a copy of K_4 with one subdivided edge at every vertex of degree 1.

Theorem 5.1 *Let G be a cubic graph with $g(G) = g$. Then*

$$\phi(G) \leq \frac{g}{4(g-1)}|G| + \frac{g-3}{2g-2}$$

if $G \notin \{K_4, Q_3, W\} \cup \mathcal{G}$ where Q_3 is the 3-cube and W is the Wagner's graph.

If $G \in \mathcal{G}$, then

$$\phi(G) = \frac{3}{8}|G| + \frac{1}{4}.$$

Corollary 5.1 *If $g(G) \geq 3$ then $\phi(G) \leq \frac{3}{8}|G|$ for $G \notin \{K_4\} \cup \mathcal{G}$. If G is a connected cubic graph with $g(G) \geq 4$ and $G \neq Q_3$ or W, then $\phi(G) \leq \frac{|G|}{3}$.*

Theorem 5.2 *Let G be a connected graph of maximum degree 3. If $G \neq K_4$ then*

$$\phi(G) \leq \left\lfloor \frac{|E(G)|+1}{4} \right\rfloor.$$

The family $\mathcal{G}$ of graphs show the sharpness of this result.

A polynomial time algorithm to decide the decycling number of any cubic graph has been found by Li and Liu in [17].

6 Random Regular Graphs

Working with random regular graphs using differential equation method, Bau, Wormald and Zhou have recently studied the expected bounds on the decycling numbers of random regular graphs. It was a little surprising that the expected upper and lower bounds for the decycling number of random cubic graphs are the same and the value is much smaller than the upper bound given by Liu and Zhao (see Section 5) and are essentially the one predicted in Problem 1.1. For general random regular graphs the expected upper and lower bounds are also strikingly close. All results in this section are cited from [7]. In the following statements, $\mathbf{P}\{E\}$ denotes the probability of event E.

Theorem 6.1 *Let G be a random cubic graph. Then*

$$\mathbf{P}\left\{\phi(G) = \left\lceil \frac{|G|}{4} \right\rceil\right\} \to 1 \quad \text{as} \quad |G| \to \infty.$$

Let b_r and B_r be the numbers given in the following table.

r	b_r	B_r
3	0.25	0.25
4	0.3787	0.411145
5	0.3786	0.507895
6	0.423	0.5739
7	0.461	0.6223

The upper bounds B_r given in this table were obtained by demonstrating the almost certain existence of an induced forest of order $|G| - B_r$ in a random r-regular graph G. Differential equation method for random algorithms developed by Wormald (see [26]) has been used to give the values of B_r. The lower bounds b_r were obtained by confirming that the expected number of induced forests in G with order greater than $|G| - b_r$ is asymptotically almost surely smaller than 1. The following theorem is the special case for small values of r of one of the main results of [7].

Theorem 6.2 *Let G be a random r-regular graph. Then for $r = 3, 4, 5, 6, 7$, we have*

$$\mathbf{P}\left\{b_r|G| \leq \phi(G) \leq B_r|G|\right\} \to 1 \quad \text{as} \quad |G| \to \infty.$$

7 Connectivity and Decycling

In this section, the dependency of decycling number of a graph on its connectivity number will be considered.

Let H and J be graphs, $S \subseteq V(H)$ and $T \subseteq V(J)$ with $|S| \leq |T|$. Let $f : S \to T$ be an injection. An *identification* of H and J *via* f is a graph G denoted $H \circ_f J$ obtained by identifying H and J via f and maintaining the adjacencies in the rest of H and J. As for the adjacencies in S and T, keep all the edges $E(H|_S) \cup E(J|_T)$. Formally, for each $x \in S$, identify x and $f(x)$, thus embedding S into T via inclusion f. Now for $G = H \circ_f J$,

$$V(H \circ_f J) = V(H) \cup V(J),$$

$$E(H \circ_f J) = E(H) \cup E(J).$$

Lemma 7.1 *Let S be a (minimum) decycling set of H and T be a (minimum) decycling set of J. If $f : S \longrightarrow T$ is an injection, then T is a (minimum) decycling set of $H \circ_f J$.*

Proof: Since S and T decycle H and J, T is a decycling set of $H \circ_f J$. $H \circ_f J - T$ is a forest that is the union of forests $H - S$ and $J - T$. If S is a mimimum decycling set of H and T is a minimum decycling set of J, then T is a minimum decycling set of $H \circ_f J$. To see this, let T' be any decycling set of $H \circ_f J$ such that $|T'| < |T|$. Then since $\phi(J) = |T|$, T' cannot decycle J. Let C be a cycle of $J - T'$. Then C must also be a cycle of $H \circ_f J - T'$ since $J - T'$ is an induced subgraph of $H \circ_f J - T'$. The proof is complete.

Denote by $\kappa(G)$ the connectivity of G.

Lemma 7.2 *If $\phi(G) = k$ then $\kappa(G) \leq k + 1$.*

Proof: Let S be a decycling set of G with cardinality $k = \phi(G)$. If $\kappa(G) \geq k + 2$, then $G - S$ is 2-connected, and hence $G - S$ is not a forest. This contradicts the choice of S. Hence $\kappa(G) \leq k + 1$, and the lemma follows.

Let G be a $(k + 1)$-connected graph with $\phi(G) = k$. Since G is not $(k + 2)$-connected, $\kappa(G) = k + 1$. Let S be a minimum decycling set of G. Then $G - S$ is a connected acyclic graph, i.e., a tree. Thus G can be obtained by joining a set S to a tree of order at least k in a way so as to make G a $(k + 1)$-connected graph.

Theorem 7.1 *Let $\phi(G) = k$. Then $\kappa(G) = k + 1$ if and only if for each $S \subset V(G)$ with $|S| = k$, $G - S$ is a tree.*

Proof: Let $\kappa(G) = k+1$ and S be a minimum decycling set of G. Since G is $(k+1)$-connected and $\phi(G) = k$, $G - S$ is connected. Since S is a decycling set, $G - S$ is acyclic. Hence $G - S$ is a tree. On the other hand, assume that $G - S$ is a tree for each $S \subset V(G)$. Then since $G - S$, being a tree, is connected, hence $\kappa(G) \geq k+1$. By Lemma 7.2, $\kappa(G) \leq k+1$ since $\phi(G) = k$. Therefore $\kappa(G) = k+1$ and the proof is complete.

8 Decycling and Codes

The capacity of error correction of a code is measured by a certain distance defined over the set of codewords (see [20] or [22]). Nonlinear codes are used in order to obtain a largest possible number of codewords with a given minumum distance (in this, nonlinear codes have advantages over linear ones). An (n, M, d)-*code* is a set of M vectors of length (dimension) n in a vector space $V(\mathcal{F})$ over a division ring $\mathcal{F}$ such that the (Hamming) distance between any two vectors is at least d, and d is the smallest number with this property. A binary code is obtained if $\mathcal{F} = GF(2)$.

A binary vector $(a_1, \cdots, a_n)$ of length n is just a vertex of the n-cube Q_n. An (n, M, d)-code $\mathcal{C}$ is a subset of $V(Q_n)$. A code with a good error correction capacity is a choice of as many vertices as possible from Q_n while the distance between every pair of vertices in $\mathcal{C}$ is as large as possible. This is a packing problem: if the code has minimum Hamming distance d, the Euclidean distance between codewords is $\geq \sqrt{d}$. Finding an (n, M, d)-code is equivalent to finding M non-overlapping spheres of radius $\frac{\sqrt{d}}{2}$ with centers at vertices of Q_n. It should be clear from the definition of Q_n that the Hamming distance between two codewords is just the distance between the two vertices in Q_n. If a set of vertex disjoint stars in Q_n can be obtained from $Q_n - S$ where S is a minimum decycling set of Q_n, then the centers of these stars give rise to an (n, M, d)-code $\mathcal{C}$ with a reasonable d and a large M. For example, in Theorem 2.3, $\phi(Q_9) \leq 237$ and there is a decycling set S of 237 vertices such that $Q_9 - S$ consists of 19 vertex disjoint stars. The set of the centers of these stars provide a $(9, 19, 4)$-code. This is a nonlinear binary code with 19 codewords each having length 9 that corrects any one error in the digits.

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