The intersection problem for graphs with six vertices, six edges and a 4-cycle subgraph.

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Abstract

In this paper the possible numbers of blocks $|B_1 \cap B_2|$ in common to two G-designs, (V, B_1) and (V, B_2), are determined, where the graph G has six vertices and six edges, contains a cycle of length four, and has two pendant edges. There are four such graphs G.

1 Introduction

The *Intersection Problem* was first considered for the combinatorial structure, Steiner triple systems, by Lindner and Rosa [7]. This initial work was extended to cover many other combinatorial structures. For a particular structure, the intersection problem asks for which values of k is it possible to find two objects of the structure that have k blocks, entries, cycles etc. in common. Both objects must be based on the same element set. A survey by Billington [1] in 1992 addresses the progress made on the intersection problem for certain combinatorial structures, such as latin squares, one-factorizations of complete graphs, cycle systems and block designs. Billington later completed the intersection problem for m-cycle systems of K_v [2]. Another structure that has been investigated is a G-design. The intersection problem for $K_4 - e$ designs was completed by Billington, Gionfriddo and Lindner in 1997 [3]. Billington and Kreher [4] completed the intersection problem for connected simple graphs G where the minimum of the number of vertices of G and the number of edges of G is less than or equal to four. The intersection problem for a graph having a cycle of length four plus a pendant edge was done by Mortimer [8]. This particular graph has five vertices and five edges, and in [8] was referred to as a "dragon".

One of the more recent problems in this area, intersection numbers of Kirkman triple systems, has been completed by Chang and Faro [5] (with only a small number of cases missing).

The structure being considered here is a particular small type of G-design. A *G-design of order n*, where G is a simple graph, is a pair (V, B) where V is the vertex set of K_n and B is an edge-disjoint decomposition of K_n into copies of the simple graph G; these copies of G are called blocks. Furthermore, if V is the vertex set of a graph H and it is possible to decompose H into copies of G, then this is called a G-decomposition of H. Thus a G-design is the special case where $H = K_n$. The number of blocks, $|B|$, is $b = \binom{n}{2} \ / \ |E(G)|$ where $|E(G)|$ is the number of edges in the graph G and n is the number of vertices in K_n.

The general type of intersection problem which we shall consider here investigates the possible numbers of blocks which two designs, based on the same element set V, may have in common. That is, for designs, (V, B_1) and (V, B_2), we determine all possible values of k for which $|B_1 \cap B_2| = k$.

The type of graph G being considered here is one with a cycle of length four, six vertices, and six edges. There are four different graphs like this; we call them A, E, S and T, and they are shown in Figure 1, together with the notation used to denote them.

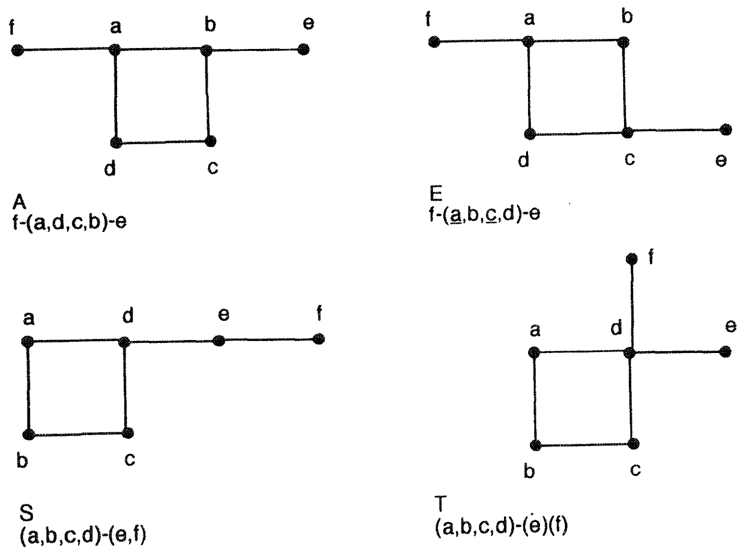

Figure 1. The types of graphs

Let $I_G(H)$ denote the set of achievable intersection values of a G-design on the graph H. When $H = K_n$ we abbrievate this to $I_G(n)$. Let $J_G(n) = \{0, 1, 2, \ldots, b-2, b\}$, which is the set of expected intersection numbers of one of our G-designs of order n.

2 Necessary Conditions and Methods

For a G-design of order n to exist, the number of edges in K_n, which is $\binom{n}{2}$, must be a multiple of the number of edges in the graph G, which is six in our case. So for the six-edged graphs, $b = \dfrac{n(n-1)}{12}$, and this must be an integer, so $n \equiv 0, 1, 4, 9 \pmod{12}$. Also $n \geq 6$ is clearly necessary since our graphs G have six vertices. So the smallest case will be of order 9.

In order to find intersection numbers, we use two techniques here: permuting vertices and trading blocks. Permuting involves applying a permutation to the vertices of the original design. If the permutation on the vertices is α, then we denote the resulting design by $G\alpha$. Trading involves replacing some of the blocks by a disjoint set of blocks which use precisely the same edges as the original blocks. A trade X consists of two sets of blocks, say T_X and T'_X where $E(T_X) = E(T'_X)$, $T_X \cap T'_X = \emptyset$, and T_X and T'_X both contain m blocks. We call m the volume of the trade. Clearly $T_X \subseteq B$, and is the original block set of the trade, while T'_X is called the final block set of the trade. If $X = \{T_X, T'_X\}$ and $Y = \{T_Y, T'_Y\}$ are two trades with edge-disjoint original block sets, we define $X \cup Y$ to be the union of the trades, with the original block set equal to $T_X \cup T_Y$ and the final block set equal to $T'_X \cup T'_Y$.

3 Small cases

The intersection numbers which can be achieved for small cases for each of the four graphs in Figure 1, that are necessary in the proof of Theorem 1 below, are given in a separate Appendix on a web page [6]. This Appendix has four sections, for small cases for the graphs A, E, S and T respectively. However, we include one example here for immediate illustrative purposes.

Example 3.1 $I_A(9) = \{0, 1, 2, 3, 4, 6\} = J_A(9)$.

For K_9 on the vertex set $V = \{1, 2, \ldots, 9\}$, one possible A-decomposition is such that $B = \{$7-(5, 6, 1, 2)-4, 3-(7, 9, 2, 6)-4, 7-(4, 5, 1, 3)-6, 1-(9, 5, 3, 8)-7, 3-(2, 7, 1, 8)-5, 1-(4, 8, 6, 9)-3$\}$.

Let $\alpha = (1\ 2)$ and $\beta = (1\ 2\ 6\ 9)(5\ 7\ 3)$. The following trades are used to establish the intersection numbers.

set	*original blocks*	*set*	*final blocks*
T_1	{7-(5,6,1,2)-4, 3-(7,9,2,6)-4, 7-(4,5,1,3)-6, 1-(4,8,6,9)-3}.	T'_1	{8-(6,1,5,4)-3 5-(6,3,7,9)-4, 8-(4,2,5,7)-6, 4-(1,3,9,2)-6}.
T_2	{7-(5,6,1,2)-4, 3-(7,9,2,6)-4, 7-(4,5,1,3)-6}.	T'_2	{1-(3,6,2,4)-7, 3-(7,9,2,5)-6, 2-(1,5,4,6)-7}
T_3	{3-(2,7,1,8)-5, 3-(7,9,2,6)-4}	T'_3	{9-(2,7,1,8)-5, 9-(7,3,2,6)-4}.

From the above trades and permutations we obtain the following intersection numbers for K_9.

$$|B \cap B\alpha| = 0$$
$$|B \cap B\beta| = 1$$
$$|B \cap ((B \backslash T_1) \cup T_1')| = 2$$
$$|B \cap ((B \backslash T_2) \cup T_2')| = 3$$
$$|B \cap ((B \backslash T_3) \cup T_3')| = 4$$
$$|B \cap B| = 6.$$

Hence $I_A(9) = J_A(9)$.

4 Intersection Numbers

Let G represent one of the graphs A, E, S or T (see Figure 1). If P is a set of non-negative integers and $h \in P$, then $h * P$ denotes the set of all integers which can be obtained by adding any h elements of P together (repetitions of elements of P allowed). If X and Y are two sets of non-negative integers then $X + Y$ denotes the set $\{x + y \mid x \in X, y \in Y\}$.

Theorem 1 *$I_G(n) = J_G(n)$ for all $n \equiv 0, 1, 4, 9$ (mod 12), $n \neq 4$.*

Proof.
Let $n = 12m + h$ where $h \in \{0, 1, 9, 16\}$, $m \geq 0$. Now $n = 12m + 16$, $m \geq 0$ covers the same values of n as $n = 12m + 4$, $m \geq 0$, $n \neq 4$.
We start with the construction of a suitable G-design.

$h \in \{0, 9, 16\}$

Let the vertex set of K_{12m+h} be $\{\infty_i \mid 1 \leq i \leq h\} \cup \{(i, j) \mid 1 \leq i \leq 2m, 1 \leq j \leq 6\}$.

For the graph K_{12m+h}, take one design on these vertices to have the following blocks:
1. The blocks in a G-design of order 12 on the set $\{(2i-1, j), (2i, j) \mid 1 \leq j \leq 6\}$ for $1 \leq i \leq m$ ([6]).

2. The blocks in a G-design of order h on the set $\{\infty_i \mid 1 \leq i \leq h\}$ ([6]).

3. The blocks in a G-design on the graph $K_{h,12}$ with vertex set $\{\{\infty_1, \ldots, \infty_h\}, \cup\{(2i-1, j), (2i, j) \mid 1 \leq j \leq 6\}\}$, for each i with $1 \leq i \leq m$ ([6]).

4. The blocks in a G-design on the graph $K_{6,6}$ with the vertex set $\{\{(i, k) \mid 1 \leq k \leq 6\} \cup \{(j, k) \mid 1 \leq k \leq 6\}\}$ for the following values of i and j:

   when i is even: for each i, j with $1 \leq i < j \leq 2m$;

   when i is odd: for each i, j with $1 \leq i < j \leq 2m$ where $j > i+1$ ([6]).

$\boxed{h = 1}$

Let the vertex set of K_{12m+h} be the same as above.
For the graph K_{12m+h}, take one design on these vertices to have the following blocks:

1. The blocks in a G-design of order 13 on the set, $\{\infty_1\} \cup \{(2i-1, j), (2i, j) \mid 1 \leq j \leq 6\}$ for $1 \leq i \leq m$ ([6]).
2. Step 4 as above.

The number of blocks, b, is $\dfrac{(12m+h)(12m+h-1)}{12}$. Then we expect the intersection numbers to be $\{0, 1, 2, \ldots, b-2, b\}$.

Having constructed our G-designs, we now establish the required intersection numbers.

Intersection numbers for K_{12m}.

From the decomposition of K_{12m} into copies of K_{12} and $K_{6,6}$ ([6]), and using their respective achievable intersection numbers, we have

$$\begin{aligned} I_G(K_{12m}) &\supseteq m * \{0, 1, \ldots, 9, 11\} + 2m(m-1) * \{0, 3, 6\} \\ &= \{0, 1, \ldots, 9, 10, 11, \ldots, 11m-2, 11m\} \\ &\quad + \{0, 3, 6, 9, 12, \ldots, 12m^2-12m-3, 12m^2-12m\} \\ &= \{0, 1, 2, \ldots, 12m^2-m-2, 12m^2-m\}. \end{aligned}$$

So the achievable intersection numbers of K_{12m} are equal to the expected intersection numbers.

Intersection numbers for K_{12m+9}.

From the decomposition of K_{12m+9} into copies of K_{12}, $K_{6,6}$, $K_{9,12}$ and K_9 ([6]), using their respective achievable intersection numbers, we have

$$\begin{aligned} I_G(K_{12m+9}) &\supseteq m * \{0, 1, \ldots, 9, 11\} + 2m(m-1) * \{0, 3, 6\} + m * \{0, 3, 6, \ldots, 15, 18\} \\ &\quad + \{0, 1, 2, 3, 4, 6\} \\ &= \{0, 1, \ldots, 9, 10, 11, \ldots, 11m-2, 11m\} \\ &\quad + \{0, 3, 6, 9, 12, \ldots, 12m^2-12m-3, 12m^2-12m\} \\ &\quad + \{0, 3, 6, \ldots, 18m-3, 18m\} + \{0, 1, 2, 3, 4, 6\} \\ &= \{0, 1, 2, \ldots, 12m^2+17m+4, 12m^2+17m+6\}. \end{aligned}$$

So we have the achievable intersection numbers of K_{12m+9} equal to the expected intersection numbers.

Intersection numbers for K_{12m+16}.

From the decomposition of K_{12m+16} into copies of K_{12}, $K_{6,6}$, $K_{16,12}$ and K_{16} ([6]), using their respective achievable intersection numbers, we have

$$
\begin{aligned}
I_G(K_{12m+16}) \supseteq\ & m * \{0,1,\dots,9,11\} + 2m(m-1) * \{0,3,6\} + m * \{0,4,8,\dots 28,32\} \\
& +\{0,1,\dots,18,20\} \\
=\ & \{0,1,\dots,9,10,11,\dots,11m-2,11m\} \\
& +\{0,3,6,9,12,\dots,12m^2-12m-3,12m^2-12m\} \\
& +\{0,4,8,\dots,32m-4,32m\} + \{0,1,\dots,18,20\} \\
=\ & \{0,1,2,\dots,12m^2+31m+18,12m^2+31m+20\}.
\end{aligned}
$$

So we have the achievable intersection numbers of K_{12m+16} equal to the expected intersection numbers.

Intersection numbers for K_{12m+1}.

From the decomposition of K_{12m+1} into copies of K_{13} and $K_{6,6}$ ([6]) and using their respective achievable intersection numbers, we have

$$
\begin{aligned}
I_G(K_{12m+1}) \supseteq\ & m * \{0,1,\dots,11,13\} + 2m(m-1) * \{0,3,6\} \\
=\ & \{0,1,\dots,11,12,13,\dots,13m-2,13m\} \\
& +\{0,3,6,9,12,\dots,12m^2-12m-3,12m^2-12m\} \\
=\ & \{0,1,2,\dots,12m^2+m-2,12m^2+m\}.
\end{aligned}
$$

So we have the achievable intersection numbers of K_{12m+1} equal to the expected intersection numbers.

We have now shown that the achievable intersections numbers of K_{12m+h}, $h \in \{0,1,9,16\}$ and $m \geq 0$, are equal to the expected intersection numbers, completing the proof of the theorem. □

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References

[1] E.J. Billington, *The intersection problem for combinatorial designs*, Congressus Numerantium **92** (1993), 33–54.

[2] E.J. Billington, *The intersection problem for m-cycle systems*, Journal of Combinatorial Designs **1** (1993), 435–452.

[3] E.J. Billington, M. Gionfriddo and C.C. Lindner, *The intersection problem for $K_4 - e$ designs*, Journal of Statistical Planning and Inference **58** (1997), 5–27.

[4] E.J. Billington and D.L. Kreher, *The intersection problem for small G-designs*, Australasian Journal of Combinatorics **12** (1995), 239–258.

[5] Y. Chang and G.L. Faro, *Intersection Numbers of Kirkman Triple Systems*, Journal of Combinatorial Theory, Series A **86** (1999), 348–361.

[6] Karen G. Harris, Appendix to this paper, `http://www.maths.uq.edu.au/~kgh/appendix1.html`

[7] C.C. Lindner and A.Rosa, *Steiner triple systems having a prescribed number of triples in common*, Canadian Journal of Mathematics, **27** (1975), 1166–1175, corrigendum: Canadian Journal of Mathematics, **30** (1978), 896.

[8] I. Mortimer *The intersection problem for a five vertex dragon*, Australasian Journal of Combinatorics **14** (1996), 109–120.

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