Properties of queens graphs and the irredundance number of Q_7

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Abstract

We prove results concerning common neighbours of vertex subsets and irredundance in the queens graph Q_n. We also establish that the lower irredundance number of Q_7 is equal to four.

1 Introduction

The rows and columns of the $n \times n$ chessboard will be numbered $1, 2, ..., n$ from the bottom left hand corner. Thus each square has *co-ordinates* (x, y), where x and y are the column and row numbers of the square, respectively. The *lines* of the board are the rows, columns, *sum diagonals* (*i.e.*, sets of squares such that $x + y = k$, where k is a constant) and *difference diagonals* (sets of squares such that $x - y = k$). These will be denoted by the symbols r, c, s, d, respectively.

The vertices of the *queens graph* Q_n are the n^2 squares of the chessboard, and two squares are adjacent if they are collinear. This graph has received much attention in the literature recently because of the well-known century-old problem of determining the smallest number of queens which will cover all the squares of the $n \times n$ board. This problem may be restated as the determination of the domination number $\gamma(Q_n)$ of the queens graph. It remains unsolved and progress is detailed in [2, 3, 9, 11].

Let X be a subset of the vertex set of a graph G. For $x \in X$, we denote the closed neighbourhood (see [8]) of x by $N[x]$, and the closed neighbourhood of X by $N[X]$. A *private neighbour of x relative to X* (denoted X-pn) is an element of

$pn(x, X) = N[x] - N[X - \{x\}]$. The set X is called *irredundant* if each vertex of X has an X-pn.

A dominating set of a graph is minimal if and only if it is also irredundant. This fact has led to much current work on the development of the theory of irredundance. The parameter $ir(G)$, known as the lower irredundance number of G, is the smallest cardinality amongst all maximal irredundant sets of G.

As was shown in [1], the irredundance number of any graph is bounded below by $ir(G) \geq (\gamma(G) + 1)/2$, where as usual $\gamma(G)$ denotes the domination number of G. This bound, together with the lower bound $\gamma(Q_n) \geq (n-1)/2$ of P. Spencer (see [5]), shows that $ir(Q_n) \geq (n + 1)/4$. The values $ir(Q_5) = ir(Q_6) = 3$ were established in [4], so it looks as though this bound is not particularly good, even for small values of n.

In Section 2 we prove some properties of Q_n for general n. Some of these, together with other results for Q_7, will be used in Section 3 to show that $ir(Q_7) = 4$. This number can also be established by an exhaustive computer search – in fact, Harborth [7] recently reported that Jens-P. Bode had verified by computer that $ir(Q_n) = \gamma(Q_n)$ for $n \leq 10$, and Rall [10] did the same for $n \leq 8$. However, our methods may assist in the evaluation of $ir(Q_n)$ for higher values of n.

The reader is referred to [8] for definitions, theory and bibliography concerning domination and irredundance in graphs. Results on domination parameters of chessboard graphs are summarized in [9].

2 Properties of Q_n

Our first results deal with common neighbours of certain vertex subsets of Q_n. A sequence of at least three squares form an *equally-spaced set* (abbreviated *ES-set*) if they are collinear and equally spaced along their line. For the square A, $r(A)$ ($c(A)$, $s(A)$, $d(A)$, respectively) will denote both the row (column, sum diagonal, difference diagonal) of A and the number of the row (column, sum diagonal, difference diagonal) of A. Thus, if A has co-ordinates (x, y), then $r(A) = y$, $c(A) = x$, $s(A) = x + y$ and $d(A) = x - y$.

Theorem 1 *Let p, q be lines of Q_n which intersect in square W. Consider $\{A_1, A_2\} \subseteq p - \{W\}$ and $\{A_3, A_4\} \subseteq q - \{W\}$. Let $\Omega \cup \{W\}$ (disjoint union) be the set of squares adjacent to all of A_1, A_2, A_3, A_4, and Σ the subset of Ω containing the squares not on p or q. Then*

(a) $|\Sigma| \leq 2$, $|\Omega| \leq 4$;

(b) *if $|\Sigma| = 2$, then the two squares of Σ are adjacent.*

Proof. We consider three cases.

Case 1 p is a sum diagonal s and q is a column c.
We re-label A_1, A_2, A_3, A_4 by S_1, S_2, C_1, C_2 to signify that S_1, S_2 are on s and C_1, C_2 are on c. Observe that

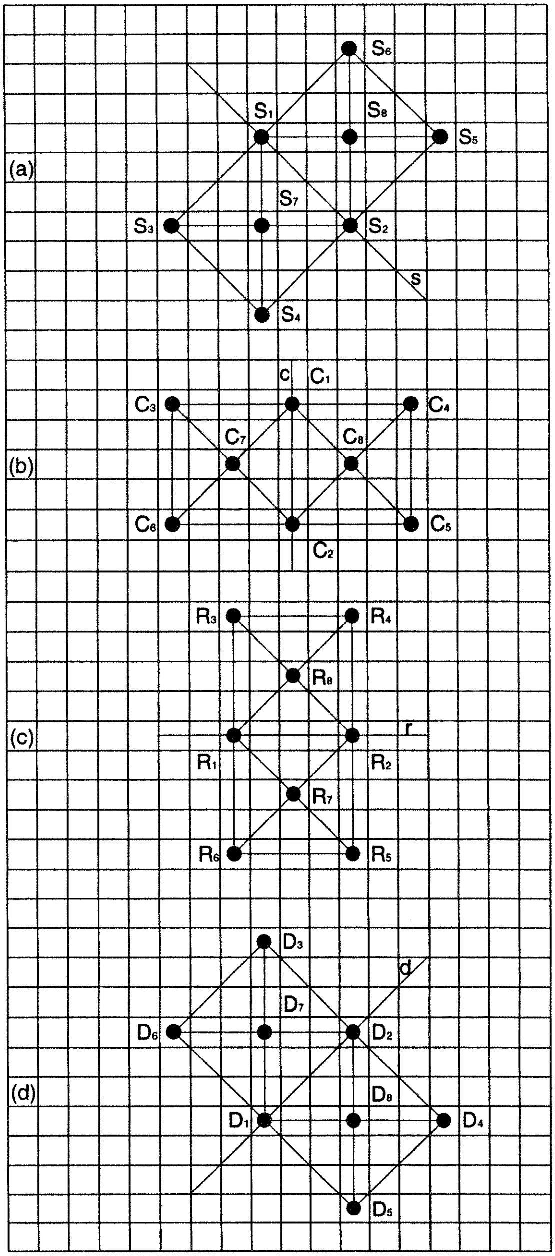
(a)
S6
S1
S8
S5
S7
S3
S2
s
S4
(b)
c
C1
C3
C4
C7
C8
C6
C5
C2
(c)
R3
R4
R8
r
R1
R2
R7
R6
R5
(d)
D3
d
D7
D6
D2
D8
D1
D4
D5

Figure 1

(i) if square P is adjacent to S_1 and S_2, then $P = W$, $P \in s - \{W\}$ or $P \in \mathcal{S} = \{S_3, ..., S_8\}$, the squares depicted in Figure 1(a);

(ii) if square P is adjacent to C_1 and C_2, then $P = W$, $P \in c - \{W\}$ or $P \in \mathcal{C} = \{C_3, ..., C_8\}$, the squares depicted in Figure 1(b), where C_7, C_8 only exist if $r(C_1) - r(C_2)$ is even.

We deduce that each $Z \in \Omega$ is of exactly one of the following types:

type 1: $Z \in (s - \{W\}) \cap \mathcal{C}$

type 2: $Z \in (c - \{W\}) \cap \mathcal{S}$

type 3: $Z \in \mathcal{C} \cap \mathcal{S} = \Sigma$.

Suppose that $Z \in \Omega$ is of type 1. If $Z \in \{C_3, C_5, C_7, C_8\}$, then, due to the geometry of $\mathcal{C}$ and $\mathcal{S}$, the line s includes C_1 or C_2, which contradicts the definition of these squares. Hence $Z \in \{C_4, C_6\}$. If $C_4 \in s$, then $C_6 \notin s$, and *vice versa*, and so there is at most one type 1 square of Ω. Observe that (say) $C_4 \in s$ implies that W is the square of c such that C_2, C_1, W form an ES-set on c.

Suppose that $Z \in \Omega$ is of type 2. If $Z \in \{S_4, S_6, S_7, S_8\}$, then c contains S_1 or S_2, a contradiction which implies that $Z \in \{S_3, S_5\}$. If S_3 is on c, then S_5 is not, and so there is at most one type 2 square of Ω. Notice that $S_3 \in c$ implies that W is the square of s such that W, S_1, S_2 form an ES-set on s.

By comparing the sets $\mathcal{C}$ and $\mathcal{S}$ in Figure 1(a) and (b), we see that it is impossible to choose positions for S_1, S_2, C_1, C_2 so that $|\mathcal{C} \cap \mathcal{S}| \geq 3$. Moreover, if $|\mathcal{C} \cap \mathcal{S}| = 2$, then the two squares of this set are collinear. This completes the proof of Case 1.

Case 2 p is a row r and q is a column c.
Re-label A_1, A_2, A_3, A_4 by R_1, R_2, C_1, C_2 respectively. If $\mathcal{R} = \{R_3, ..., R_8\}$ and $\mathcal{C} = \{C_3, ..., C_8\}$ are the sets of squares depicted in Figure 1(c) and (b) (existence of R_7, R_8, C_7, C_8 depend on parity), then each $Z \in \Omega$ has one of the following types:

type 1: $Z \in (c - \{W\}) \cap \mathcal{R}$

type 2: $Z \in (r - \{W\}) \cap \mathcal{C}$

type 3: $Z \in \mathcal{R} \cap \mathcal{C} = \Sigma$.

Suppose that $Z \in \Omega$ is a type 1 square. If $Z \in \{R_3, R_4, R_5, R_6\}$, then R_1 or R_2 is on c, which is impossible. Therefore $Z \in \{R_7, R_8\}$. Both R_7 and R_8 are type 1 squares if R_1, W, R_2 form an ES-set, and there are no type 1 squares otherwise.

By symmetry, C_7 and C_8 are the only type 2 squares if C_1, W, C_2 form an ES-set, and there are no type 2 squares otherwise.

If there are two type 1 and two type 2 squares, then $\mathcal{R} \cap \mathcal{C} = \emptyset$ and the result holds. (1)

The geometry of $\mathcal{C}$ and $\mathcal{R}$ prevents $|\mathcal{R} \cap \mathcal{C}| \geq 3$, and if $|\mathcal{R} \cap \mathcal{C}| = 2$, these two vertices are adjacent. If $|\mathcal{R} \cap \mathcal{C}| > 0$, there cannot be both type 1 and type 2 squares (by statement (1)). Therefore $|\Omega| \leq 4$ as required.

Case 3 p is a sum diagonal s and q is a difference diagonal d.
Re-lable A_1, A_2, A_3, A_4 by S_1, S_2, D_1, D_2 respectively. If $\mathcal{D} = \{D_3, ..., D_8\}$ and $\mathcal{S} = \{S_3, ..., S_8\}$ are the sets of squares depicted in Figure 1(a) and (d), then each $Z \in \Omega$ has one of the following types:

type 1: $Z \in (s - \{W\}) \cap \mathcal{D}$
type 2: $Z \in (d - \{W\}) \cap \mathcal{S}$
type 3: $Z \in \mathcal{D} \cap \mathcal{S} = \Sigma$.

Notice that if $D_3 \in s - \{W\}$, then $D_2 \in s$, a contradiction. Hence D_3 (and similarly D_4, D_5, D_6) is not a type 1 square. Both D_7 and D_8 are type 1 squares if W is the square of d such that D_1, W, D_2 form an ES-set, and there is no type 1 square otherwise.

By symmetry, S_7 and S_8 are the only type two squares if W is the square of s such that S_1, W, S_2 form an ES-set, and there is no type 2 square otherwise.

If there are two type 1 and two type 2 squares, then $\mathcal{S} \cap \mathcal{D} = \emptyset$ and the result holds. (2)

The geometry of $\mathcal{S}$ and $\mathcal{D}$ prevents $|\mathcal{S} \cap \mathcal{D}| \geq 3$, and if $|\mathcal{S} \cap \mathcal{D}| = 2$, these two squares are adjacent. If $|\mathcal{S} \cap \mathcal{D}| > 0$, there cannot be both type 1 and type 2 squares (by statement (2)). Therefore $|\Omega| \leq 4$ as required. ■

Theorem 2 (a) *There are at most five squares which are adjacent to each of three independent squares Z_1, Z_2, Z_3.*

(b) *There are at most four squares which are adjacent to each of four independent squares Z_1, Z_2, Z_3, Z_4.*

Proof. (a) Suppose to the contrary that each of $A_1, ..., A_6$ is adjacent to the three independent squares Z_1, Z_2, Z_3. Let M be the 6×3 matrix with entries in $L = \{r, c, s, d\}$, where for $p \in L$, $m_{ij} = p$ if A_i and Z_j are on the same line p. Note that the independence of Z_1, Z_2, Z_3 implies that the elements of each row of M are distinct. We need two lemmas.

Lemma 2.1 *No element of L appears more than twice in a column of M.*

Proof of Lemma 2.1. Suppose to the contrary that for some $l \in L$, $m_{11} = m_{21} = m_{31} = l$, and that A_1, A_2, A_3 is the order of these squares on l. Note that Z_2, Z_3 are the independent squares not on l which are adjacent to each of A_1, A_2, A_3. The existence of such squares requires that A_1, A_2, A_3 form an ES-set. In this case exactly two such squares exist. However, these are adjacent on the line through A_2 perpendicular to l. Thus Z_2, Z_3 cannot exist. □

Lemma 2.2 *No two elements p, q of L are duplicated in a column of M.*

Proof of Lemma 2.2. Suppose to the contrary that $m_{11} = m_{21} = p$ and $m_{31} = m_{41} = q$. Note that Z_2, Z_3 are independent squares not on either p or q, which are adjacent to A_1, A_2, A_3 and A_4. This is impossible by Theorem 1(b). □

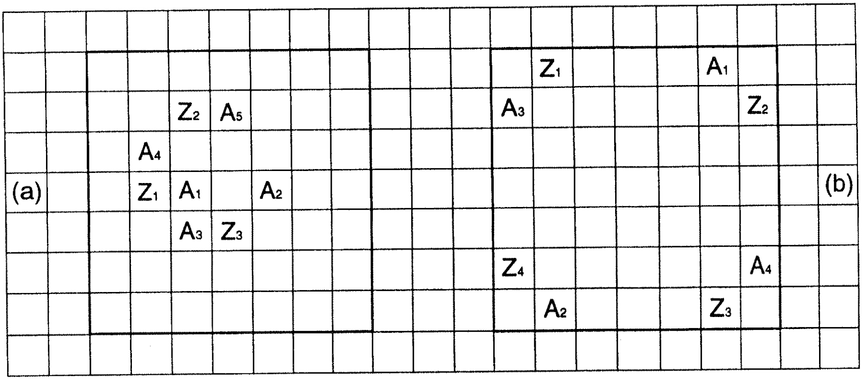

Figure 2

By Lemmas 2.1 and 2.2, a column of M has elements from $\{r, c, s, d\}$ with at most one element appearing more than once. This is impossible and hence part (a) of the theorem is established.

(b) Suppose to the contrary that each of $A_1, \ldots, A_5$ is adjacent to the four independent squares $Z_1, \ldots, Z_4$. Let M' be the 5×4 matrix formed similar to M in (a). In each column of M', some element of L appears more than once; say $m_{11} = m_{21} = l$. Then Z_2, Z_3 and Z_4 are independent squares not on l that are adjacent to each of A_1 and A_2. But as is apparent from Figure 1, each of the graphs induced by $\mathcal{C}$, $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{D}$, respectively, contains $2K_3$ as spanning subgraph, and so Z_2, Z_3 and Z_4 cannot be independent. ∎

In Figure 2(a) (respectively 2(b) we depict three independent squares Z_1, Z_2, Z_3 (respectively four independent squares $Z_1, \ldots, Z_4$) which have common neighbours $A_1, \ldots, A_5$ (respectively $A_1, \ldots, A_4$). Further properties of such configurations may be obtained by more detailed analysis of the matrix M. Note that five independent squares have no common neighbour.

Proposition 3 *Let Z_1 and Z_2 be squares of Q_n, where $|N[Z_1] \cap N[Z_2]| = m$. Then*

$$m \leq \begin{cases} n+6 & \text{if } Z_1, Z_2 \text{ are adjacent} \\ 12 & \text{otherwise.} \end{cases}$$

Proof. If Z_1 and Z_2 are both on line l, then, noting that $|l| \leq n$, the result is immediate from Figure 1. Otherwise, each of the four lines of Z_1 meets at most three of the lines of Z_2 and the result follows. ∎

Proposition 4 *Let Z_1, Z_2, Z_3 be squares of Q_n, where Z_1, Z_2 are on the line l, and $|N[Z_1] \cap N[Z_2] \cap N[Z_3]| = m$. Then*

$$m \leq \begin{cases} n+2 & \text{if } Z_3 \in l \\ 7 & \text{otherwise.} \end{cases}$$

Proof. If Z_1, Z_2 and Z_3 are on l, then there are at most n squares on l and at most two squares off l which are adjacent to each of Z_1, Z_2 and Z_3. Otherwise, there are at most six squares off l adjacent to both Z_1 and Z_2, and any Z_3 off l is adjacent to at most four of these, or equal to one and adjacent to at most three. (See Figure 1.) Further, Z_3 is adjacent to at most three squares on l, and so $m \leq 7$. ■

Subsequent results require further definitions from the theory of irredundance. For $X \subseteq V = V(G)$, define $R = V - N[X]$. The maximality of an irredundant set X is characterized in the following result.

Theorem 5 [6] *The irredundant set X is maximal irredundant if and only if for each $v \in N[R]$, there exists $x \in X$ such that $pn(x, X) \subseteq N[v]$.*

For $v \in V - X$ and $x \in X$, v is an *annihilator* of x if $pn(x, X) \subseteq N[v]$, and so Theorem 5 may be restated as

Theorem 5′ *The irredundant set X is maximal irredundant if and only if each vertex of $N[R]$ is an annihilator of some $x \in X$.*

The following three results were proved in [4].

Proposition 6 [4] *If X is maximal irredundant in G and $|X| < i(G)$ (the independent domination number of G), then X is not independent.*

Proposition 7 [4] *Let X be a maximal irredundant set of G with $|X| = \gamma(G) - k$, where $k \geq 1$. Then there does not exist $Y \subseteq V - X$ with $|Y| \leq k$ such that Y dominates R.*

Theorem 8 [4] *If X is a maximal irredundant set of Q_n with $|X| = \gamma(G) - k$, where $k \geq 1$, then R contains*

(a) *exactly four squares; their coordinates are (x_1, y_1), (x_1, y_2), (x_2, y_1) and (x_2, y_2), where $|x_1 - x_2| \neq |y_1 - y_2|$, or*

(b) *squares in (without loss of generality) exactly two rows and at least three columns, and if R is contained in exactly three columns, the squares with coordinates (say) (x_1, y_1), (x_2, y_1), (x_2, y_2) and (x_3, y_2) are in R, where $|x_1 - x_2| \neq |y_1 - y_2|$ or $|y_1 - y_2| \neq |x_2 - x_3|$, or*

(c) *three squares, no two of which are in the same row or column.*

Two of the possibilities for R given in the conclusion of Theorem 8 may be eliminated, and the other one strengthened, if $k \geq 2$.

Proposition 9 *If X is a maximal irredundant set of Q_n with $|X| = \gamma(G) - k$, where $k \geq 2$, then R contains three independent squares.*

Proof. By hypothesis one of the conclusions (a), (b) or (c) of Theorem 8 occurs. If (a) or (b) is true, then there exist two squares, one on each of the two rows of R, which dominate R. This contradicts Proposition 7 and so (c) holds. If two squares are on the same diagonal l, then any square on l together with the third square dominates R, also contradicting Proposition 7. ■

We now improve the trivial lower bound $ir(Q_n) \geq (\gamma(Q_n) + 1)/2$ for $n = 8, 9, 10, 11$.

Theorem 10 *For $n \geq 8$, Q_n has no maximal irredundant set of size three.*

Proof. Suppose to the contrary that $X = \{B, B_1, B_2\}$ is a maximal irredundant set of Q_n, $n \geq 8$. We first show that no square of X has exactly one X-pn. Suppose B has exactly one X-pn. If neither B_1 nor B_2 is on $r(B)$ (respectively $c(B)$), then B has an X-pn on its row (column). Hence we may assume without loss of generality that $B_1 \in r(B)$. Now suppose $B_2 \notin c(B)$. Then B_1, B_2 are adjacent to at most five squares of $c(B) - \{B\}$, and B has at least two X-pns on $c(B)$, a contradiction which shows that $B_2 \in c(B)$. Thus, without loss of generality the co-ordinates of the three squares are

$$B = (x, y), \quad B_1 = (x_1, y) \quad \text{and} \quad B_2 = (x, y_2),$$

where $x_1 > x$ and $y_2 > y$.

If $(x-2, y-2)$ is on the board, then it, together with $(x-1, y-1)$, are X-pns of B. We deduce (without loss of generality) that $x \leq 2$. Suppose that $x = 2$ and $y \geq 2$. Then $(x-1, y-1)$ is an X-pn of B and so neither $(3, y-1)$, nor $(1, y+1)$ is an X-pn. Therefore $x_1 \in \{3, 4\}$ and $y_2 \in \{y+1, y+2\}$. But $(5, y-3)$ or $(5, y+3)$ is on the board and is a second X-pn of B. This is impossible and shows that if $x = 2$, then $y = 1$. In this case, $|d(B) - \{B\}| \geq 6$. However, $\{B_1, B_2\}$ dominates at most four squares of $d(B) - \{B\}$ and so B has at least two X-pns on $d(B)$, a contradiction.

Therefore $x = 1$ and so B_1 dominates $W_1 \subseteq (s(B) \cup d(B)) - \{B\}$, where $|W_1| \leq 4$, while B_2 dominates $W_2 \subseteq d(B) - \{B\}$, where $|W_2| \leq 2$, and no square of $s(B) - \{B\}$. Since $|(s(B) \cup d(B)) - \{B\}| = n - 1$ and B has exactly one X-pn, we deduce that

$$n = 8, \quad |W_1| = 4 \tag{3}$$

and

$$|W_2| = 2, \quad W_1 \cap W_2 = \emptyset. \tag{4}$$

But (3) implies that $x_1 = 3$ and $y \geq 3$, while (4) implies that $(1, y+6)$ is on the board. Hence $y + 6 \leq 8$, *i.e.*, $y \leq 2$, a contradiction.

Hence each square of X has at least two X-pns. By Proposition 3, each set of two X-pns has at most $n+6$ common neighbours, one of which is the element of X. Hence each element of X has at most $n+5$ annihilators, so that there are at most $3(n+5)$ annihilators in total. Further, $\gamma(Q_n) \geq 5$ and so Proposition 9 holds. Let

Z_1, Z_2, Z_3 be independent squares in R. By counting the squares on the rows and columns of the Z_i, we obtain

$$\left|\bigcup_{i=1}^{3}(r(Z_i) \cup c(Z_i))\right| = 6n - 9. \tag{5}$$

For $i \neq j$, the row and column of Z_i intersect the diagonals of Z_j in at most four squares. If the rows and columns of (say) Z_2 and Z_3 intersect the diagonals of Z_1 in at most six squares, then, noting that $n \geq 8$ and thus $|(s(Z_1) \cup d(Z_1)) - \{Z_1\}| \geq 7$, we see that there is a square of $N[R]$ on a diagonal of Z_1 not counted in (5). If the rows and columns of Z_2 and Z_3 intersect the diagonals of Z_1 in seven or eight squares, then the row and column of (say) Z_2 intersect the diagonals of Z_1 in four squares. But then it is easy to see that Z_1 is not on the edge (first or last row or column) of Q_n, hence $|(s(Z_1) \cup d(Z_1)) - \{Z_1\}| \geq 9$ and again there is a square of $N[R]$ on a diagonal of Z_1 not counted in (5). In either case

$$|N[R]| \geq 6n - 8.$$

By Theorem 5$'$, each square of $N[R]$ is an annihilator and so $3(n+5) \geq 6n - 8$, *i.e.*, $n \leq 7$, the final contradiction which proves the result. ■

3 Irredundance in Q_7

The remaining work of the paper will show that Q_7 has no maximal irredundant set of size three. We require several preliminary results concerning properties of an assumed counterexample $X = \{B, B_1, B_2\}$.

Lemma 11 *Let $X = \{B, B_1, B_2\}$ be maximal irredundant in Q_7. If B is adjacent to neither B_1 nor B_2 in Q_7, then B has at least three X-pns.*

Proof. Observe that B is an X-pn for B and that by Proposition 6 and the fact that $\gamma(Q_7) = 4$ (*cf.* [9]), B_1 is adjacent to B_2. First suppose that B_1 and B_2 are on the same column, say

$$B = (x, y), \quad B_1 = (x_1, y_1) \quad \text{and} \quad B_2 = (x_1, y_2),$$

where $y_2 > y_1$ and $x_1 > x$. Now $\{B_1, B_2\}$ dominates at most five squares on $r(B)$, hence $r(B) - \{B\}$ contains at least one X-pn of B. Suppose that there is no X-pn of B on $c(B) - \{B\}$. Then without loss of generality the possibilities are

$$x_1 = x + 1, \quad y_2 - y = y - y_1 = 2;$$
$$x_1 = x + 1, \quad y_2 - y_1 = 3, \quad y_1 - y = 2;$$
$$x_1 = x + 2, \quad y_2 - y_1 = 1, \quad y_1 - y = 3.$$

In each of these three situations there are at least two X-pns on $r(B) - \{B\}$. Hence in all cases there are at least two X-pns on $(r(B) \cup c(B)) - \{B\}$. In addition, B is also an X-pn of B. Thus B has at least three X-pns.

Secondly, suppose that B_1 and B_2 lie on the same diagonal. Then $\{B_1, B_2\}$ dominates at most five squares on each of $r(B) - \{B\}$ and $c(B) - \{B\}$, and so each of these contains an X-pn of B. Since B is also an X-pn, the result follows. ■

Lemma 12 *Let $X = \{B, B_1, B_2\}$ be maximal irredundant in Q_7. If $B_1 \in s(B) \cup d(B)$ and $B_2 \notin r(B) \cup c(B)$, then B has at least three X-pns.*

Proof. Without losing generality assume that $B_1 \in s(B)$ and $c(B_1) > c(B)$. Then

$$B \text{ has at least one } X\text{-pn on each of } r(B) - \{B\},\ c(B) - \{B\}. \tag{6}$$

If both bounds of (6) are attained, then B_1 is not adjacent to B_2 (since no line of B_2 coincides with a line of B_1), $c(B_1) - c(B) \leq 3$, $|c(B_2) - c(B)| \leq 3$ and $|r(B_2) - r(B)| \leq 3$. However, an investigation of the three relative positions of B and B_1 shows that there is no B_2 which enables both bounds of (6) to be attained. ■

Corollary 13 *If B has at most two X-pns, then (say) $B_1 \in r(B) \cup c(B)$.*

With Corollary 13 in mind, we make additional definitions. A square B on Q_7 with at most two X-pns is of exactly one of two types. Such a square B is an

X_α-square if both $r(B) - \{B\}$ and $c(B) - \{B\}$ contain another square of X;

X_β-square if exactly one of $r(B) - \{B\}$ and $c(B) - \{B\}$ contains another square of X.

Lemma 14 *For an X_α-square B, the positions of the squares in $X = \{B, B_1, B_2\}$ are rotationally equivalent to*

$$B = (1, y),\quad B_1 = (x_1, y),\quad B_2 = (1, y_2),\quad x_1 > 1,\quad y_2 > y.$$

Proof. By symmetry, the positions of the squares in X, where B is an X_α-square, are equivalent to

$$\begin{aligned} B &= (x, y), \text{ where } x \leq y, \\ B_1 &= (x_1, y), \text{ where } x_1 > x, \\ B_2 &= (x, y_2), \text{ where } y_2 > y. \end{aligned}$$

It remains to prove that $x = 1$. If $x \geq 3$, then $y \geq 3$ and both $(x-1, y-1)$ and $(x-2, y-2)$ are X-pns of B. Since B has at most two X-pns, $(x-3, y-3)$ (which is not adjacent to either B_1 or B_2) is off the board, and we may assume that $x = 3$. If $y > 3$, then no positions for B_1, B_2 can prevent two of $(4, y-1)$, $(5, y-2)$, $(6, y-3)$ being X-pns of B. Hence $y = 3$. Since $(7, 7)$ is not an X-pn, we may assume without loss of generality that $B_2 = (3, 7)$. However, this means that $(2, 4)$ is an X-pn of B, a contradiction showing that x is at most 2.

Suppose $x = 2$ and $y \geq 4$. Then $(1, y-1)$ and two squares of $s(B)$ are X-pns of B. If $B = (2, 3)$ and $x_1 > 4$, then $(1, 2)$, $(3, 2)$, and at least one of $(1, 4)$, $(5, 6)$, $(6, 7)$ are X-pns. If $B = (2, 3)$ and $x_1 \in \{3, 4\}$, then $(1, 2)$ and two of $(1, 4)$, $(5, 6)$, $(6, 7)$ are X-pns. A similar argument eliminates $B = (2, 2)$ and the result follows. ■

Lemma 15 *An X_β-square has exactly two private neighbours on either its row or its column, and no private neighbour on a diagonal.*

Proof. If a maximal irredundant set Y of Q_7 with $|Y| = 3$ has a Y_β-square, then Y is rotationally equivalent to $X = \{B, B_1, B_2\}$, where $B = (x, y)$ $(x \le y)$, $B_1 = (x_1, y)$ $(x_1 > x)$ and $B_2 = (x_2, y_2)$, where $x \neq x_2$ and $y_2 \ge y$.

If there is exactly one X-pn on $c(B)$, then $x_1 - x \le 2$, $|x_2 - x| \le 3$ and B_1, B_2 dominate disjoint sets of sizes two and three, respectively, on $c(B) - \{B\}$. Investigation of the two relative positions of B, B_1 shows that for each possible B_2, B has at least two more X-pns on its diagonals, a contradiction. There is at least one X-pn on $c(B)$. Thus we deduce that there are exactly two X-pns on $c(B)$ and none on $s(B) \cup d(B)$. ∎

Lemma 16 *If a 3-square maximal irredundant set Y of Q_7 has a Y_β-square, then Y may be rotated into $X = \{B, B_1, B_2\}$, where*

(a) *$B = (1, y)$ is an X_β-square, $B_1 = (x_1, y)$, $B_2 = (x_2, y_2)$, where $x_2 > 1$ and $y_2 \ge x_1$;*

(b) *$y \le 8 - x_1$ or $y \ge x_1$.*

Proof. Y is equivalent to $X = \{B, B_1, B_2\}$, where $B = (x, y)$ is an X_β-square, $B_1 = (x_1, y)$ with $x_1 > x$, and $B_2 = (x_2, y_2)$, with $x \neq x_2$ (definition of X_β-square) and $y_2 \ge y$.

Suppose that $x > 1$ and $y > 1$. Then $(x-1, y-1)$ is on the board. If $y = 7$, then B, B_1, B_2 are all on row 7 and B_1, B_2 dominate at most four squares of $s(B) \cup d(B)$, contrary to Lemma 15. Hence $y \le 6$, and so $(x-1, y+1)$ is also on the board.

If $x_1 - x \ge 3$, then $(x-1, y-1)$, $(x-1, y+1)$, $(x+1, y-1)$, $(x+1, y+1)$ are on diagonals of B, are not dominated by B_1 and (by Lemma 15) are not X-pns. These squares are dominated by B$_2$ and so $B_2 \in \{(x-1, y+1), (x+1, y+1)\}$. In each case there exists an X-pn on $s(B) \cup d(B)$, contrary to Lemma 15.

Therefore $B_1 \in \{(x+1, y), (x+2, y)$. Since B_2 is adjacent to $(x-1, y-1)$ and $(x-1, y+1)$, we have $B_2 \in W_1 \cup W_2 \cup W_3$ (disjoint union), where

$$W_1 = \{(x-1, y), (x-1, y+2), (x-1, y+4), (x-2, y)\},$$
$$W_2 = \{(x-1, y+1), (x+1, y+1)\}, \text{ and}$$
$$W_3 = \{(x-3, y+1), (x-1, y+3)\}.$$

If $B_2 \in W_1$, then the column $x+3$ does not intersect the board, for otherwise $(x+3, y+3)$ or $(x+3, y-3)$ is an X-pn. Hence the column $x-3$ intersects the board and so $(x-3, y+3)$ or $(x-3, y-3)$ is an X-pn, a contradiction. If $B_2 \in W_2$, then for each of the two possible positions for B_1, there are three X-pns of B on $c(B)$, which is impossible. If $B_2 \in W_3$, then for each position of B_1, Lemma 15 is also contradicted. We have established that $x = 1$ or $y = 1$.

To complete the proof of (a), we must eliminate the case $x > 1$ and $y = 1$, so assume that X satisfies these conditions. Observe that $(x-1, 2)$ is on $s(B)$.

Also note that B_1 (respectively B_2) dominates exactly one square C_1 (respectively exactly three squares C_2, C_3, C_4) on $c(B) - \{B\}$, where $C_1 \notin \{C_2, C_3, C_4\}$. This implies $r(B_2) \geq 3$. To satisfy these conditions and to ensure that $(x-1, 2)$ is not an X-pn of B, B_2 is restricted to the following possibilities:

$$B_2 \in W_4 = \{(x+1, 4), (x+2, 5)\}$$
$$B_2 \in W_5 = \{(x-1, y) : y = 3, 4, 5, 6\}.$$

If $B_2 \in W_4$, then $x = 2$, otherwise $(x-2, 3)$ is an X-pn of B. Since $(7, 6)$ is not an X-pn, $B_1 = (1, 7)$ and for each choice of B_2, there is an X-pn on $d(B)$, contrary to Lemma 15. Similar contradictions may be obtained for $B_2 \in \{(x-1, y) : y = 5, 6\} \subseteq W_5$. (These elements of W_5 also do not dominate $(x-2, 3)$ and it follows that $x = 2$.) If $B_2 = (x-1, 3)$, then to facilitate two X-pns on $c(B)$, we require $x_1 \geq x+3$. Therefore $(x+1, 2)$ is an X-pn, which is impossible. Finally, let $B_2 = (x-1, 4)$. Since $c(B)$ has exactly two X-pns, $x_1 \in \{x+1, x+5\}$. In the former case at least one of $(x+2, 3)$ and $(x-4, 5)$ is an X-pn of B. In the latter case $x = 2$ and $(4, 3)$ is an X-pn. These contradictions show that $x = 1$, and (a) holds.

The relation (b) is true because it is the condition for B_1 to dominate at least one square of $c(B) - \{B\}$. ■

Lemma 17 *Suppose that B is an X_α-square of the maximal irredundant set $X = \{B, B_1, B_2\}$ of Q_7. Then each of B_1, B_2 has at least three X-pns.*

Proof. Without loss of generality assume that X is positioned as specified in Lemma 14. By definition, neither B_1 nor B_2 is an X_α-square.

If B_1 is an X_β-square, then by Lemma 16(a), $x_1 = 7$, and by Lemma 16(b), $y \in \{1, 7\}$. But $y = 7$ is impossible because $y_2 > y$, and if $y = 1$, then $\{B, B_2\}$ dominates at most two squares of $c(B_1) - \{B_1\}$. Thus B_1 has four X-pns on $c(B_1)$, a contradiction.

If B_2 is an X_β-square, then it has exactly two X-pns on $r(B_2)$ (Lemma 15). By Lemma 16(a), $B_2 = (1, 7)$, and since B dominates exactly one square of $r(B_2) - \{B_2\}$, B_1 dominates exactly three squares of $r(B_2) - \{B_2\}$. This implies that $y \in \{5, 6\}$, $B_1 \notin s(B_2)$, and $c(B_1) \neq 7$. Therefore $(7, 1)$ is an X-pn of B_2 on $s(B_2)$, contrary to Lemma 15.

We have thus shown that $\{B_1, B_2\}$ contains neither X_α- nor X_β-squares. By definition each of B_1 and B_2 has at least three X-pns. ■

Lemma 18 *Suppose that B is an X_β-square of the maximal irredundant set $X = \{B, B_1, B_2\}$ of Q_7. Then each of B_1, B_2 has at least three X-pns.*

Proof. Without loss of generality assume that X is positioned as specified in Lemma 16. By Lemma 17 and the definition of X_α- and X_β-squares, neither B_1 nor B_2 is an X_α-square. Suppose that B_1 is an X_β-square. Then by Lemma 16(a), $x_1 = 7$ and by Lemma 16(b), $y \in \{1, 7\}$. If $y = 7$, then B, B_1, B_2 are all on row 7 and B_1 has four X-pns on $c(B_1)$, which is impossible. If $y = 1$ (*i.e.*, $B = (1, 1)$ and $B_1 = (7, 1)$), then by Lemma 15, B_2 dominates exactly three squares of $\{(7, y') : y' = 2, ..., 6\}$. In all cases $(2, 6)$ is an X-pn of B_1 on $s(B_1)$, contrary to Lemma 15.

If B_2 is an X_β-square, then by Lemma 16(a), B_2 is not on $r(B)$ (by the same proof as the previous paragraph), hence $B_2 \in c(B_1)$. By Lemma 16(a), $B_2 = (x_1, 7)$. But B_1 (respectively B_2) dominates at most two (respectively one) squares of $c(B) - \{B\}$ and so B has at least three X-pns on $c(B)$, a contradiction.

Therefore $\{B_1, B_2\}$ contains neither X_α- nor X_β-squares, and so each of B_1, B_2 has at least three X-pns. ■

Lemma 19 *Let R be the set of vertices of Q_7 not dominated by a 3-square maximal irredundant set. Then $|N[R]| \geq 29$.*

Proof. Since $\gamma(Q_7) = 4$ (*cf.* [9]), we can apply Theorem 8 with $k = 1$. If R satisfies (b) or (c) of that theorem, then R occupies (without loss of generality) at least two rows and three columns. By counting the squares of $N[R]$ on these lines only, we obtain $|N[R]| \geq 29$.

Now suppose Theorem 8(a) applies and R contains precisely the squares at the intersections of rows y_1, y_2 and columns x_1, x_2. Without loss of generality we may assume that $x_1 < x_2$, $y_1 < y_2$ and $y_2 - y_1 > x_2 - x_1$. (Note that Theorem 8(a) insists that $y_2 - y_1 \neq x_2 - x_1$.) Observe that $N[R]$ has 24 squares on these rows and columns. Let W be the set of squares of $N[R]$ which are not on those lines, $\overline{x} = x_2 - x_1$ and $\overline{y} = y_2 - y_1$.

Case 1 $\overline{x} \geq 3$.
Then $\overline{y} \geq 4$ and W contains at least six squares (x, y), where $x_1 < x < x_2$ and $y_1 < y < y_2$.

Case 2 $\overline{x} = 2$.
Then $\overline{y} \geq 3$ and W contains at least two squares $(x_1 + 1, y)$ where $y_1 < y < y_2$. Without loss of generality columns $x_2 + 1$, $x_2 + 2$ exist and each contains at least two squares of W.

Case 3 $\overline{x} = 1$.
Then $\overline{y} \geq 2$ and without loss of generality columns $x_2 + 1$, $x_2 + 2$ and $x_2 + 3$ exist. If $\overline{y} \geq 4$, then W contains at least six squares (x, y), where $x_2 + 1 \leq x \leq x_2 + 3$ and $y_1 < y < y_2$. If $\overline{y} = 3$, then without loss of generality W contains $(x_2 + i, y_1 + j)$, for any $i, j \in \{1, 2\}$, and also $(x_2 + 1, y_2 + 1)$. Finally, if $\overline{y} = 2$, we may assume that rows $y_2 + 1$, $y_2 + 2$ also exist, so that R is in the corner of a 5×5 sub-board of Q_7 which contains seven squares of W.

In all cases $|W| \geq 5$ and $|N[R]| \geq 29$ as required. ■

Theorem 20 *Q_7 contains no maximal irredundant set of size three.*

Proof. Suppose to the contrary that X is a maximal irredundant set of size three. If no square in X has exactly one X-pn, then no more than one square has exactly two X-pns (Lemmas 17 and 18). If $B \in X$ has at least three X-pns, then Theorem 2 or Proposition 4 applies. Now B itself is a common neighbour of the three X-pns and

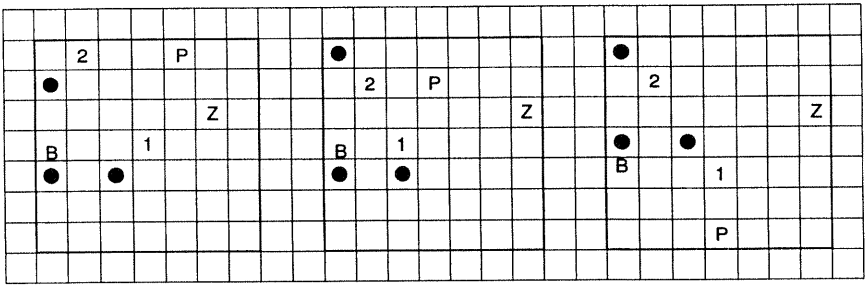

Figure 3

is not an annihilator. Hence there are at most $n + 1 = 8$ annihilators of B and the total number of annihilators of the three squares in X is at most $12 + 8 + 8 = 28$. However, by Theorem 5′, each vertex of $N[R]$ is an annihilator, and so $|N[R]| \leq 28$, contrary to Lemma 19.

Therefore $B \in X$ has exactly one X-pn and is an X_α-square (Lemma 15). Without losing generality we may assume X is positioned as in Lemma 14. If $y \geq 5$, then $|s(B) - \{B\}| \geq 4$. But B_2 (respectively B_1) dominates zero (respectively at most two) squares of $s(B) - \{B\}$ and so B has at least two X-pns, a contradiction. If $y = 1$, then $B_1 \cup B_2$ dominates at most four of the six squares of $d(B) - \{B\}$. If $y = 2$, any choice of B_1 and B_2 which dominates the maximum number, *i.e.*, four, of the five squares of $b(B)$, leaves the one square of $s(B)$ undominated and again B has two X-pns. We conclude that $y \in \{3, 4\}$. Figure 3 depicts the only (up to symmetry) sets X (black dots) which have X_α-squares B with exactly one X-pn (labelled P). In each diagram the square Z is in $N[R]$ but is not an annihilator since it is not adjacent to P, nor to squares 1 and 2, which are X-pns of B_1 and B_2 respectively. Thus in each case X is not maximal irredundant and the proof is complete. ■

Corollary 21 $ir(Q_7) = 4$.

Proof. Immediate from Theorem 20, the bounds $(\gamma(G) + 1)/2 \leq ir(G) \leq \gamma(G)$ and the fact that $\gamma(Q_7) = 4$. ■

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