Arc- and circle-visibility graphs

Joan P. Hutchinson*

Department of Mathematics and Computer Science
Macalester College
St. Paul, MN 55105
U.S.A.

In memory of Neal H. McCoy, 1905–2001

Abstract

We study polar visibility graphs, graphs whose vertices can be represented by arcs of concentric circles with adjacency determined by radial visibility including visibility through the origin. These graphs are more general than the well-studied bar-visibility graphs and are characterized here, when arcs are proper subsets of circles, as the graphs that embed on the plane with all but at most one cut-vertex on a common face or on the projective plane with all cut-vertices on a common face. We also characterize the graphs representable using full circles and arcs.

1 Introduction

Bar-visibility graphs (BVGs), introduced in [TT1, Wi], have been studied by graph theorists [MR, T, CJLW] and by theoretical computer scientists interested in graph drawing. BVGs are graphs that can be represented in the plane, vertices by horizontal line segments and edges by vertical visibility between segments. Generalizations are obtained by considering such layouts on the sphere and cylinder [TT2, TT3], on the torus [MR], and on the Möbius band [D], and also by considering rectangles in the plane with horizontal and vertical visibilities [BDHS, DH1, DH2, HSV].

In contrast we consider arcs of concentric circles (arcs that are proper subsets of a circle) with radial visibility, including visibility through the origin, the center of all the concentric circles. We show that these graphs, though arising naturally from visibility in the plane, correspond to graphs that embed on the (real) projective plane, the nonorientable surface of Euler characteristic 1. We call graphs with such layouts polar visibility graphs (PVGs) and characterize these as the planar graphs

* Research supported in part by NSA Grant #MDA904-99-1-0069

that can be drawn in the plane with all but at most one cut-vertex on a common face plus the graphs that can be embedded on the projective plane with all cut-vertices on a common face. We also consider the variation in which full circles are allowed along with arcs, and characterize the graphs so representable (CVGs) in terms of their block-cutpoint tree.

Preliminary results of this work are announced in [H1], and [H2] contains a proof of a simplified version of Theorem 2.2. Here we create more complex layouts of PVGs than in [H2] in order to allow extension to CVGs.

2 Background

A *bar-visibility graph* is one whose vertices can each be represented by a closed horizontal line segment in the plane, with segments having pairwise disjoint relative interiors, and with two vertices adjacent in the graph if and only if the corresponding segments are vertically visible. Two segments are considered *vertically visible* when there is a nondegenerate rectangle R such that R intersects only these two segments, and the horizontal sides of R are subsets of these two segments. These graphs are characterized in [TT1, Wi] as those planar graphs that can be embedded in the plane with all cut-vertices on a common face. Figure 1a shows a bar-visibility layout (a *bv layout*) of $K_{2,3}$ with two additional vertices of degree 1 appended, one each to a vertex of degree two; we call this graph $K_{2,3}+2e$. There exist linear-time algorithms to detect and then layout BVGs; see [O]. We use graph theoretic terminology as in [We] and topological notions as in [MT].

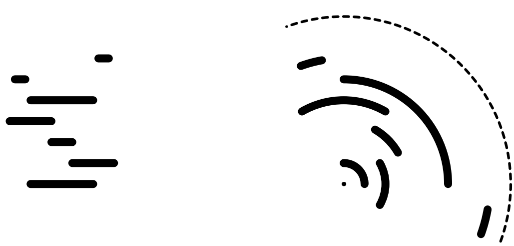

Figure 1a. $K_{2,3}+2e$ as BVG; 1b. same graph as PVG

As was suggested by a Macalester student, Michael McGeachie, we shift from rectangular to polar-coordinate representations and visibility, using arcs of circles (and later full circles) all centered at the origin with radial visibility. Throughout this paper, by an arc we mean a proper subset of a circle. Just as visibility wider than along a line is required for BVGs, we now ask that radial visibility be available through a nondegenerate cone, rather than just along a line. Define a (nondegenerate) *cone* in the plane to be a 4-sided region of positive area with two opposite sides being arcs of circles, centered at the origin, and the other two sides, possibly intersecting, being radial line segments on lines through the origin. Thus, both $\{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi/6\}$ and also $\{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le \pi/6 \ \text{ or } \ \pi \le$

$\theta \leq 7\pi/6\} = \{(r,\theta) : -1 \leq r \leq 1,\ 0 \leq \theta \leq \pi/6\}$ are considered to be cones, respectively, not containing and containing the origin. Given a set of arcs, all centered at the origin, two of these arcs a_1 and a_2 are said to be *radially visible* if there is a cone that intersects only these two arcs and whose two circular ends are subsets of the two arcs; the same definition holds for visibility between an arc and a circle and between two circles. A graph is called a *polar visibility graph* if its vertices can be represented by arcs, including endpoints, of circles centered at the origin, having pairwise disjoint relative interiors, so that two vertices are adjacent if and only if the corresponding arcs are radially visible. If x is a vertex of a PVG, we typically let a_x denote its arc in the layout, and conversely x_a is the vertex corresponding to an arc a. (If this model is used, but without visibility through the origin, the graphs arising are one of the cylindrical types characterized in [TT2]. In [D] PVGs are shown to be equivalent to a layout of bars on the Möbius band.)

In Fig. 1b is a polar visibility layout (or *pv layout*) of the graph of 1a, and shown in Figure 2a is a pv layout of K_6. Note that for a 2-connected graph there is no loss in taking arcs as proper subsets of circles since a full circle can be cut down to a smaller arc, leaving the same visibilities, for otherwise it represents a cut-vertex. Arcs in a pv layout spanning more than half its circle will provide interesting variations, full circles even more.

Figure 2a. A pv layout L of K_6;   2b. $I(L)$ and $I(L)^*$ on the projective plane

Figure 2c. For $G = K_6$, $I(L_G) = (I(L))_G$ on the projective plane

Similarly a graph is called a *circular visibility graph* if its vertices can be represented by arcs and circles with radial visibility between arcs, arcs and circles, and circles determining edges as for PVGs. When possible we prefer, but do not require, arcs over circles; that is, in a layout we will decrease a circle to become a proper arc if no additional visibilities are introduced. As above we let $\{x, c_x\}$ and $\{x_c, c\}$ be corresponding pairs of vertices and circles in a CVG and its layout. We shall see that some planar and projective planar graphs with cut-vertices on an arbitrary number of faces are CVGs, but not PVGs, but that these faces must be nested appropriately. In Figure 3 is shown such a planar CVG. In that layout the inner circle contains one arc; if instead, four mutually visible arcs are placed within that circle, encircling the origin, to form a K_5 with the innermost circle, the example becomes a nonplanar CVG.

Note that in a pv or cv layout L of a graph G, we may draw each arc and circle on a distinct circle, and we may take these circles to have radii $1, 2, \ldots, n$ where $n = |V(G)|$. This naturally leads to another layout of the graph in a disc of radius $n+1$ and centered at the origin by inverting each circle and arc through the circle of radius $(n+1)/2$. That is, each point with polar coordinates (r, θ), $0 < r < n+1$, is mapped by the inversion to the point $(n+1-r, \theta)$. This inversion preserves circles, arcs, and the angles defining these arcs. If the original layout was L, we denote this inverted layout by $I(L)$; see Fig. 2b.

Recall that the (real) projective plane can be obtained by taking a circular disc and identifying opposite (or antipodal) points. Thus if we identify opposite points of the circle of radius $n+1$, we create a projective plane. Two arcs in $I(L)$ (or an arc and a circle or two circles) that were previously radially visible in a cone, not containing the origin, are still radially visible, and a pair visible in a cone through the origin are now visible in a "generalized cone" that crosses the boundary of the projective plane, reemerging on the other side. The coordinates of such a generalized cone are given by $\{(r, \theta), r^* \leq r \leq n+1 \text{ or } -(n+1) \leq r \leq -s^*,\ \theta_1 \leq \theta \leq \theta_2\}$ where r^*, s^*, $\theta_1 < \theta_2$ are constants, $0 \leq r^*, s^* < n+1$. In addition, the interior of no two of these new cones intersect. Fig. 2b shows the inverted layout of K_6 on the projective plane with dashed lines indicating a conical area of visibility. The first proposition is then clear since each inverted arc and circle on the projective plane can be replaced by a single vertex. Then the visibility cones can each be shrunk and transformed to a set of nonintersecting edges on the projective plane.

Proposition 2.1 *A PVG or a CVG embeds on the projective plane.*

Consequently all PVGs and CVGs embed either on the plane or on the projective plane. For consistency we primarily describe the layouts as arc and circle layouts in the plane with radial visibility, including visibility through the origin.

In the next section we prove the following, more definitive characterization of PVGs. Recall that a graph G is said to *embed* on a surface S if it can be drawn there without any edge crossing, and that each maximal connected component of $S \setminus \{V(G), E(G)\}$ is called a *face* of the embedding (we do not require that faces be simply connected). A graph is said to be *plane* (respectively, *projective plane*) if it is planar and embedded in the plane (resp., if it can be and is embedded in the projective plane.) For a plane graph one face is the infinite, exterior face and the others are finite, interior faces. If the graph were considered to be embedded on the sphere, all faces would be interior, but in the plane and similarly for pv and cv layouts in the plane there is one distinguished face, the external and infinite one. For graphs on the projective plane, all faces are interior though one will be the representative of the external face of the layout.

Theorem 2.1 *A graph G is a PVG if and only if either a) G has an embedding in the plane with all but at most one cut-vertex on a common face, or else b) G has an embedding on the projective plane with all cut-vertices on a common face.*

Note that condition a) allows for the representation of planar graphs that are not BVGs; for example $K_{2,3}$ with three additional vertices of degree 1 appended, one each to a vertex of degree two ($K_{2,3} + 3e$), is a PVG. Similarly $K_4 + 4e$ is a PVG (see Fig. 7); these are the smallest graphs that are not BVGs. Condition b) also allows for more planar graphs; for example, two vertices joined by three internally disjoint paths of length three (i.e., three edges each) plus six vertices of degree 1, each adjacent to a different vertex of degree two, satisfies b), but not a).

Recall that a graph is said to be 2-*connected* if it contains at least three vertices and the deletion of any vertex and its incident edges leaves the graph connected. Then every graph G can be decomposed into its blocks and their connecting cut-vertices (a *block* is either an edge or a maximal 2-connected subgraph; see [We]), and that these connections determine a tree, called the *block-cutpoint tree* of the graph, BC(G). This tree has a vertex for each block and for each cut-vertex of G, and two vertices of BC(G) are adjacent if and only if they correspond to an incident cut-vertex and block. We call a block *planar* if it represents a planar graph.

Theorem 2.2 *A graph G is a CVG if and only if the vertices of $BC(G)$ can be partitioned into three sets P, Q, and R, where*

- *0) $P = (b_1, b_2, \ldots, b_{2k+1})$, $k \geq 0$, is a path with each b_{2i} representing a planar block, $i = 1, \ldots, k$;*
- *1a) Q is a nonempty block adjacent to b_1, representing a (2-connected) projective planar graph, or*
- *1b) Q is a set of one or more (nonempty) planar blocks, all adjacent to b_1; and*
- *2) R is an arbitrary tree structure adjacent to b_{2k+1}, such that $R \cup \{b_{2k+1}\}$ represents a planar graph that can be drawn in the plane with all cut-vertices, except possibly for that representing b_{2k+1}, on a common face.*

Note that when $k = 0$, $b_1 = b_{2k+1}$, and these conditions reduce to those of Theorem 2.2. On the other hand, it may be that each cut-vertex of G, represented by $b_1, b_3, \ldots, b_{2k+1}$, lies on a different face, as in Figure 3. The example of Fig. 3 is the first of an infinite family with an increasing number of cut-vertices on different faces. Draw K_4 in the plane with vertices labeled $\{1, 2, 3, 4\}$, placing 4 in the interior, and add a pendant edge to 1, making it a cut-vertex. Subdivide the triangle $\{2, 3, 4\}$ with a new vertex 5 adjacent to all of these. Vertex 5 becomes the second cut-vertex by joining it to all vertices of a triangle on $\{6, 7, 8\}$, placed within face $\{2, 3, 5\}$ with 8 lying inside the triangle $\{5, 6, 7\}$. Then add 9 adjacent to all vertices of $\{6, 7, 8\}$ plus a pendant edge to 9 giving the graph represented in Fig. 3. If instead 9 is joined to a triangle on $\{10, 11, 12\}$ with 12 in the interior of $\{9, 10, 11\}$, then 9 is a cut-vertex, and additions can be made to 12 so that it becomes a cut-vertex, and in this graph all cut-vertices lie on different faces in any planar embedding. The layout of Fig. 3 is easily extended with concentric layouts to form such a family of CVGs.

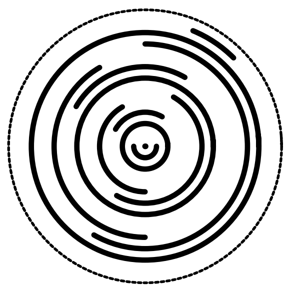

Figure 3. A CVG that is not a PVG.

As described in [O, We], planar layouts and the block-cutpoint tree of a graph can be determined in linear time. Projective planar graphs can also be recognized and embedded in linear time [M]. It can quickly be determined whether all cut-vertices of a graph lie on a common simple cycle and, if so, whether there is an embedding in either surface in which this cycle bounds a face. More details can be found in [M, MT].

3 Main results on PVGs

Of course, all planar graphs can be embedded in the projective plane, in a contractible region, and we first consider such graphs and their layouts as PVGs with no visibilities through the origin. Then using arcs that span more than one half of a circle, we see how to get more planar graphs than those representable as BVGs, namely those with one additional cut-vertex. The main concern in these cases is to avoid unwanted visibilities. Finally we turn to the case when the pv layout corresponds to an embedding on the projective plane. We develop theory to be useful also with CVGs, but study these in detail in Section 4.

We focus on simple graphs, and in Theorems 2.2 and 2.3 we are primarily characterizing the simple graphs that are PVGs and CVGs. More precisely, in our definition of radial visibility and its application to PVGs and CVGs, we say two arcs are radially visible if there is at least one maximal cone providing mutual visibility. Thus we allow multiple visibility and in some cases self-visibility. One can learn more precise results for multigraphs by keeping track of distinct multiple and self-visibility as they arise or are avoided in the various proofs.

First we need more precise topological and geometric definitions. Let L be a pv or cv layout of a graph G and $I(L)$ the inverse layout on the projective plane. We let L^* (respectively, $I(L)^*$) denote the visibility depiction obtained by shrinking each maximal visibility cone of L (resp., $I(L)$) to a distinct line segment by reducing its angles $b_1 \leq \theta \leq b_2$ to some constant $\theta = b$, $b_1 < b < b_2$; strict inequality ensures distinct visibility segments. For $G = K_6$, $I(L)^*$ is shown in Fig. 2b.

Also let L_G (resp., $I(L))_G$) denote the graph obtained from L^* (resp., $I(L)^*$), as in the proof of Prop. 2.1, by replacing each arc or circle by a vertex, and transforming each visibility line segment to an edge that intersects no other edge except possibly at the origin (resp., an edge that intersects no other edge on the projective plane). Thus $(I(L))_G$ is a graph embedded on the projective plane; see Fig. 2c for an example when $G = K_6$. Note that L^*, $I(L)^*$, L_G, and $(I(L))_G$ have visibility segments and edges for each distinct, maximal visibility cone so that multiple edges and loops may be present in these depictions; however, a pair of multiple edges will not form an embedded interior digon (a 2-sided face) with empty interior. We shall see that there can be loops incident with at most one vertex in a PVG or CVG.

Note that the complement of the arcs, circles, and lines of L^* divide up the plane into faces; similarly $I(L)^*$ divides up the projective plane. One face of L^* is the infinite, exterior face, possibly containing the origin; this exterior face is the one in which most cut-vertices of a PVG and their blocks can be placed. We say that an arc or circle of a layout L lies on the exterior face if it lies on the exterior face of L^*.

In a simple graph G, the *contraction* of an edge $e = \{x, y\}$ produces the graph G/e with x and y identified to become a new vertex x^*, with e removed, and any set of multiple edges replaced by a single edge. In a multigraph an edge $e = \{x, y\}$ is said to be *simple* (respectively, *multiple*) if vertices x and y are not (resp., are) joined by an additional edge. If G is a multigraph embedded on any surface and e a nonloop edge, G/e is the multigraph embedded on the same surface, obtained by contracting e on the surface to become a new vertex x^* and removing e. Depending on the situation, multiple edges at x^* (formed when e lies on a 3-cycle) are retained or eliminated by deleting multiple edges. No new loops are introduced in the formation of G/e if and only if e is a simple edge.

We need the following variation on an elementary (non-topological) lemma, whose proof appears in [H2] and is similar to that of [MT, Lemma 1.4.5]; a similar contraction proof for general 2-connected graphs is even more easily found; see [We, p. 174, Exer. 4.2.15].

Lemma 3.1 *Let G be a loopless, 2-connected graph with at least 4 vertices, embedded in the plane (respectively, on the projective plane) with at most one interior digon face. Then G contains an edge e such that G/e is loopless, 2-connected, and plane* *(resp., projective plane) with at most one interior digon face.*

We apply Lemma 3.1 in Proposition 3.6 to embedded loopless multigraphs on the plane or projective plane, and with the exceptions described there, we delete any multiple edge that forms a digon with empty interior after the contraction of e, occurring when e lies on a triangular face.

We use the following combinatorial description of an embedded graph and of a pv or cv layout. If a graph is embedded on any surface, then for each vertex there is naturally defined a local cyclic rotation of its incident edges, given by the order, say clockwise, of its edges in the embedding; such a collection of rotations, one for each vertex, is called a *rotation system.* (See, for example, [MT, Wh] where it is shown that an embedding in an orientable surface is equivalent to a rotation system.) Two graphs embedded on the same orientable surface are said to be *equivalent* if at each vertex the corresponding rotations agree. For graphs on the (nonorientable) projective plane it is not possible to assign an orientation consistently throughout the surface. In any one depiction of a graph on this surface (with all vertices located within the disc representing the projective plane), a clockwise direction can be assigned at each vertex to give a rotation system, but there are other "equivalent" representations of this embedding. In addition, a *signature* is needed, an assignment of ± 1 to each edge — for example, in the disc depiction we may assign $+1$ to each edge contained wholly within the disc and -1 to each edge that crosses the boundary of the disc. Then in general an embedding in any surface is equivalent to a set of rotations of incident edges, one at each vertex, and a signature assignment to edges representing local consistency [MT]. An edge $e = \{x, y\}$ is assigned $+1$ (respectively, -1) if in a local, contractible neighborhood of e the rotation orientation at x and y agree (resp., are reversed). Such an assignment is called an *embedding scheme.* Two graphs embedded on the same surface are said to be *equivalent* if, by a series of local reversals at a vertex and its incident edges, the embedding scheme of one can be transformed into the other's. (See [H2] and [MT] for more details.)

We obtain similar embedding schemes for layouts of PVGs and CVGs. Given a pv layout L in the plane (respectively, an inverse layout $I(L)$ in the projective plane), one can define the *arc-rotation scheme* to be the set of cyclic rotations of neighbors about each arc of its visibilities to other arcs plus the assignment of $+1$ or -1 to each visibility cone according as it does not or does pass through the origin (resp., does not or does cut across the disc boundary); note that the rotations at the arcs of L and of $I(L)$ are inverses of each other. We say that an embedding of a PVG graph G in the plane or on the projective plane and its polar visibility layout L are *equivalent* if the arc-rotation scheme of $I(L)$, when translated into a set of vertex-neighbor cycles and an edge-signature, yields the embedding scheme of the embedded graph; see Fig. 2. Given a circle in a cv layout L or $I(L)$, the neighbors divide into two cyclic rotations of the inner and outer visibilities, called the *circle-rotation scheme.* Then a drawing of a CVG and its layout L are *equivalent* if the arc/circle-rotation scheme of $I(L)$ agrees with that of the embedded graph. These embedding schemes and their equivalences will be needed in Proposition 3.6.

Proposition 3.2 *A connected graph has a polar visibility layout with no visibilities through the origin if and only if the graph is a BVG.*

Proof. If G is a BVG with a (rectangular) layout L in the plane, we may assume without loss of generality that the layout lies within the rectangular region $\mathcal{R}$ bounded by $0 \le x \le 1$ and $0 \le y \le n+1$ where $n = |V(G)|$; see Fig. 1a. The resulting PVG is easily obtained by mapping horizontal lines $y = r$, $r = 1, 2, \ldots, n$, (and the bars on them) into circles (and their subarcs) of radius r. Specifically, map each point (x, y) in $\mathcal{R}$ to the point with polar coordinates $(r, \theta) = (y, (1-x)\pi/2)$, giving a pv layout in the first quadrant; see Fig. 1b. The resulting layout has no visibility through the origin.

Conversely if G is a PVG laid out with no visibilities through the origin, then we may suppose the arcs of the layout lie on circles of radius $r = 1, 2, \ldots, n$ where $n = |V(G)|$. Then (perhaps after a rotation of the entire layout) there are angles $0 < \theta_1 < \theta_2 < \ldots < \theta_{2k+1} \le \pi$ such that all arcs of the layout span angles lying in $[0, \theta_1] \cup [\theta_2, \theta_3] \cup \ldots \cup [\theta_{2k}, \theta_{2k+1}] \cup [\pi+\theta_1, \pi+\theta_2] \cup [\pi+\theta_3, \pi+\theta_4] \cup \ldots \cup [\pi+\theta_{2k+1}, 2\pi]$ with no arcs in the intervening angles. But notice that this layout can be redrawn, depicting the same graph, by reflecting the arcs lying within $[\pi, 2\pi]$ through the origin. Hence in this case we may assume that arcs of the layout span angles lying between 0 and π. Then the inverse map of the one above, taking each point (r, θ) to $(x, y) = (1 - \theta/\pi, r)$, will map the pv layout into a BVG representation within the rectangle bounded by $0 \le x \le 1$ and $0 \le y \le n+1$. □

Of course there are planar graphs with layouts as PVGs including visibilities through the origin and with arcs representing cut-vertices on the exterior face. Note that whenever there are visibilities through the origin in a layout L, then the equivalent graph $(I(L))_G$ is embedded on the projective plane. It turns out that in some pv layouts there is a (sneaky) hiding place for a cut-vertex and its connecting blocks, but the resulting graphs turn out to be planar. In a pv layout we call an arc a^* a *long arc* if its angular span is greater than π. Suppose $a^* = \{(r^*, \theta),\ 0 \le \theta \le \pi + x\}$, for some x, $0 < x < \pi$. Then the cone defined by $C(a^*) = \{(r, \theta),\ -r^* \le r \le r^*,\ 0 \le \theta \le x\}$ is an area in which interior arcs can see the arc a^* and possibly no others. (For an example, see Fig. 7.) If $C(a^*)$ is empty, a^* has self-visibility (and so a loop could be added to x_{a^*} to depict this); if $C(a^*)$ is nonempty, the arcs within can be extended to block self-visibility for a^*.

Let a^* be a long arc at radius 1, spanning $\theta_1 \le \theta \le \theta_1 + \pi + x$ for some $x > 0$. Arcs a^* and b^* are called a *long-arc pair at the origin* if they are mutually visible, together they span at least 2π, and if b^* lies at radius $r^* > 1$, no arcs intersect the *long-arc cone* $\{(r, \theta) : 0 \le r < r^*,\ \theta_1 + \pi + x < \theta < \theta_1 + 2\pi\}$. (For example, when $r^* = 2$, no arcs can meet the designated cone when all arcs have integral radii.) Similarly if a^* is a long arc at the outermost radius $n = |V(G)|$, spanning $\theta_2 \le \theta \le \theta_2 + \pi + y$ for some $y > 0$, then a^* and b^* are a *long-arc pair at infinity* if they are mutually visible, together span 2π, and if b^* lies at radius $r^* < n$, no arcs intersect the long-arc cone $\{(r, \theta) : r^* < r < n,\ \theta_2 + \pi + y < \theta < \theta_2 + 2\pi$. For examples, see Fig. 5. The arc a^* of a long-arc pair at the origin always has visibility to itself, but later we will be using this configuration when arcs are to be added to $C(a^*)$. Notice that in long-arc pairs the long arc at radius 1 or at radius n could be extended to form a full circle without changing visibilities; such extensions will be needed for our representation of CVGs.

Proposition 3.3 *Let G be laid out as a PVG L including a long arc a^* that represents a cut-vertex x^*, not lying on the exterior face, and let B be a block of G incident with x^* and whose representation lies within $C(a^*)$ in L. Then G is a planar graph and can be drawn in the plane with one face including all vertices whose arcs lie on the exterior face of L.*

Proof. Since a^* represents a cut-vertex x_{a^*} to which B is connected, then within $C(a^*)$ lies a cone $C' = \{(r,\theta),\ -r^* \le r \le r^*,\ x_1 \le \theta \le x_2,\ \text{ with } 0 < x_1 < x_2\}$ in which lie only arcs representing vertices of B. Note that in $I(L)$ the arc $I(a^*)$ will also be a long arc, and in forming $(I(L))_G$, a^* and the cone C' are transformed into vertex x_{a^*} and a "generalized" cone on the projective plane; more specifically the projective plane is divided by x_{a^*} and C' into two connected regions as shown in Figure 4 with no vertex or edge of $(I(L))_G$ lying on or crossing the regional boundary. A graph so represented is clearly planar, and the corresponding planar embedding has a face that was derived from the face containing the origin in $I(L)$, which corresponds to the exterior face of L, but may include additional vertices. □

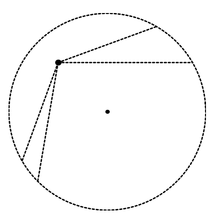

Figure 4. A division of the projective plane into two planar pieces

If G and L are as in Prop. 3.3, note that the embedding of G that is equivalent to L is actually an embedding of G in the projective plane, but this proof shows how the embedding can be redone, giving a planar embedding of G.

For the layout of graphs with cut-vertices we need the following special sorts of layout, as given by the algorithm described in [O] for 2-connected graphs.

Proposition 3.4 *Let G be an edge or a 2-connected, loopless, plane graph with no interior digon face, and c a designated vertex on the exterior face F. If d is a neighbor of c with edge (c,d) lying on F, then there is a bar-visibility layout L equivalent to G with bottom-most bar representing c, top-most bar representing d, with both bars extending the entire width of the layout, and with F represented as the external face of L. In addition if $F = (c, d, x_1, \ldots, x_k)$, $k \ge 1$, the bars representing the vertices on F can all have collinear right (respectively, left) endpoints, and visibility between c and d is along their left (resp., right) endpoints.*

The importance of this proposition is that only bars c and d have visibilities to the outside of the layout. And such a bv layout can be transformed into a pv layout as in the proof of Prop. 3.2, spanning whatever angle is needed, and with F becoming the exterior face of the layout.

Proposition 3.6 will be the necessary topological argument needed to characterize planar and projective planar PVGs (and later CVGs); it is a contraction proof similar to that of [H2, MR, T] and is the central result that allows for the PVG and CVG characterizations. For this result we consider special classes of multigraphs, those embedded in the plane (respectively, on the projective plane) with at most one (resp., with no) interior digon face with empty interior as designated below. In embedded graphs equivalent to pv or cv layouts there are three types of possible multiple edges: those bounding the infinite face of the plane, those bounding an interior region with nonempty interior, and a noncontractible 2-cycle on the projective plane. For these classes of multigraphs we achieve layouts with a 1–1 correspondence between distinct, maximal visibility cones and edges of G with possibly a loop created by a long arc at the origin in some cases. The layouts are unusual with the presence of the long-arc pairs, pairs needed for extension to CVGs; in [H2] less constrained pv layouts are obtained for 2-connected planar and projective planar graphs. Note that a pv layout L may have (a_1, b_1) as a long-arc pair at the origin or at infinity though vertices x_{a_1} and x_{b_1} lie on all non-digon faces; e.g., if a_1 spans $0 \leq \theta \leq 3\pi/2$ and b_1 $17\pi/12 \leq \theta \leq 2\pi$, the arcs for the other vertices on a face incident with x_{a_1} and x_{b_1} can be incident with the line $\theta = 2\pi$.

There is flexibility in the layout of PVGs, although sometimes there seems to be less. It is not hard to see (by experimentation or as in [H2] or [LMW]) that a bar-visibility or polar visibility representation of an interior face with four or more edges, must have bars or arcs with repeated or *collinear* endpoints. For example, the layouts of Fig. 1a and b have, respectively, collinearities among the bars (resp., arcs) with y-coordinates (resp., radii) 2, 3, and 4, for these represent vertices lying on a face of four edges in the embedded graphs, but the collinearities between the bars (resp., arcs) with y-coordinates (resp., radii) 1 and 5 can be easily eliminated by extending or contracting the corresponding elements. In fact, collinearities on the external face can always similarly be eliminated, but collinearities must always arise to represent an interior face with more than three edges. To prepare for one problematic type of collinearity [S], we prove the following.

Lemma 3.5 *Let L be a polar visibility layout, equivalent to a plane or projective plane graph G, and suppose there are arcs $A = (a_1, \ldots, a_k)$ that are consecutive and collinear along a ray $\theta = \theta^*$. Then there is an $\epsilon > 0$ so that L and the arcs of A can be altered to have collinearities along the line $\theta = \theta^* + \epsilon$ or $\theta^* - \epsilon$.*

Proof. We may assume that A is a maximal collection of collinear arcs, and we do not assume that all arcs of A lie on the same side of the ray. For a sufficiently small angle ϵ, if there are no arcs in the (infinite) cone $C^* = \{(r, \theta),\ \theta^* \leq \theta \leq \theta^* + \epsilon\}$, then the arcs can each be extended or contracted (depending on which side of the ray they lie) by ϵ so that they remain collinear along the ray $\theta = \theta^* + \epsilon$, without changing any other visibilities. Otherwise there is an arc a_0 that enters and crosses the cone C^* at radius less than that of a_1 or an arc a_{k+1} that similarly crosses C^* at radius greater than that of a_k. Again the arcs of A can each be extended or contracted by ϵ to become collinear along the ray $\theta = \theta^* + \epsilon$. Arc a_1 cannot gain unwanted visibility to a_0 by expansion, for then A was not a maximal collection, and cannot lose visibility to a_0 by contraction when ϵ is chosen sufficiently small. Similarly arc a_k will not become incorrectly visible or invisible to a_{k+1}. This same alternation may also be done to move the collinearity to the line $\theta = \theta^* - \epsilon$ for ϵ sufficiently small. □

Proposition 3.6

(i) Let G be a loopless 2-connected plane multigraph with exterior face F (possibly a digon), and let c be a vertex of G. When c does not lie on F, G may have at most one interior digon face, which must then be incident with c. Then $G' = G$ plus a loop at c has a polar visibility layout L' in which all vertices of F are represented on the exterior face of L' and (a_c, a_k) is a long-arc pair at the origin for some neighbor x_k of c. In addition, G' has an embedding on the projective plane that is equivalent to L'.

(ii) Let G be a loopless 2-connected plane multigraph with v_1 and v_2 designated, distinct vertices and with no interior digon face. Then $G' = G$ plus a loop at v_2 has a polar visibility layout L' with v_i represented by arc a_i, $i = 1, 2$, with (a_1, b_1) a long-arc pair at infinity, and with (a_2, b_2) a long-arc pair at the origin, where for $i = 1$ and 2, arc b_i corresponds to some neighbor of v_i. Also G' has an embedding on the projective plane that is equivalent to L'.

(iii) Let G be a loopless 2-connected multigraph with a 2-cell embedding on the projective plane with face F and with no digon face except possibly for F (respectively, with no digon face and with v_1 a designated vertex). Then G has a polar visibility layout L that is equivalent to the embedding of G with exterior face corresponding to F (resp., with v_1 represented by arc a_1 with (a_1, b_1) a long-arc pair at infinity and with arc b_1 corresponding to a neighbor of v_1.)

Proof of *(i)*. Let G be as given and suppose c lies on $F = (c, d, x_1, \ldots, x_k)$, $k \geq 0$. The graph G is a BVG and by Prop. 3.4 and Prop. 3.2 can be laid out as a PVG in the first quadrant with a_c at radius 1, with a_k representing x_k visible to a_c along the ray $\theta = 0$, and with exterior face corresponding to F. Without loss of generality suppose a_c spans $0 \leq \theta \leq \pi/2$ and is visible to a_k through $0 \leq \theta \leq x$. Then a_c can be shifted and extended to span $x \leq \theta \leq 3\pi/2$ and a_k can be extended to include the span $-\pi/2 \leq \theta \leq x$ so that we have (a_c, a_k) retain the same number of mutual visibilities and become the needed long-arc pair at the origin with long-arc cone spanning $-\pi/2 \leq \theta \leq x$.

Otherwise c does not lie on F and we proceed by induction on n. When $n = 3$, the graph can have at most 5 edges, and layouts of all possibilities are either in Figure 5 or are close variations to these. In general suppose $n \geq 4$ and let F' be the face incident with c with the fewest boundary edges, possibly only two. By Lemma

3.1, there is a simple edge $e = \{x, y\}$ of G so that G/e is 2-connected. We contract e to form a new vertex x^*. Set F^* (respectively, F'^*) to be F or F/e if e lies on F (resp., F' or F'/e if e lies on F'). Let $c^* = c$ or x^* if x or y is c. If e lies on a 3-sided face $\{x, y, z\}$, other than F or F', in G/e, x^* and z are joined by a single edge, but on F/e or F'/e this face becomes a digon.

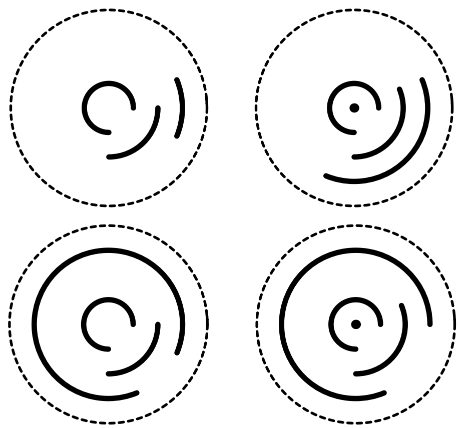

Figure 5. Layouts of planar PVGs with $n = 3$.

Then G/e satisfies the inductive hypothesis with face F^* and vertex c^* and so has a pv layout L_e with vertices of F^* on the exterior face, and long-arc pair (a_{c*}, a_k) at the origin for some neighbor x_k of c^* in G/e. Let G'' be G/e plus a loop at c^*, embedded on the projective plane equivalent to L_e. Let a^* be the arc representing x^* at, say, radius r. First suppose $a^* \neq a_{c^*}$ or a_k so that a^* is not part of L_e's long-arc pair at the origin, and consider the arc-rotation at a^*. Arc a^* will be replaced by two mutually visible arcs a_x and a_y at radii $r - .5$ and $r + .5$ (or vice versa), representing vertices x and y of G, so that together they span the same angle as did a^*, their visibilities give all edges incident with x and y and preserve the arc-rotations at x and y in G: consider a^* as shown in Figure 6a. There the visibilities to a_x are shown with dashed lines and those to a_y with solid lines. In that example a_x would be placed at radius 3.5, spanning (roughly) $\pi/4$ to $\pi/2$, and a_y at radius 2.5 spanning 0 to $\pi/3$. In general, because the embeddings of G'' and L_e are equivalent, the lines of visibility to arcs representing vertices adjacent to x in G are consecutive in the rotation of visibility lines about a^*.

There is one possibly difficult case in determining the splitting points for a_x and a_y and their overlap [S]; these were $\pi/4$ and $\pi/3$ in Fig. 6a. Note that, as in Fig. 6a, it may be that both a_x and a_y have visibility to another arc, there an arc at radius 5, and so there is choice throughout an interval of where one's visibility outwards ends and the other's begins, and overlap between a_x and a_y can be achieved within that interval. If in contrast as in Fig. 6b, a_x had visibility to an arc like a_4 at radius 4, but not to a_5 at radius 5, whereas a_y had visibility to a_5, but not to a_4, then the division point for a^* is uniquely determined at some fixed angle θ^*. If on both sides of a^*, inward and outward as shown in Fig. 6b, the division point between a_x and a_y, dictated by their visibilities, is exactly θ^*, then a^* must be divided into a_x and a_y with no overlap between the two new arcs. In fact, such precise division happens when the edge (x, y) in G lies on two faces, each of four or more vertices so that in the layout of G, collinearities must exist among the corresponding arcs to achieve these two faces. If (by bad luck) the collinearities on both sides of a^* lie on the same line $\theta = \theta^*$, we apply Lemma 3.5, and move the collinearities on one side to, say, $\theta = \theta^* + \epsilon$. Then the new arcs a_x and a_y can be drawn to overlap through an angle of ϵ, from θ^* to $\theta^* + \epsilon$. This alteration and then positioning of a_x and a_y gives us the desired pv layout L for G. Furthermore, whether or not e lies on F, all vertices of F are represented by arcs on the exterior face. The long-arc pair $(a_{c^*}, a_k) = (a_c, a_k)$ at the origin is as needed, and the graph so represented is G plus a loop at c.

The same substitution and possible alteration can be made if $a^* = a_k$, except that a long-arc pair may not be created. If a^* lies at radius r, the visibilities of a_x and of a_y dictate which should be placed on the inner radius $r - .5$ and which on $r + .5$. Since all visibilities from a^* in the angular span of the long-arc cone are outward except for the $a_c - a_k$ visibility, the inner arc, say a_x, can be extended within a_y to form a long-arc pair at the origin. If a_y loses needed visibility to a_c, a_c can be extended and a_x contracted within the angular span of the long-arc cone to create this visibility radially, not through the origin, keeping (a_c, a_x) a long-arc pair.

If $a^* = a_{c^*}$, then the division of a^* into a_x and a_y probably won't leave a long arc or a long-arc pair at the origin. Again since all but one visibility from a^* in L_e is outward, either the inner arc, say a_x, can take the place of a_{c*} and a_y can be extended inside a_k to form a long-arc pair (a_x, a_y) at the origin, if y and x_k are adjacent, or, if not, since x and x_k are then adjacent, a_x can be extended inside a_k so that again (a_x, a_y) form the correct long-arc pair.

Figure 6. The visibilities of a^*, (a) without collinearities and (b) with collinearities

(ii) This case is similar with v_2 playing the role of c in case *(i)*. Suppose that v_1 and v_2 lie on a common face, and if they are not adjacent, add an edge joining them so that they now lie on $F = (v_1, v_2, x_1, \ldots, x_k)$, $k \geq 1$. Then by Prop. 3.4 and Prop. 3.2 the graph has a a pv layout in the first quadrant with a_1 the outermost arc, a_2 the innermost arc, representing v_1 and v_2, respectively, with both arcs spanning $0 \leq \theta \leq \pi/2$ and with F represented as the exterior face. Since all interior facial

cycles of the graph have length at least three, there are arcs $b_1, \dots b_k$ representing $x_1, \dots, x_k$ with a_2 and b_1 visible along $\theta = 0$, and also b_k and a_1 visible along $\theta = 0$; possibly $b_1 = b_k$. Arcs a_1 and a_2 are mutually visible along $\theta = \pi/2$. First expand the whole layout to span $0 \leq \theta \leq 3\pi/2$, then extend all the b_i to include $-\pi/2 \leq \theta \leq 0$. If b_1 spans $-\pi/2 \leq \theta \leq x$ for some $x > 0$, then delete the interval $0 \leq \theta \leq x$ from a_2's span so that (a_2, b_1) retain the same number of mutual visibilities. If v_1 and v_2 were not originally adjacent in G, other arcs can be extended to the line $\theta = 3\pi/2$ to block $a_1 - a_2$ visibility. This gives the desired layout of G' and of G. (If the infinite face is a digon, determined by v_1 and v_2 and a pair of multiple edges joining them, the same procedure works, only a_1 can be extended to $-\pi/4$ and a_2 to $7\pi/4$.)

Otherwise we proceed by induction. When $n = 3$, the graph is a triangle or a triangle with one edge doubled to form an exterior digon with different choices for v_1 and v_2. These are represented as one of the layouts in Fig. 5 or as minor variations of these. For $n > 4$, by Lemma 3.1 we find $e = \{x, y\}$ so that G/e is 2-connected with new vertex x^*. If x or y is v_1 (respectively, v_2) set $v_1 = x^*$ (resp., $v_2 = x^*$). If e lies on a triangular face $\{x, y, z\}$, the face becomes the single edge $\{x^*, z\}$ so that no interior digon face is formed. By induction G/e has a pv layout L_e with long-arc pair (a_{v_1}, b_1) at infinity and long-arc pair (a_{v_2}, b_2) at the origin for x_{b_1} a neighbor of v_1 and x_{b_2} a neighbor of v_2, and let arc a^* represent vertex x^*. If x^* is not part of a long-arc pair, we may proceed as in the proof (of the easiest case) of *(i)* to split a^* into arcs a_x and a_y possibly altering the layout to avoid collinearities so that the desired pv layout L of G is obtained.

Suppose a^* is one of the arcs in a long-arc pair, $a^* \epsilon \{a_{v_1}, a_{v_2}, b_1, b_k\}$. If $a* = b_1$ (respectively, b_2), the same substitution can be made as above, and if the needed long-arc pair is not created, we proceed as we did in case *(i)*. Since all visibilities are inward (resp., outward) within the span of the long-arc cone, the arc a_x can be placed on the greater (outer) radius (resp., lesser (inner) radius) and extended outside (resp., inside) a_y so that a_{v_1} (resp., a_{v_2}) is part of a long-arc pair at infinity (resp., at the origin). If a_y has lost needed visibility, a_{v_1} (resp., a_{v_2}) can be extended and a_x contracted to replace this visibility, keeping the long-arc pair. This procedure works even when $b_1 = b_2$ since the final adjustments are made independently. If $a^* = a_{v_1}$ (respectively a_{v_2}), then $a_{v_1} \neq a_{v_2}$, and as in *(i)* either a_x or a_y can become a long arc, paired with b_1 (resp., b_2).

(iii) If G is embedded on the projective plane with a region that is not a 2-cell, then G is embedded in a contractible region of the projective plane and case *(i)* applies. Otherwise the main statement is essentially equivalent to that proved in [H2, Theorem 1b], and the variation needed for the designated vertex v_1 is covered just as the proof for *(i)* was varied to give *(ii)*. □

Proposition 3.7 *If G has a polar visibility layout L, then the embedding $(I(L))_G$ of G on the projective plane has cut-vertices on at most two faces. If the embedding has cut-vertices on two faces, then on one face there is only one cut-vertex, represented in L by a long arc.*

Proof. Let the layout have exterior face E, and in the graph $(I(L))_G$ on the projective plane suppose E has become E', a face containing the origin. Suppose in $(I(L))_G$ there are two additional faces F_1 and F_2 containing, respectively, cut-vertices v_1 and v_2 that do not lie on E' or on F_{3-i}, $i = 1$ or 2, respectively. In L both F_1 and F_2 correspond to interior faces bounded by arcs and radial lines. Then arcs, corresponding to vertices within a block incident with v_i and lying within F_i, will see at least two boundary arcs unless one boundary arc is a long arc. Suppose both a_1 and a_2 are long arcs representing v_1 and v_2, respectively, not lying on E, and lying at radius r_1 and r_2 with $r_1 < r_2$. Then arcs corresponding to a block incident with v_2 will see at least two arcs, a contradiction. If there is one additional face F, besides E', it can contain at most one cut-vertex since this must be represented as a long arc in L on the face corresponding to F. □

Corollary 3.8 *If G has a polar visibility layout L with a long arc a^*, representing a cut-vertex x^* and not lying on the exterior face, then G has a planar embedding with all cut-vertices except for x^* lying on a common face.*

Proof. By Prop. 3.3, G is planar, and by Prop. 3.7 all other cut vertices of G lie on a common face E' in the embedding of $(I(L))_G$ on the projective plane with E' corresponding to the exterior face E of L. Then Prop. 3.3 shows that G can be drawn in the plane with E' as a face. □

Theorem 3.9 *A simple planar graph G has a PVG representation if it has a planar embedding with all but at most one cut-vertex on a common face.*

Proof. Let G be drawn in the plane with cut-vertices lying on the exterior face F_1 and an additional cut-vertex c lying on F_2; if c does not exist, then G is a BVG and by Prop. 3.2 has a pv layout. Consider the block-cutpoint tree BC(G) of G; c may lie on several blocks, but at least one, call it B_0, contains a cut-vertex $c' \neq c$ lying on F_1. Both of the faces F_i are bounded by a facial walk W_i, and each W_i contains a unique simple subcycle C_i, lying in B_0 and containing c' and c, respectively. If G has cut vertices $c_1, \ldots, c_i$ lying on F_1, we label the blocks other than B_0 incident with c_1, …, or c_i, $B_1, B_2, \ldots, B_j$, and the blocks incident with c, $D_1, \ldots, D_k$. We prove by induction on j that there is a pv layout L of G with F_1 represented by the exterior face of L, with (a_c, a_d) a long-arc pair at the origin for some neighbor d of c, and with the blocks incident with c represented within $C(a_c)$. (Note that when $n = 8$, $G = K_4 + 4e$ can be so laid out with one of the added edges represented with double visibility, as shown in Figure 7a, as can $K_{2,3} + 3e$. The smallest simple PVGs that are not BVGs and are represented with exact visibility are $K_4 + 3e + t$ and $K_{2,3} + 2e + t$ where t is a triangle of three edges incident with one degree-2 vertex of the first graph; see Fig. 7b.)

With the base case of $j = 0$ when there is no cut-vertex on F_1 and G has only one cut-vertex c, we begin our layout technique even though alternatively G is a BVG and so a PVG. We remove the vertices and the edges of blocks $D_1 \setminus \{c\}, \ldots, D_k \setminus \{c\}$ leaving the 2-connected graph, call it B_0, with new face F_2' formed by the removal of blocks D_i from F_2. We apply Prop. 3.6*(i)* with $F = F_1$ to get the desired layout

L' of $B' = B_0$ plus a loop at c, with F corresponding to the exterior face and with (a_c, a_d) a long-arc pair at the origin for some neighbor d of c. For each block D_i, $i = 1, \ldots, k$, we use Prop. 3.4 to obtain a bv layout of D_i for some neighbor d_i of c, with c represented by the top-most bar and d_i as the bottom-most bar. Then we use Prop. 3.2 to obtain pv layouts L_i of D_i. These can each be added to exactly fill the angular span $C(a_c)$ of the long arc a_c by placing each arc representing c as a consecutive subarc of a_c so that the appropriate arcs see a_c and nothing else; the arc from the bottom-most bar sees a_c through the origin. This gives the desired layout of G.

Otherwise $j \geq 1$, and we may renumber so that B_j is a leaf in the block-cutpoint tree of G, incident with cut-vertex c_i. We delete all vertices and edges of $B_j \setminus \{c_i\}$, and by induction lay out the remaining graph as L' so that L' has exterior face F_1' containing the arc a_i representing c_i, with the long-arc pair (a_c, a_d) at the origin for some neighbor d of c, and with the bv layouts for the blocks at c filling the angular span of $C(a_c)$. We use Prop. 3.4 to obtain a bv layout of B_j with c_i represented by the bottom-most bar, and then use Prop. 3.2 to obtain the corresponding L_j. Here L_j is added to the exterior face of L' placing c_i's arc as a subarc of a_i in L' so that appropriate arcs see a_i and no others. This gives the desired layout of G. □

Notice that when any of the blocks D_i consists of a single edge, then it will be represented with double visibility. If a block has at least 3 vertices, then it is possible to use Prop. 3.4 and Prop. 3.2 to layout the block minus an edge from c to a neighbor and to have that missing visibility achieved through the origin; e.g., see Fig. 7.

Theorem 3.10 *If a simple graph has an embedding on the projective plane with all cut-vertices on a common face, then it is a PVG.*

Proof. Let G have an embedding on the projective plane $\boldsymbol{P}$ with all cut-vertices on a common face F. When $n = |V(G)| < 5$, the graph has a bv layout and so a PVG one by Prop. 3.2; each such graph containing a cycle also has a 2-cell embedding on the projective plane and an equivalent pv layout. We prove by induction on $n \geq 5$ that G has a pv layout with arcs representing cut-vertices on the exterior face and with its embedding equivalent to that of G.

If G has no cut-vertex, then we apply Prop. 3.6*(iii)* for graphs on the projective plane to get the pv layout of G. If G has a cut-vertex, consider the block-cutpoint tree $\mathrm{BC}(G)$ of G, and, if possible, let c be a cut-vertex incident with a leaf of $\mathrm{BC}(G)$ with that leaf-block planar and embedded in a contractible region of $\boldsymbol{P}$; call this block B. Deleting the vertices and edges of $B \setminus \{c\}$ leaves G' on the projective plane with face F now a face F', containing all remaining cut-vertices. By induction, G' has a pv layout L' that is equivalent to G' and with exterior face representing F'. By Prop. 3.4 there is a bv layout of B with the bar representing c bottom-most, and by Prop. 3.2, B has a corresponding pv layout L_B. Then a_c in L_B can be inserted as a subarc of a_c on the exterior face of L' so that L_B together with L' gives the desired layout of G.

Otherwise every leaf-block B is embedded in a noncontractible region of $\boldsymbol{P}$ and contains a noncontractible cycle in its embedding. If blocks B and B' are two such leaves, they must intersect at a cut-vertex c since every pair of noncontractible cycles on $\boldsymbol{P}$ intersect. If there are additional blocks, there are additional leaves which must also all meet at c so that BC(G) is a star $K_{1,i}$ with the degree-i vertex of BC(G) representing c, the only cut-vertex of G, and each block is embedded in a wedge of $\boldsymbol{P}$, all wedges meeting at, say, the origin. Such a graph is planar with one cut-vertex c and so by Theorem 3.9 is a PVG. □

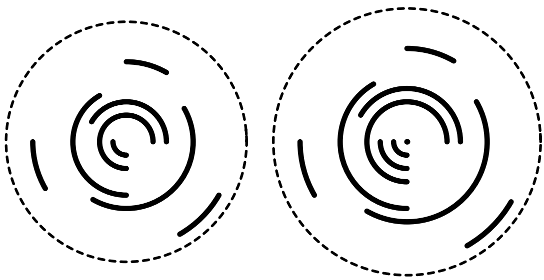

Figure 7. $K_4 + 4e$ as a PVG; $K_4 + 3e + t$ as a PVG.

Proof of Theorem 2.2 for simple graphs. By Theorems 3.9 and 3.10 the graphs described are PVGs. Conversely if L is a layout of a PVG G, then G has an embedding on the projective plane by Prop. 2.1 with embedding $(I(L))_G$. If L has no visibility through the origin, then by Prop. 3.2 G is a BVG and so embeds in the plane with all cut-vertices on a common face. Otherwise, if L contains a long arc, satisfying the conditions of Cor. 3.8, then G embeds in the plane with all but one cut-vertex on a common face. Otherwise G embeds in the projective plane with all cut-vertices on a common face by Prop. 3.7. □

4 Results on CVGs

As the example in Fig. 3 and its extensions demonstrate, cut-vertices on many faces can be achieved using circles in layouts. We characterize CVGs in this section, as given in Theorem 2.3.

Suppose G has a layout L with circles $c_1, c_2, \ldots, c_k$ at radii $r_1 < r_2 < \ldots < r_k$ respectively and with no circle replaceable by an arc so that the same visibilities are achieved; that is, in L, circles have been replaced with an arc or long arc when possible, leaving $k > 0$ circles. The circles c_i divide up the plane into annular regions and one projective planar region; note that neither the interior of c_1, denoted int(c_1), nor the exterior of c_k, ext(c_k), is empty in L since neither circle can be replaced by an arc. Then the corresponding vertices $v_1, v_2, \ldots, v_k$ of G are cut-vertices, and G is the union of graphs whose layouts lie in the annular regions plus the innermost region: $G = G_1 \cup G_2 \cup \ldots \cup G_k \cup G_{k+1}$ where G_1 is the subgraph whose layout in L

lies on $c_1 \cup \operatorname{int}(c_1)$, G_{k+1} lies on $c_{k+1} \cup \operatorname{ext}(c_{k+1})$, and for $i = 2, \ldots, k$, G_i lies on the annulus given by $c_{i-1} \cup c_i \cup \{\operatorname{int}(c_i) \cap \operatorname{ext}(c_{i-1})\}$. Thus $G_2, \ldots, G_{k+1}$ are each planar. In addition for $i = 2, \ldots, k$, G_i is 2-connected since each block of G_i contains some vertices adjacent to v_{i-1} and some to v_i. Thus the block-cutpoint tree for G, $\mathrm{BC}(G)$, contains a path of $2k - 1$ vertices, representing consecutively $v_1, G_2, v_2, \ldots, G_k, v_k$. What sorts of graphs are possible for G_1 and for G_{k+1}, and what additional tree structure in $\mathrm{BC}(G)$ is possible at the two ends of this path?

Consider G_1, laid out on $c_1 \cup \operatorname{int}(c_1)$, with c_1 opened up to become an arc so that this is a pv layout of G_1 with a_1 on the exterior face. If G_1 is planar, by Prop. 3.7, G_1 can have at most one cut-vertex that is represented by a long arc a^* at the origin (possibly $a^* = a_1$). If there is no long arc a^* besides a_1, then v_1 may be attached to an arbitrary positive number, say i_1, of planar blocks. If there is a long arc $a^* \neq a_1$, then each block represented between a^* and a_1 sees these two arcs and so there is only one block lying in this annular region. Inside and attached to a^* may be any number $i_a \geq 0$ of 2-connected, planar graphs, but in any case, $\mathrm{BC}(G)$ has attached to the path-end v_1 either $i_1 > 0$ leaves (when $a^* = a_1$), or else one additional block vertex b, representing part or all of G_1, then a vertex for a^* that is also adjacent to $i_a > 0$ vertices of degree one. (Thus the latter case corresponds to having v_3 represented by c_1 and v_1 by a^*.) If G_1 is not planar, by Prop. 3.7 and Cor. 3.8 it is 2-connected so that the path of $\mathrm{BC}(G)$ is extended at v_1 by one additional vertex representing G_1.

The layout for the planar graph G_{k+1} lies in the infinite region, $c_k \cup \operatorname{ext}(c_k)$. In this layout of G_{k+1} the circle c_k can be opened up to a long arc with empty interior to form a pv layout; by Prop. 3.7, G_{k+1} has all its cut-vertices on a common face, the exterior face, and so can have arbitrarily many cut-vertices with arbitrarily many connected blocks, provided all cut-vertices lie on the infinite face. Thus attached to v_k in $\mathrm{BC}(G)$ is any tree representing a planar graph with all cut-vertices, except possibly for v_k, on a common face. These remarks prove the necessity of Theorem 2.3.

Lemma 4.1 *Let L be a layout of a PVG G with n vertices and with a long-arc pair at infinity or at the origin (or both). Then G can be laid out as a CVG with a circle on the exterior face at radius n or a circle about the origin at radius 1 (or both).*

Proof. Since the paired arcs span 2π and have an empty long-arc cone, a long arc at radius 1 or at n can be extended to a full circle, changing no visibilities. □

Theorem 4.2 *A simple graph G is a CVG if the vertices of $BC(G)$ can be partitioned into three sets P, Q, and R, where*

0) *$P = (b_1, b_2, \ldots, b_{2k+1})$, $k \geq 0$, is a path with each b_{2i} representing a planar block, $i = 1, \ldots, k$;*

1a) *Q is a nonempty block adjacent to b_1, representing a (2-connected) projective planar graph, or*

1b) *Q is a set of one or more (nonempty) planar blocks, all adjacent to b_1; and*

2) *R is an arbitrary tree structure adjacent to* b_{2k+1}*, such that* $R \cup \{b_{2k+1}\}$ *represents a planar graph that can be drawn in the plane with all cut-vertices, except possibly for that representing* b_{2k+1}*, on a common face.*

Proof. Suppose G has BC(G) satisfying 0), 1a), and 2) so that BC(G) is $(b_0, b_1, b_2, \ldots b_{2k+1}, R)$ where for $i = 1, \ldots, k$, each b_{2i-1} represents a cut-vertex v_i of G, each b_{2i} represents a 2-connected planar graph, b_0 is a 2-connected projective planar graph, and R represents a plane graph with all cut-vertices on a face F. It is not hard to see that such a graph embeds on the projective plane; that result follows also from our cv layout of G. In the layout each cut-vertex v_i will be represented by a circle c_i.

By Prop. 3.6*(iii)* the projective planar subgraph of G corresponding to b_0 has a pv layout L_0' with the arc a_1 representing v_1 as long-arc pair at infinity, paired with some neighbor of v_1. By Lemma 4.1, L_0' can be changed to the CVG L_0 so that a_1 becomes a circle surrounding L_0. By Prop. 3.6*(ii)* the planar subgraph of G corresponding to b_2 can be represented as a PVG L_1' with a_1, representing v_1, part of a long-arc pair at the origin and with a_2, representing v_2, part of a long-arc pair at infinity. By Lemma 4.1 L_1' can be changed to the CVG L_1 so that a_1 and a_2 each become circles inside and surrounding L_1 respectively. Then L_1 is joined with L_0 by identifying the two copies of a_1, placing L_1 wholly outside of L_0. This process of expansion can be repeated for $b_4, \ldots, b_{2k}$. Finally by Prop. 3.6*(i)* R can be laid out as a PVG with v_k represented by a_k, part of a long-arc pair at the origin. Again by Lemma 4.1 a_k can be extended to a full circle inside of R's layout and can be identified with the circle representing a_k on the exterior of the layout previously constructed. In this way G is laid out.

Suppose BC(G) satisfies 1b) and 2). Begin by laying out the planar block corresponding with b_2, as in the preceding paragraph, to get L_1 with circles c_1 and c_2, representing v_1 and v_2, as bounding inner and outer circles. Extension on the exterior of c_2 to layout the graph represented by $b_3, b_4, \ldots, R$ is also done just as in the preceding paragraph; it's the interior of c_1 where the difference occurs. Since v_1 is incident with one or more planar blocks, we can lay these out in radial segments as within the angular span of a long arc at the origin. Each planar block is represented as a BVG with v_1 represented top-most and a neighbor bottom-most, then as a PVG via Prop. 3.2, and then inserted with v_1's arc as a subarc of c_1 within a distinct wedge of, say, $0 \leq \theta \leq \pi$, giving the desired visibilities. Also the pv layouts can be expanded to fill this whole interval so that c_1 is not self-visible. Thus in all cases the graph can be laid out as a CVG. □

These observations and Theorem 4.2 prove Theorem 2.3 for simple graphs, and more details on multigraphs can be deduced from the proofs.

5 Concluding thoughts

It is clear that more complex graphs can be achieved in the polar visibility model by allowing visibility through the origin and diagonally across the boundary of a disc with antipodal points identified. These correspond naturally to graphs that embed on the Klein bottle, the nonorientable surface of Euler characteristic 0. Some preliminary results in this context are given in [H1]. Also there and in [H2] some comments are given on the algorithmic aspects of laying out PVGs and CVGs.

Acknowledgements.

The author wishes to thank Therese Biedl, Alexandru Burst, Alice Dean, Michael McGeachie, Bojan Mohar, Joe O'Rourke, William Owens, Tom Shermer, Carsten Thomassen, and Stan Wagon for useful conversations and insightful examples concerning this work, and an anonymous referee for helpful suggestions.

References

[BDHS] P. Bose, A. Dean, J. Hutchinson, and T. Shermer, On rectangle visibility graphs, *Lecture Notes in Computer Science* #1190 (S. North, ed.), Springer, Berlin (1997), 25–44.

[CJLW] Y-W. Chang, M.S. Jacobson, J. Lehel, and D.B. West, The visibility number of a graph (preprint).

[DH1] A. Dean and J.P. Hutchinson, Rectangle-visibility representations of bipartite graphs, *Discrete Applied Math.* 75 (1997), 9–25.

[DH2] ———, Rectangle-visibility layouts of unions and products of trees, *J. Graph Algorithms and Applications* 2 (1998), 1–21.

[D] A. Dean, Bar-visibility graphs on the Möbius Band, *Graph Drawing, 8th International Symposium, GD 2000,* Colonial Williamsburg, VA, Sept. 2000, Proceedings, (J. Marks, ed.), *Lecture Notes in Computer Science* #1984, Springer, Berlin, 2001, pp. 350–359.

[HSV] J.P. Hutchinson, T. Shermer and A. Vince, On representations of some thickness-two graphs (extended abstract), *Computational Geometry, Theory and Applic.,* 13 (1999), 161–171.

[H1] J.P. Hutchinson, On polar visibility representations of graphs, *Graph Drawing, 8th International Symposium, GD 2000,* Colonial Williamsburg, VA, Sept. 2000, Proceedings, (J. Marks, ed.),*Lecture Notes in Computer Science* #1984, Springer, Berlin, 2001, pp. 63–76.

[H2] ———, A note on rectilinear and polar visibility graphs, preprint.

[LMW] F. Luccio, S. Mazzone and C. Wong, A note on visibility graphs, *Discrete Math.* 64 (1987), 209–219.

[M] B. Mohar, Projective planarity in linear time, *J. Algorithms* 15 (1993), 482–502.

[MR] B. Mohar and P. Rosenstiehl, Tessellation and visibility representations of maps on the torus, *Discrete Comput. Geom.* 19 (1998), 249–263.

[MT] B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins Press, Baltimore, 2001.

[O] J. O'Rourke, *Art Gallery Theorems and Algorithms*, Oxford University Press, Oxford, 1987.

[S] T. Shermer, personal communication.

[TT1] R. Tamassia and I.G. Tollis, A unified approach to visibility representations of planar graphs, *Discrete Comput. Geom.*, 1 (1986), 321–341.

[TT2] ———, Tesselation representations of planar graphs, *Proc. 27th Annual Allerton Conf. on Communication, Control, and Computing* (J.V. Medanic and P.R. Kumar, eds.), 1989, 48–57.

[TT3] ———, Representations of graphs on a cylinder, *SIAM J. Disc. Math.* 4 (1991), 139–149.

[T] C. Thomassen, Planar representations of graphs, *Progress in Graph Theory* (J.A. Bondy and U.S.R. Murty, eds.), 1984, 43–69.

[We] D. West, *Introduction to Graph Theory*, Prentice-Hall, Upper Saddle River, NJ, 1996.

[Wh] A. White, *Graphs, Groups and Surfaces* (revised ed.), North-Holland, Amsterdam, 1984.

[Wi] S. Wismath, Characterizing bar line-of-sight graphs, *Proc. 1st Symp. Comp. Geom.*, ACM (1985), 147–152.

(Received 23 Jan 2001)