Path achievement games

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Abstract

Starting with the empty graph on n vertices, two players alternately add edges until the graph contains a p-path. The last player to move wins. Assuming both players play optimally, the winner will depend only on p and n. We analyse the game for $p \leq 6$ and arbitrary n, determining the winner and providing a winning strategy. The results illustrate how deceptive computing the results of small cases can be.

§1. Introduction

The p-path achievement game is played by two players as follows. The *game graph* G starts off as $\overline{K}_n$ (that is, n isolated points), for some $n > p$. The first player to move is designated Player A (or simply A), and the other player is known as Player B (or B). These two players take turns to add a single undistinguished edge to G, with play halting when G first contains a path of length at least p. The player who made the last move wins.

Since the complete graph K_n contains a p-path the game must finish (after finitely many moves) in a win to one of the players. In fact, by a famous theorem of von Neumann, one player will have a winning strategy. Call this player W and call W's opponent L. Whatever moves L makes, W can always choose moves in reply which eventually lead to W being victorious.

For fixed p we can view W as a function from the set of integers greater than p to $\{A, B\}$. If this function is eventually periodic, with period τ, then we say that the p-path achievement game has period τ. This notion of period was introduced in [3] where the periods of a number of graph games were studied. The computational complexity of identifying W is considered in [1], for a number of games involving the construction of a path within a graph, although the games are not of the form to be discussed here.

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In a path achievement game, let M refer to the player whose turn it is to move and let N be M's opponent. We can assume that M never considers a move which would allow N to win immediately. Specifically, M never creates a $(p-1)$-path; nor two vertex disjoint paths whose combined length is $p-1$. Furthermore, we assume that M will concede defeat when all available moves breach this criterion.

§2. Terminology and Notation

A *path* of length p (also called a p-path) is a sequence $v_0, v_1, \ldots, v_p$ of $p+1$ distinct vertices of G, such that v_{i-1} and v_i are adjacent in G for each $i = 1, 2, \ldots, p$. The vertices v_0 and v_p are the *endpoints* of the path. For $c \geq 3$, a *cycle* of length c (or *c-cycle*) is a c-path, except that $v_0 = v_p$. A *chord* of a cycle is an edge which is itself not part of the cycle but which connects two vertices on the cycle.

Analogous to the terminology of [2], we say that a player who adds an edge between isolated vertices in G is *moving conservatively* and the move itself is a *conservative move.*

We denote by $\varepsilon(G)$ the *essence* of G, being the isomorphism class of G once all isolated vertices have been removed. It should be clear that the state of the game is entirely determined by knowledge of n, p and $\varepsilon(G)$. By a *configuration* we mean a class representative of $\varepsilon(G)$ for some game graph G. In each configuration C we distinguish a subset $d(C)$ of the vertices. If C and D are configurations then $C + D$ and CD are also configurations defined as follows:

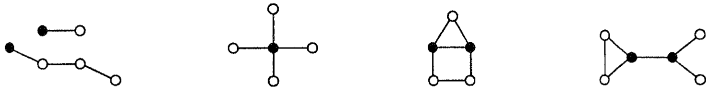
- $C + D$ is a graph whose vertices, edges and distinguished vertices are the disjoint union of the corresponding sets in C and D (relabelling vertices and edges if necessary to avoid a conflict).
- CD is derived from $C + D$ by identifying all the vertices in $d(C + D)$. By choice, $d(CD)$ is always the single vertex corresponding to the vertices in $d(C + D)$.

We use the obvious shorthands of mC for $\underbrace{C + C + \ldots + C}_{m \text{ times}}$ and C^m for $\underbrace{CC \ldots C}_{m \text{ times}}$.

The configurations we will need include

(1) P_a is an a-path with one endpoint distinguished.
(2) C_a is an a-cycle with any one point distinguished.
(3) C_a^* is the same as C_a except that any number of chords may be included.
(4) K_a is the same as C_a except that all chords are included.

There remains only one type of configuration which we need to define. Let $a_1, a_2, \ldots a_m$ be positive integers and C, D be any two configurations. Then $P'_{a_1, a_2, \ldots a_m}$ denotes the configuration consisting of two distinguished vertices v_1 and v_2 together with m edge disjoint paths (of respective lengths a_1, a_2, $\ldots a_m$) between v_1 and v_2. Moreover, $C \circ P'_{a_1, a_2, \ldots a_m} \circ D$ is formed from $C + P'_{a_1, a_2, \ldots a_m} + D$ by identifying v_1 with the vertices in $d(C)$ and then identifying v_2 with the vertices in $d(D)$.



$P_3 + P_1$ $\quad$ P_1^4 $\quad$ $P'_{2,1,3}$ $\quad$ $C_3 \circ P'_1 \circ P_1^2$

Figure 1: Some examples of configurations.
$\bullet$ = Distinguished vertex $\quad \circ$ = Other vertex

Configurations such as those in Figure 1 will be used to describe progress in our path achievement games. Recalling the assumption at the end of the introduction, we stress that configurations such as $P_2^3P_1$, $(C_3P_1) \circ P'_1 \circ P_1^2$ and $(P_2P_1) \circ P'_1 \circ P_1^2$ need never be considered.

§3. Some Endgames

The game will be analysed in terms of three phases; the opening, midgame and endgame. These phases will not be rigorously defined but rather serve only for convenience of terminology. We look first at endgames.

Endgame 1. *For $p \geq 4$, the winner of p-path achievement from C^*_{p-1} is Player A if $p \equiv 0, 3 \bmod 4$ and Player B if $p \equiv 1, 2 \bmod 4$.*

Proof: The only edges which can be added are chords, so play will proceed to K_{p-1} at which point $\binom{p-1}{2}$ edges have been played and M concedes defeat. $\odot$

Endgame 2. *For $p \geq 5$ and $e \geq 1$ the winner of p-path achievement from $C^*_{p-2} + eP_1$ is Player A if*

$$\begin{cases} p \equiv 0, 1 \bmod 4 \text{ and } n - p \equiv 2, 3 \bmod 4; \text{ or} \\ p \equiv 2, 3 \bmod 4 \text{ and } n - p \equiv 0, 1 \bmod 4 \end{cases}$$

and Player B otherwise.

Proof: Only chords of the C^*_{p-2} or extra disjoint edges can be added. Hence play will proceed to $K_{p-2} + \lfloor \frac{n-p+2}{2} \rfloor P_1$. $\odot$

Endgame 3. *For $p \geq 4$ and $a \geq 2$ the winner of p-path achievement from $K_{p-2}P_1^a$ is Player A if*

$$\begin{cases} p \equiv 0, 1 \bmod 4 \text{ and } n - p \text{ is even; or} \\ p \equiv 2, 3 \bmod 4 \text{ and } n - p \text{ is odd} \end{cases}$$

and Player B otherwise.

Proof: Play proceeds to $K_{p-2}P_1^{n-p+2}$ at which point M concedes defeat. $\odot$

§4. The p-path achievement game; $p \leq 5$

We first observe that p-path achievement is trivial for $p \leq 3$; being won on the p^{th} move of the game. Obviously, each of these games has period 1, with Player A winning 1-path and 3-path achievement and Player B winning 2-path achievement.

The 4-path achievement game is very simple too. Since A wins from C_3 (Endgame 1), B will not open with P_2. Also, after the second move of the game any non-conservative move loses immediately. We conclude that both players will play conservatively as long as possible, and that A will win if $n \equiv 2, 3 \bmod 4$, whereas B wins if $n \equiv 0, 1 \bmod 4$.

To discuss 5-path achievement we divide into 3 cases.

Case 1 : n even (B wins)

Player B opens with P_2. Whatever move A then makes, B can reach either C_3P_1 or $P_1^2P_2$. Then, assuming that A avoids C_4^* (Endgame 1), Player B can play to $C_3P_1^3$ which is handled by Endgame 3.

Case 2 : $n \equiv 1 \bmod 4$ (B wins)

Player B is content to play conservatively until Player A makes a non-conservative move (which A will be forced to do eventually since $n \equiv 1 \bmod 4$). If A creates a P_2 component then B makes it into a C_3 and wins by Endgame 2. Player A's only other option is to open with P_3, which allows B to win via C_4 (Endgame 1).

Case 3 : $n \equiv 3 \bmod 4$ (A wins)

This case is similar to the previous case with the roles of A and B reversed. Player B must make the first non-conservative move, and unless it is done immediately Endgame 2 will be reached. On the other hand if B opens with P_2 then A plays to C_3 and B must choose either $C_3 + P_1$ (losing by Endgame 2) or C_3P_1 which A converts to $C_3P_1^2$ (a win to A by Endgame 3).

We note at this point that both 4-path achievement and 5-path achievement have period 4.

§5. 6-path achievement

We begin our solution to 6-path achievement by looking at some more endgames.

Endgame 4. *For any $a \geq 0$, the winner of 6-path achievement from $C_3^2P_1^a$ is Player A if n is even and Player B if n is odd. The same ending is reached from $P_2^2P_1^a$ for $a \geq 2$.*

Proof: Play proceeds to $C_3^2P_1^{n-5}$ which contains $n+1$ edges. Note that if $a = 0$ then A also has the option from C_3^2 of moving to C_5^*, but this is a win for B by Endgame 1. ⊙

Endgame 5. *For $a, b \geq 1$ Player A wins 6-path achievement from $P_1^a \circ P'_{2,2,1} \circ P_1^b$ and $P_1^a \circ P'_{2,2} \circ P_1^b$.*

Proof: Let v_1 and v_2 be the two distinguished vertices. Separate 2-paths can be constructed from v_1 to v_2 via each other vertex of G. The only other edge allowable is directly between v_1 and v_2. Hence an odd numbered total of $2(n-2)+1$ edges will be played to reach $P'_{1,2,2,2,\ldots,2}$, where B concedes defeat. ⊙

Endgame 6. *Let $a, b \geq 2$. Then M wins 6-path achievement from $C_3P_1^a + bP_1$ if and only if $n - a - 2b \not\equiv 0 \bmod 3$.*

Proof: The number ψ of isolated points remaining in G is $\psi = n - 3 - a - 2b$. The only two legitimate moves are to $C_3P_1^{a+1} + bP_1$ or $C_3P_1^a + (b+1)P_1$, reducing ψ by 1 or 2 respectively. The winning strategy is to ensure your opponent always moves from a state where $\psi \equiv 0 \bmod 3$. That way your opponent will have to concede defeat when $\psi = 0$. $\odot$

Endgame 7. *Let $a \geq 2$, $b \geq 2$ and $c \geq 1$ be integers. Then M wins from $P_1^a \circ P_1' \circ P_1^b + cP_1$ if and only if $n - a - b - 2c \not\equiv 2 \bmod 3$.*

Proof: Similar to the previous endgame. The only available moves decrease the number of isolated points by 1 or 2, whilst remaining in this Endgame. $\odot$

Next we study a few midgames.

Midgame 1. *From P_4 or $P_1^2P_3$ play will reach Endgame 3 (without loss of generality).*

Proof: If A plays $P_1P_2^2$ from P_4 then B creates the configuration in Figure 2 and A must immediate concede defeat. Since B also wins Endgame 1, Player A has only one move from P_4 which might avoid defeat, namely $P_1^2P_3$. Now Player B's only move which avoids defeat via Endgame 5 is $C_3 \circ P_1' \circ P_1^2$. Endgame 3 is now inevitable. $\odot$

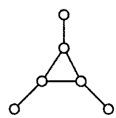


Figure 2: A win for Player B.

Midgame 2. *Suppose that $a \geq 2$ and $e \geq 2$. Then M wins from $P_1^aP_2 + eP_1$ if and only if $n - a - 2e \not\equiv 2 \bmod 3$.*

Proof: If $n - a - 2e \equiv 0 \bmod 3$ then M wins by moving to Endgame 6 whereas M wins by moving to Endgame 7 when $n - a - 2e \equiv 1 \bmod 3$. The only other options are to play to $P_1^{a+1}P_2 + eP_1$ or $P_1^aP_2 + (e+1)P_1$, but N will win from both by the same logic. $\odot$

Midgame 3. *Suppose that play has reached eP_1 for some $e \geq 3$ and that M loses Endgame 2. Then N can choose between Endgame 6 and Endgame 7.*

Proof: M cannot choose a move which will stop N reaching $P_3 + (e-1)P_1$. From there the only move which prevents Endgame 2 is $P_1^2P_2 + (e-1)P_1$. Endgames 6 and 7 can each be reached in the next move. $\odot$

We can now solve 6-path achievement on an odd number of points.

Case 1: $n \equiv 1 \bmod 4$ (B wins)

Player B opens with $2P_1$, then play is similar to Midgame 3. B forces to $P_3 + P_1$. Now A must avoid Endgame 2 by playing either $P_1^2P_2 + P_1$ or $P_2^2P_1$. From either B can reach $P_2^2P_1^2$ which leads inexorably to Endgame 4 and a win to B.

Case 2: $n \equiv 3 \bmod 4$ (A wins)

Player A opens by playing 3-path achievement. From P_3 Player B has a number of options. However, the only one which avoids both Endgame 2 and Midgame 1 is C_3P_1 after which A plays to C_4^*. Since A wins Endgame 2, Endgame 3 and Endgame 5 there is no escape now for B.

The game on an even number of points is more complicated. Our first observation is that, without loss of generality, the opening will be $3P_1$. For suppose that instead, B opened with P_2 and that A then moved to P_1^3. Whatever B then played, A could reach either $C_3P_1^2$ or $P_1^3P_2$. There would then be no way B could prevent A reaching Endgame 4 via $C_3^2P_1$ or $P_2^2P_1^3$ or Endgame 5 via $P_1^3 \circ P_2' \circ P_1^2$, $P_1^2 \circ P_{2,2}' \circ P_1$ or $P_1^2 \circ P_{1,2}' \circ P_1^2$. We conclude that B will open with $2P_1$. On the third turn if A chooses any move other than $3P_1$ then B can move to P_4, which A loses via Midgame 1. It follows immediately that

Case 3: $n \equiv 2 \bmod 4$ and $n \not\equiv 2 \bmod 3$. A wins by employing Midgame 3 from $3P_1$.

Case 4: $n \equiv 0 \bmod 4$ and $n \not\equiv 1 \bmod 3$. B opens with $4P_1$ then wins by Midgame 3.

Next we contend that when $n \equiv 2 \bmod 12$, Player B can be assumed to reach whichever of $C_3 + P_2 + P_1$ and $C_3 + P_2 + 3P_1$ is most advantageous.
Proof: We have already seen that play reaches $3P_1$. If B now plays $4P_1$ then A will play $5P_1$ and win by Midgame 3. Alternately, if B plays $P_3 + P_1$ then A wins by moving to Endgame 2. So B is forced to play to $P_2 + 2P_1$. Now A must prevent B from reaching $P_1^2P_2 + 2P_1$ and can only do this via $2P_2 + P_1$ or $C_3 + 2P_1$. However, the former is no good since B can just convert it to $P_1^2 \circ P_1' \circ P_1^2 + P_1$ which A loses (Endgame 7). Hence play reaches $C_3 + 2P_1$ and B may choose to move straight to $C_3 + P_2 + P_1$. The other options are $C_3P_1 + 2P_1$ (which A will win by playing to Endgame 2) and $C_3 + 3P_1$. From the latter A cannot afford to play to $C_3P_1 + 3P_1$ because B wins from $C_3P_1^2 + 3P_1$ (Endgame 6). It follows that from $C_3 + 3P_1$ Player B can force to $C_3 + P_2 + 3P_1$. It only remains to show that B cannot do better. From $C_3 + 3P_1$ Player A can choose $C_3 + 4P_1$. If B is avoiding $C_3 + P_2 + 3P_1$ and Endgame 2 (which A wins) the only alternative now is $C_3 + 5P_1$. However, A simply plays $C_3P_1 + 5P_1$ and B loses; either by playing to $C_3P_1^2 + 5P_1$ (Endgame 6) or allowing A to reach Endgame 2. ⊙

When $n \equiv 4 \bmod 12$ Player A will choose to reach whichever of $C_3 + P_2 + 2P_1$ and $C_3 + P_2 + 4P_1$ is most advantageous.
Proof: If play reaches $4P_1$ then it continues exactly as it did in the previous example with the roles of A and B reversed (and an extra P_1). The only question

is whether B can profit by making a different move from $3P_1$. Any move other than $4P_1$ allows A to play to $P_1^2P_2 + P_1$. Now B has no attractive choice. Playing to $P_1^2P_2 + 2P_1$ (Midgame 2) or $P_1^2 \circ P_1' \circ P_1^2 + P_1$ (Endgame 7) both lose, and any other play leads to Endgame 4. ⊙

Thus we see that in the two as yet unsolved cases, play can be assumed to reach a state $C_3 + P_2 + eP_1$ for some $e \in \{1, 2, 3, 4\}$. Notice that in such a state the game has effectively become one of 3-path avoidance. That is, the first player to create a 3-path will lose immediately (and such a path must be created before anyone can win).

§6. The 3-path avoidance phase

It was shown in [3] that 3-path avoidance has period 7. We therefore need to establish who wins from $C_3 + P_2 + eP_1$ for each residue class of n modulo 7, and for $1 \leq e \leq 4$. This will be done by a series of ten lemmas below. A reference of the form [3,Prop.X] will refer to Proposition X in paper [3].

We note that the vertices in a 3-cycle can have no further edges joined to them without a 3-path being created. It follows that a 3-cycle can be excised from G to reach an equivalent game G' of 3-path avoidance on three fewer points. The only difference in the new game is that the roles of the two players will be interchanged because an odd number of edges have been removed. Whenever we determine the winner of a game by means of this observation we will say that we are performing a *reduction* or that G *reduces* to G'.

Lemma 1: Suppose that it is M's move from $aC_3 + bP_2 + cP_1$ (a, b and c being positive integers) and that $n - 3a - 5b - 6c \equiv 5 \bmod 7$ and $n \geq 3a + 5b + 6c + 5$. Then M must play to $(a+1)C_3 + (b-1)P_2 + cP_1$ to avoid defeat.
Proof: If M plays any move other than $(a+1)C_3 + (b-1)P_2 + cP_1$ then N will be able to reach either $aC_3 + (b-1)P_2 + (c+1)P_1 + P_1^3$ or $aC_3 + bP_2 + (c-1)P_1 + P_1^3$, both of which result in M losing by [3,Prop.3(i)]. ⊙

Lemma 2: If $n+e \equiv 0, 3, 5 \bmod 7$ and $n - 6e \geq 7$ then M wins from $C_3 + P_2 + eP_1$.
Proof: M wins by playing to $C_3 + P_1^3 + eP_1$ which is handled by [3,Prop.3]. ⊙

Lemma 3: If $n + e \equiv 6 \bmod 7$ and $n - 6e \geq 20$ then N wins from $C_3 + P_2 + eP_1$.
Proof: [3,Prop.8]. ⊙

Lemma 4: If $n+e \equiv 0, 2 \bmod 7$ and $n - 6e \geq 18$ then M wins from $C_3 + P_2 + eP_1$.
Proof: M wins by playing to $2C_3 + eP_1$, again using [3,Prop.8]. ⊙

Lemma 5: Player A wins from $C_3 + P_2 + P_1$ if $n \equiv 0 \bmod 7$ and $n \geq 35$.
Proof: Player A moves to $C_3 + 2P_2$ which reduces to [3,Prop.10(b)]. ⊙

Lemma 6: Let $k + e \equiv 4 \bmod 7$ and $k \geq 32 + 6e$. If Player A wins from $C_3 + P_2 + eP_1$ when $n = k - 3$ then B wins from $C_3 + P_2 + eP_1$ when $n = k$.
Proof: Assume that $n + e \equiv 4 \bmod 7$ and that play has reached $C_3 + P_2 + eP_1$. We split into two cases depending on the parity of e. If e is even then B moves to

$2C_3 + eP_1$. Whatever move A now makes, B can reach $2C_3 + P_2 + eP_1$. On the other hand if e is odd then it is A's turn to move and

- If A moves to $C_3 + P_1^3 + eP_1$ then B wins via $C_3 + P_1^4 + eP_1$ ([3,Prop.3(ii)]).
- If A moves to $2C_3 + eP_1$ then B moves to $2C_3 + (e+1)P_1$. Now A can move to $2C_3 + P_2 + eP_1$ or to $2C_3 + (e+2)P_1$. From the latter B wins via $2C_3 + (e+3)P_1$ and [3,Prop.8 or Prop.7(ii)].
- If A plays to either $C_3 + 2P_2 + (e-1)P_1$ or $C_3 + P_2 + (e+1)P_1$ then B can play to $C_3 + 2P_2 + eP_1$. Now by Lemma 1, A must play to $2C_3 + P_2 + eP_1$.

In conclusion, we see that in every case Player B can guide play to $2C_3 + P_2 + eP_1$ which reduces to $C_3 + P_2 + eP_1$. ⊙

Lemma 7: Let $e \in \{2, 4\}$, $k + e \equiv 1 \bmod 7$ and $k \geq 24 + 6e$. If Player B wins from $C_3 + P_2 + (e-1)P_1$ when $n = k - 3$ then A wins from $C_3 + P_2 + eP_1$ when $n = k$.

Proof: Whatever move B makes from $C_3 + P_2 + eP_1$, Player A can always reach one of the three states $2C_3 + (e+1)P_1$, $C_3 + P_1^4 + eP_1$ or $2C_3 + P_2 + (e-1)P_1$. Assuming that $n + e \equiv 1 \bmod 7$, Player A wins from the first two by [3,Prop.8] and [3,Prop.3(i)] respectively, whilst the third reduces to $C_3 + P_2 + (e-1)P_1$. ⊙

We notice that by combining Lemmas 5, 6 and 7 the winner from $C_3 + P_2 + eP_1$ can be determined for $e \in \{1, 2\}$ and $n + e \equiv 1, 4 \bmod 7$. Lemma 8 will use several reductions to these games.

Lemma 8: Player A wins from $C_3 + P_2 + 3P_1$ provided $n \equiv 5 \bmod 7$ and $n \geq 47$.

Proof: Player A's first move is to $C_3 + 2P_2 + 2P_1$. If B immediately plays to $2C_3 + P_2 + 2P_1$ then A wins by reduction and if B allows A to play to $C_3 + P_1^3 + 2P_2 + P_1$ then A wins by [3,Prop.3(ii)]. Hence B is forced to play to $C_3 + 2P_2 + 3P_1$ and A then chooses $C_3 + 3P_2 + 2P_1$. If B next plays to $2C_3 + 2P_2 + 2P_1$ then A wins via a reduction from $3C_3 + P_2 + 2P_1$ and if B chooses $C_3 + P_1^3 + 2P_2 + 2P_1$ instead then A wins from $C_3 + P_1^4 + 2P_2 + 2P_1$ [3,Prop.3(ii)]. It follows that B is forced to make a move which allows A to reach $C_3 + 4P_2 + 2P_1$. At this stage B has to play to $2C_3 + 3P_2 + 2P_1$ (Lemma 1) and A replies with $3C_3 + 2P_2 + 2P_1$. Player B cannot now play $3C_3 + P_1^3 + P_2 + 2P_1$ because A would win with $4C_3 + P_1^3 + 2P_1$ [3,Prop.3(iii)] and nor can B afford to let A reduce to $4C_3 + P_2 + 3P_1$ (Lemma 3). After B selects the only other option, $3C_3 + 3P_2 + P_1$, A moves to $4C_3 + 2P_2 + P_1$ and B is again forced by Lemma 1 to play to $5C_3 + P_2 + P_1$. Since A wins this reduction as well (Lemma 5), B has no recourse. ⊙

The eight Lemmas above suffice to solve 3-path avoidance from $C_3 + P_2 + eP_1$ for all large n and $e \in \{1, 2, 3, 4\}$. A summary of the results can be found in the table below (Figure 3).

e	n mod 7						
	0	1	2	3	4	5	6
1	${}_5\mathrm{A}_{35}$	${}_4\mathrm{A}_{29}$	${}_2\mathrm{A}_{16}$	${}_6\mathrm{B}_{38}$	${}_2\mathrm{A}_{18}$	${}_3\mathrm{B}_{26}$	${}_2\mathrm{A}_{20}$
2	${}_4\mathrm{B}_{35}$	${}_2\mathrm{B}_{22}$	${}_6\mathrm{B}_{44}$	${}_2\mathrm{B}_{24}$	${}_3\mathrm{A}_{32}$	${}_2\mathrm{B}_{26}$	${}_7\mathrm{A}_{41}$
3	${}_2\mathrm{A}_{28}$	${}_6\mathrm{B}_{50}$	${}_2\mathrm{A}_{30}$	${}_3\mathrm{B}_{38}$	${}_2\mathrm{A}_{32}$	${}_8\mathrm{A}_{47}$	${}_4\mathrm{A}_{41}$
4	${}_6\mathrm{B}_{56}$	${}_2\mathrm{B}_{36}$	${}_3\mathrm{A}_{44}$	${}_2\mathrm{B}_{38}$	${}_7\mathrm{A}_{53}$	${}_4\mathrm{B}_{47}$	${}_2\mathrm{B}_{34}$

Figure 3: Winner from $C_3 + P_2 + eP_1$ for sufficiently large n. An entry in the table of the form ${}_x\mathrm{Y}_z$ means that Lemma x can be used to show that Player Y wins whenever $n \geq z$.

There remain only a few small cases to attend to.

Lemma 9: Suppose that $n = 2e + 8$ or $n = 2e + 12$. Then M wins from $C_3 + P_2 + eP_1$.

Proof: If $n = 2e+8$ then M moves to $C_3+2P_2+(e-1)P_1$ and wins by [3,Prop.4(i)], whereas if $n = 2e + 12$ then M moves to $2C_3 + eP_1$ and wins by [3,Prop.4(vi)]. $\odot$

Lemma 10: Suppose that $n = 6e + 4$ or $n = 6e + 16$ where $e \geq 2$. Then M wins from $C_3 + P_2 + eP_1$.

Proof: M moves to $2C_3 + eP_1$, winning by [3,Prop.5] or [3,Prop.7(iii)] when $n = 6e + 4$ or $n = 6e + 16$ respectively. $\odot$

§7. Wrapping up:

We are ready to establish the winner of 6-path avoidance in the two cases that were left unsolved in Section 5.

Case 5: $n \equiv 2 \bmod 12$.

As shown already, Player B chooses between $C_3 + P_2 + P_1$ and $C_3 + P_2 + 3P_1$. By inspecting Figure 3 we see that when n is large enough B will win if $n \equiv 1, 3, 5 \bmod 7$ whereas A will win in all other cases. The only small case which is not covered by this result is when $n = 14$, which Lemma 9 tells us conforms to the overall pattern (A wins).

Case 6: $n \equiv 4 \bmod 12$.

Player A chooses between $C_3 + P_2 + 2P_1$ and $C_3 + P_2 + 4P_1$. By inspecting Figure 3 we see that when n is large enough A will win if $n \equiv 2, 4, 6 \bmod 7$ whereas B will win in all other cases. The only small cases which are not covered by this result are when $n = 16$, 28 or 40. Lemma 9 tells us that $n = 16$ does not conform to the overall pattern (B wins), whereas both $n = 28$ and $n = 40$ do conform by Lemma 10 (B wins both).

Collating the results from all six cases gives the following theorem.

Theorem 1: The game of 6-path achievement on n points ($n \geq 7$) has period 84. Player B wins when $n = 16$, $n \equiv 1 \bmod 4$, $n \equiv 0 \bmod 12$, $n \equiv 8 \bmod 12$, or $n \equiv b \bmod 84$ for some $b \in \{26, 28, 38, 40, 50, 52, 64\}$. Player A wins otherwise.

If nothing else, this theorem is a warning to empiricists. In small cases the winner of 6-path achievement can be computed by playing all possible games. A program would find the sequence of winners begins

$$\{\text{A, B, B, A, A, B, B, A, A, } \mathbf{B}\text{, B, A, A, B, B, A, A, B, B,} \ldots\}. \qquad (*)$$

There is only one obvious conclusion, especially since 4-path and 5-path achievement both have period 4 (as do most of the graph games in the literature [3]). The first hint of the true pattern would not be found until $n = 26$, well out of reach of a simple exhaustive algorithm. Even then it might be just an isolated anomaly. Indeed, the sequence $(*)$ already includes just such a glitch at $n = 16$. Did you spot it?

§8. References:

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