Indecomposable Skolem and Rosa sequences

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Abstract

We introduce indecomposable (hooked) Skolem sequences and we show their existence for all admissible orders. We then introduce a new construction for two-fold triple systems from two-fold Skolem sequences, and use this to motivate the concept of a two-fold Rosa sequence, whose existence we show for all admissible orders.

1 Introduction

In 1957, T. Skolem [13], when studying Steiner triple systems, considered the possibility of distributing the numbers $1, 2, \ldots, 2n$ in n pairs (a_r, b_r) such that $b_r - a_r = r$ for $r = 1, 2, \ldots, n$. For example, for $n = 4$, the pairs $(1, 2), (5, 7), (3, 6)$, and $(4, 8)$ will be such a partition. Later, this partition was written as a sequence; the previous partition can be written as $1, 1, 3, 4, 2, 3, 2, 4$, which is now known as a Skolem sequence of order 4.

A number of authors, for example, Billington [5], Stanton and Goulden [14], and Rosa [10], considered generalizations of such sequences for the purpose of constructing various types of designs. For more details, the reader may see [7, 12].

A Steiner triple system of order v, $STS(v)$ is a pair (V, B), where V is a set of points, $|V| = v$ and B is a collection of 3-subsets (triples or blocks) from V such that every pair of elements from V occurs in exactly one triple. It is well known that a $STS(v)$ exists if and only if $v \equiv 1$ or $3 \pmod 6$. A $STS(v)$ is called cyclic if its automorphism groups contains a v-cycle.

Formally, a *Skolem sequence of order* n is a sequence $S = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers that satisfy the following conditions:

(1) For every $k \in \{1, 2, \ldots, n\}$ there exists exactly two elements s_i, s_j such that $s_i = s_j = k$.

(2) If $s_i = s_j = k, i < j$, then $j - i = k$.

An *extended Skolem sequence* of order n is a sequence $ES = (s_1, s_2, \ldots, s_{2n+1})$ of $2n + 1$ integers that satisfy conditions (1), (2), and:

(3) There is exactly one $i \in \{1, \ldots, 2n + 1\}$ such that $s_i = 0$.

The $s_i = 0$ is also known as the hook $(*)$ of the sequence, and if $s_{2n} = 0$, then the sequence is called *hooked Skolem* sequence. It is known that the necessary conditions for the existence of (hooked) (extended) Skolem sequences are sufficient.

Theorem 1.1 [Skolem] [13] *A Skolem sequence of order n exists if and only if $n \equiv 0, 1 (mod\ 4)$.*

[O'Keefe] [8] *A hooked Skolem sequence of order n exists if and only if $n \equiv 2, 3 (mod\ 4)$.*

[Abrham & Kotzig] [1] *An extended Skolem sequence of order n exists for all n.*

The existence of a (hooked) Skolem sequence of order n implies the existence of a cyclic $STS(6n+1)$ [13], and the existence of an extended (hooked) Skolem sequence with $s_{n+1} = 0$ implies the existence of a cyclic $STS(6n + 3)$ [10].

For example, the extended Skolem sequence of order 4; $1, 1, 3, 4, 0, 3, 2, 4, 2$ gives rise to the pairs $(a_r, b_r), r = 1, \ldots, 4, \{(1, 2), (7, 9), (3, 6), (4, 8)\}$ which gives the base blocks $\{0, i, b_i + 4\}$ (or $\{0, a_i + 4, b_i + 4\}$), $i = 1, \ldots, 4(\{0, 1, 6\}, \{0, 2, 13\}, \{0, 3, 10\}, \{0, 4, 12\})$mod 27. With the addition of the base block $\{0, 9, 18\}$mod 27, we get the blocks of $STS(27)$.

An *m-fold Skolem sequence* of order n is a sequence $mS = (s_1, s_2, \ldots, s_{2mn})$ with the following condition:

$(1)'$ For every $k \in \{1, 2, \ldots, n\}$ there exist m disjoint pairs $(i, i + k), i, i + k \in \{1, \ldots, 2mn\}$ such that $s_i = s_{i+k} = k$.

An *m-fold extended Skolem sequence* of order n is a sequence $mES = (s_1, s_2, \ldots, s_{2mn+1})$ with property $(1)'$ and $(2)'$ there exists exactly one $s_i = 0, 1 \leq i \leq 2mn + 1$. If $s_{2mn} = 0$, the extended sequence is called an m-fold hooked Skolem sequence.

In [3, 4], it is shown that the necessary conditions are sufficient for the existence of m-fold (hooked) (extended) Skolem sequences.

Theorem 1.2 *An m-fold Skolem sequence of order n exists if and only if*

(1) $n \equiv 0, 1 (mod\ 4)$, or

(2) $n \equiv 2, 3 (mod\ 4)$ and m even,

and a hooked m-fold Skolem sequence of order n exists if and only if $n \equiv 2$ or $3 (mod\ 4)$ and m is odd.

Theorem 1.3 *Let m, n, k be positive integers. There exists an extended m-fold Skolem sequence of order n with $s_k = 0$ if and only if one of the following conditions hold:*

(1) $n \equiv 0$ or $1 (mod\ 4)$, and k is odd;

(2) $n \equiv 2$ or $3 (mod\ 4), m$ is even and k is odd;

(3) $n \equiv 2$ or $3 (mod\ 4), m$ is odd and k is even.

For example, $2, 3, 2, 2, 3, 2, 1, 1, 3, 1, 1, 3$ is a 2-fold Skolem sequence of order 3 and $2, 2, 2, 2, 2, 0, 2, 1, 1, 1, 1, 1, 1$ is a 3-fold extended Skolem sequence of order 2.

In section 2, we introduce the notion of an indecomposable (hooked) Skolem sequence and we determine their spectrum. Then, in Section 3, we focus our attention on $m = 2$ and introduce two-fold Rosa sequences.

2 Indecomposable Skolem Sequences

In this section, we introduce indecomposable (hooked) Skolem sequences and show their existence for all admissible orders.

Let t be a positive integer such that $t \leq m$. A t-indecomposable m-fold (hooked) Skolem sequence of order n is an m-fold (hooked) Skolem sequence of order n such that for all subscripts $i, j, 1 \leq i < j \leq 2mn (1 \leq i < j \leq 2mn + 1)$, the proper subsequence $(s_i, s_{i+1}, \ldots, s_j)$ is not a t-fold (hooked) Skolem sequence of order r where $1 < r \leq n$. We use the convention that the 0 in a subsequence need not occur in the same position as the 0 in the original sequence. For example, the hooked sequence 1123203 contains the hooked subsequence 11202.

If an m-fold (hooked) Skolem sequence of order n is t-indecomposable for all $t \leq m$, then it is called indecomposable. We call the m-fold sequence *simple* indecomposable if it does not contain a proper subsequence in which the 0 occurs in the same position as the original sequence. Thus, for example, the hooked sequence 1123203 is simple indecomposable. We restrict our discussion in this section to indecomposability of an m-fold (hooked) Skolem to a subsequence of the same kind.

We will first consider the case $n = 2$. Since $n = 2 \equiv 2 (\text{mod } 4)$, then m must be even. Clearly, $1, 1, 1, 1, 2, 2, 2, 2$; $2, 2, 2, 2, 1, 1, 1, 1$, and $1, 1, 2, 2, 2, 2, 1, 1$, are exactly the indecomposable 2-fold Skolem sequences of order 2. Moreover, for $m > 2$, any m-fold Skolem sequence of order 2 is decomposable because it has at least one of these sequences as a subsequence.

Now we assume that $n > 2$, and denote by E_n, O_n the largest even and odd number not exceeding n, respectively. Let $O'_n = \lfloor \frac{n}{2} \rfloor + 1$. Furthermore, we define the following sequences:

$$A = (O_n, O_n - 2, \ldots, 3, 1, 1, 3, \ldots, O_n - 2, O_n),$$

$$B = (E_n, E_n - 2, \ldots, 4, 2, O'_n, 2, 4, \ldots, E_n, O'_n).$$

$C = (O_n - 2, \ldots, 3, 1, 1, 3, \ldots, O_n - 2, 1, 1), n \geq 5$. For $n = 3, 4, C = (1, 1, 1, 1)$.

$D = (O_n, \ldots, O'_n + 4, O'_n + 2, O'_n - 2, \ldots, 3, 1, 1, 3, \ldots, O'_n - 2, 1, 1, O'_n + 2, O'_n + 4, \ldots, O_n), n > 5$, $n \equiv 0$ or $1 \bmod 4$. For $n = 4, D = (1, 1, 1, 1)$; for $n = 5, D = (5, 1, 1, 1, 1, 5)$.

$X = (E_n, \ldots, 4, 2, E_n, 2, 4, \ldots, E_n, E_n - 2, \ldots, 4, 2, E_n, 2, 4, \ldots, E_n - 2), n \geq 4$. For $n = 3, X = (2, 2, 2, 2)$.

$Y = (O_n, \ldots, 5, 3, O_n, O_n, 3, 5, \ldots, O_n, \ldots, 5, 3, O_n, O_n, 3, 5, \ldots, O_n - 2)$, $n \geq 5$. For $n = 3, 4, Y = (3, 3, 3, 3, 3, 3)$.

$Z = (1, 1, O_n - 2, \ldots, 3, 1, 1, 3, \ldots, O_n - 2, 1, 1), n \geq 5$. For $n = 3, 4, Z = (1, 1, 1, 1, 1, 1)$.

We start with a lemma that is helpful for the proof of the next theorem. We use square brackets [ ] to enclose portions of sequences that are increasing or decreasing by twos.

Lemma 2.1 *For all $n > 2, n \equiv 0, 1$ (mod 4), there exists an indecomposable three-fold Skolem sequence of order n.*

Proof.

Case 1: $n \equiv 0$ (mod 4).

For $n = 4$, an indecomposable sequence is: $(4, 2, 4, 2, 4, 2, 4, 2, 1, 1, 3, 1, 1, 3, 1, 1, 2, 4, 2, 3, 3, 4, 3, 3)$. For $n = 8$, an indecomposable sequence is formed by a copy of X, followed by (5,3,1,1,3,5), followed by a copy of A, followed by $(6, 4, 1, 1, 7, 4, 6, 8, 5, 3, 7, 7, 3, 5, 2, 8, 2, 7)$.

For $n > 8, n \equiv 4$ (mod 8), an indecomposable sequence is obtained by taking one copy of X, followed by $[O_n - 2, \ldots, 1], [1, \ldots, O_n - 2]$, followed by a copy of A, followed by $([E_n - 2, E_n - 4, \ldots, 4], 1, 1, O_n, [4, \ldots, E_n - 4, E_n - 2], O_n, E_n, (O_n - 4, O_n - 2, O_n - 8, O_n - 6, \ldots, \frac{E_n}{2} + 5, \frac{E_n}{2} + 7)^*, [\frac{E_n}{2} + 1, \frac{E_n}{2} - 1, \ldots, 5, 3], \frac{E_n}{2} + 3, O_n, [3, 5, \ldots, \frac{E_n}{2} - 1, \frac{E_n}{2} + 1], 2, \frac{E_n}{2} + 5, 2, (\frac{E_n}{2} + 9, \frac{E_n}{2} + 7, \ldots, O_n - 4, O_n - 6, O_n, O_n - 2)^*, E_n, \frac{E_n}{2} + 3)$. *Omit when $n = 12$.

For $n > 8, n \equiv 0$ (mod 8), an indecomposable sequence is obtained by using the same first three segments used in case of $n \equiv 4$ (mod 8) and replacing the last segment by $([E_n - 2, E_n - 4, \ldots, 4], 1, 1, O_n, [4, \ldots, E_n - 4, E_n - 2, E_n], O_n - 2, O_n, (O_n - 6, O_n - 4, \ldots, \frac{E_n}{2} + 5, \frac{E_n}{2} + 7)^{**}, [\frac{E_n}{2} + 1, \frac{E_n}{2} - 1, \ldots 5, 3], \frac{E_n}{2} + 3, O_n, [3, 5, \ldots, \frac{E_n}{2} - 1, \frac{E_n}{2} + 1], 2, \frac{E_n}{2} + 5, 2, (\frac{E_n}{2} + 9, \frac{E_n}{2} + 7, \frac{E_n}{2} + 13, \frac{E_n}{2} + 11, \ldots, O_n - 6, O_n - 8, O_n - 2, O_n - 4)^{**}, E_n, O_n, \frac{E_n}{2} + 3)$. **Omit when $n = 16$.

Case 2: $n \equiv 1$ (mod 4).

For $n = 5$, the following is a solution: $(4, 2, 4, 2, 4, 2, 4, 2, 5, 3, 1, 1, 3, 5, 3, 1, 1, 3, 4, 2, 5, 2, 4, 3, 5, 5, 3, 1, 1, 5)$.

For $n \equiv 1$ (mod 4), $n > 5$, an indecomposable sequence is obtained by taking a copy of X, followed by a copy of A, followed by $[O_n - 2, \ldots, 1], [1, \ldots, O_n - 2]$, followed by $([E_n, E_n - 2, \ldots, 4, 2], O_n, [2, 4, \ldots, E_n - 2, E_n], [O_n, O_n - 2, \ldots, \frac{E_n}{2} + 5], [\frac{E_n}{2} + 1, \frac{E_n}{2} -$

$1, \ldots, 5, 3], \frac{E_n}{2}+3, O_n, [3, 5, \ldots, \frac{E_n}{2}-1, \frac{E_n}{2}+1], 1, 1, [\frac{E_n}{2}+5, \ldots, O_n-2, O_n], \frac{E_n}{2}+3)$.

□

Theorem 2.2 *For all $n > 2$ and all $m \geq 1$, with m even when $n \equiv 2, 3$ (mod 4), there exists an indecomposable m-fold Skolem sequence of order n.*

Proof.

Case 1: m even.

For $m \equiv 0 \pmod 6$, suppose that $m = 6k$. An indecomposable sequence is obtained if we take $3k$ copies of X, followed by $2k$ copies of Y, followed by $2k$ copies of Z.

For $m \equiv 2 \pmod 6$, suppose that $m = 2 + 6k$. An indecomposable sequence is obtained if we take $3k + 1$ copies of X, followed by $2k$ copies of Y, followed by 2 copies of A, followed by $2k$ copies of Z.

For $m \equiv 4 \pmod 6$, suppose that $m = 4 + 6k$. For $k = 0$, an indecomposable sequence is obtained if we take 2 copies of X, followed by Y, followed by A, followed by Z. For $k > 0$, an indecomposable sequence is obtained if we take $3k + 2$ copies of X, followed by $2k$ copies of Y, followed by 4 copies of A, followed by $2k$ copies of Z.

Case 2: m odd (and $n \equiv 0$ or 1 mod 4).

For $m = 1$, the solution in [12] is an indecomposable one-fold Skolem sequence.

For $m \equiv 1 \pmod 6$, suppose that $m = 1+6k$, $k > 0$. An indecomposable sequence is obtained if we take $3k$ copies of X, followed by B, followed by $2k$ copies of Y, followed by $2k - 1$ copies of Z, followed by C, followed by D.

For $m = 3$, Lemma 2.1 provides a solution.

For $m \equiv 3 \pmod 6$, suppose that $m = 3+6k, k > 0$. An indecomposable sequence is obtained if we take $3k + 1$ copies of X, followed by B, followed by $2k$ copies of Y, followed by 2 copies of A, followed by $2k - 1$ copies of Z, followed by C, followed by D.

Finally, for $m \equiv 5 \pmod 6$, suppose that $m = 5 + 6k$. For $k = 0$, an indecomposable sequence is obtained if we take 2 copies of X, followed by B, followed by Y, followed by A, followed by C, followed by D. For $k > 0$, an indecomposable sequence is obtained if we take $3k + 2$ copies of X, followed by B, followed by $2k$ copies of Y, followed by 4 copies of A, followed by $2k - 1$ copies of Z, followed by C, followed by D. □

For the hooked indecomposable Skolem sequences, m must be odd and $n \equiv 2, 3 \pmod 4$. We note that for the case n = 2 it is not possible to construct an indecomposable m-fold hooked Skolem sequence of order 2.

Theorem 2.3 *For all $n \equiv 2, 3 \pmod 4$, $n > 2$ and m odd, there exists an indecomposable m-fold hooked Skolem sequence of order n.*

Proof.

For $n = 3$ and $m = 2k + 1$, an indecomposable hooked sequence is obtained if we take 3 copies of $(1,1)$, followed by $2(k-1)$ copies of $(3,1,1,3)$, followed by $(3,3,3,3,3,3)$, followed by k copies of $(2,2,2,2)$ followed by $(2,0,2)$.

For $n = 6$ and $m = 1$, an indecomposable sequence is:
$(4,2,5,2,4,3,6,5,3,1,1,0,6)$.

For $n \equiv 2 (\mathrm{mod}\ 4), n > 6$, and $m = 1$, an indecomposable hooked sequence is obtained from the solution in [7].

For $n = 7$ and $m = 1$, an indecomposable sequence is:
$(5,1,1,3,6,5,3,7,4,2,6,2,4,0,7)$.

For $n \equiv 3 (\mathrm{mod}\ 4), n > 7$, and $m = 1$, an indecomposable hooked sequence is obtained from the solution in [7].

For $n \equiv 2 (\mathrm{mod}\ 4), n > 2$, and $m > 1$, we define the following sequences:

$$A' = (E_n, O_n - 2, \ldots, \tfrac{E_n}{2} + 2, \tfrac{E_n}{2} - 2, \ldots, 5, 3, O_n, \tfrac{E_n}{2}, 3, 5, \ldots, \tfrac{E_n}{2} - 2, 1, 1, \tfrac{E_n}{2} + 2, \ldots, O_n - 2, 0, E_n), n > 7 \text{ , for } n = 6, A' = (6,5,3,1,1,0,6),$$

$$B' = (O_n - 2, \ldots, 5, 3, O_n, O_n, 3, 5, \ldots, O_n - 2, \tfrac{E_n}{2}),$$

$$C' = (E_n - 2, \ldots, 4, 2, O_n, 2, 4, \ldots, E_n - 2).$$

$$D' = (1,1),$$

$$A = (O_n, \ldots, 3, 1, 1, 3, \ldots, O_n),$$

$$X = (E_n, \ldots, 4, 2, E_n, 2, 4, \ldots, E_n, \ldots, 4, 2, E_n, 2, 4, \ldots, E_n - 2).$$

Let $m = 2k+1$. An indecomposable hooked sequence is obtained if we take k copies of X, followed by $2k-1$ copies of A, followed by D', followed by C', followed by B', followed by A'.

For $n \equiv 3 (\mathrm{mod}\ 4), n > 3$, and $m > 1$, we define the following sequences:

$$A'' = (O_n, E_n - 2, \ldots, 4, 2, E_n, 2, 4, \ldots, E_n - 2, 0, O_n),$$

$$B'' = (E_n - 2, \ldots, 4, 2, E_n, 2, 4, \ldots, E_n - 2),$$

$$C'' = (E_n, \ldots, 4, 2, E_n, 2, 4, \ldots, E_n),$$

$$D'' = (O_n - 2, \ldots, \tfrac{E_n}{2} + 2, 1, 1, \tfrac{E_n}{2} - 2, \ldots, 5, 3, \tfrac{E_n}{2}, E_n, 3, 5, \ldots, \tfrac{E_n}{2} - 2, \tfrac{E_n}{2} + 2, \ldots, O_n - 2, \tfrac{E_n}{2}), n > 7, \text{ for } n = 7, D'' = (5,1,1,3,6,5,3),$$

$$A = (O_n, \ldots, 3, 1, 1, 3, \ldots, O_n),$$

$$X = (E_n, \ldots, 4, 2, E_n, 2, 4, \ldots, E_n, \ldots, 4, 2, E_n, 2, 4, \ldots, E_n - 2).$$

Let $m = 2k+1$. An indecomposable hooked sequence is obtained if we take $2k$ copies of A, followed by $k-1$ copies of X, followed by D'', followed by C'', followed by B'', followed by A''. □

3 Rosa Sequences

As indicated in Section 1, it is well known that a (hooked) Skolem sequence of order n can be used to generate the base blocks of a cyclic STS $(6n+1)$, and that an extended (hooked) Skolem sequence of order n with $s_{n+1} = 0$ can be used to generate the base blocks of a cyclic STS $(6n+3)$. Sequences of the latter type have come to be known as (hooked) Rosa sequences.

Lemma 3.1 *A (hooked) Rosa sequence of order n is equivalent to a (hooked) Skolem sequence of order $n+1$ with $s_1 = n+1$.*

Proof.
Given a (hooked) Skolem sequence of order $n+1$ with $s_1 = n+1$, we simply delete the two $(n+1)$s from the sequence, replacing the rightmost one (i.e. the $n+1$ in position $n+2$) with a 0. The result is a (hooked) Rosa sequence of order n. The construction is clearly reversible. □

Remark 3.2 *It is well known that there exists a (hooked) Rosa sequence of order n if and only if $n \geq 2$ [10].*

We now introduce a construction for a cyclic two-fold triple system on $6n+1$ points, starting from a two-fold Skolem sequence of order n. We then modify the construction to produce cyclic two-fold triple systems on $6n+3$ points, starting from a 'certain' two-fold Skolem sequence of order $n+1$ which we then use to define a two-fold Rosa sequence of order n.

Suppose now that we have a two-fold Skolem sequence of order n; that is, a sequence $2S = (s_1, s_2, \ldots, s_{4n})$ such that for every $k \in \{1, 2, \ldots, n\}$ there exists 2 disjoint pairs $(i, i+k), i, i+k \in \{1, 2, \ldots, 4n\}$ such that $s_i = s_{i+k} = k$. From Theorem 1.2, such a sequence exists for every order n (in fact, by Theorem 2.2, we have *indecomposable* sequences of every order $n > 1$). We can use such a sequence to generate the base blocks of a cyclic two-fold triple system $TS_2(6n+1)$ on $6n+1$ points, as follows.

For each $r = 1, 2, \ldots, n$, construct the pairs (a_r, b_r) and $(c_r, d_r)(a_r, b_r, c_r, d_r \in \{1, 2, \ldots, 4n\})$ such that $b_r - a_r = d_r - c_r = r$. Then the set $\{r, a_r + n, b_r + n), (6n + 1 - r, d_r + n, c_r + n) : r = 1, 2, \ldots, n\}$ partitions the set $\{1, 2, \ldots, 6n\}$ into $2n$ triples (a, b, c) such that $a + b \equiv c (\mathrm{mod}\ 6n+1)$. Hence, the set of triples $\{\{0, r, b_r + n\}, \{0, r, d_r + n\}; r = 1, 2, \ldots, n\}$ will form the base blocks for a cyclic two-fold triple system on $6n+1$ points.

For example, the two-fold Skolem sequence of order 2 given by 11112222 yields the pairs $(1,2), (3,4), (5,7), (6,8)$, which in turn give rise to the base blocks $\{0,1,4\}$, $\{0,1,6\}, \{0,2,9\}, \{0,2,10\}$ for a cyclic $TS_2(13)$. On the other hand, the two-fold Skolem sequence of order 2 given by 11222211 yields the pairs $(1,2), (7,8), (3,5), (4,6)$, and so the base blocks $\{0,1,4\}, \{0,1,10\}, \{0,2,7\}, \{0,2,8\}$ for a (second) $TS_2(13)$. That these two triple systems really are different can be seen by noting that the latter system decomposes into two cyclic $STS(13)$s, i.e. $\{0,1,4\}, \{0,2,7\} (\mathrm{mod}\ 13)$

and $\{0,1,10\},\{0,2,8\}$ (mod 13), while the former system does not (i.e. the former system is *cyclically* indecomposable).

We now detail a construction for cyclic two-fold triple systems on $6n+3$ points. Suppose that we have a two-fold Skolem sequence $2S$ of order $n+1$ in which $s_1 = s_{n+2} = n+1$ and $s_{3n+3} = s_{4n+4} = n+1$ (i.e. the sequence begins and ends with $n+1$). Delete the four $(n+1)$s from the sequence and replace the two middle ones with 0s. The result is a sequence $2T = (t_1, t_2, \ldots, t_{4n+2})$ such that (i) for every $k \in \{1,2,\ldots,n\}$ there exist 2 disjoint pairs $(i, i+k), i, i+k \in \{1,2,\ldots,4n+2\}$ such that $t_i = t_{i+k} = k$, and (ii) $t_{n+1} = t_{3n+2} = 0$. We will call this a two-fold Rosa sequence of order n. Clearly, if we are given a two-fold Rosa sequence of order n, we can add a pair of $(n+1)$s at each of the beginning and end of the sequence and so obtain a two-fold Skolem sequence of order $n+1$ which begins and ends with $n+1$.

Now suppose that we are given a two-fold Rosa sequence $2T$ of order n. For each $r = 1,2,\ldots,n$, construct the pairs (a_r, b_r) and (c_r, d_r) such that $b_r - a_r = d_r - c_r = r$. The set $\{(r, a_r+n, b_r+n), (6n+3-r, d_r+n, c_r+n) : r = 1,2,\ldots,n\}$ partitions the set $\{1,2,\ldots,6n+2\} \backslash \{2n+1, 4n+2\}$ into $2n$ triples (a,b,c) such that $a+b \equiv c(\text{mod } 6n+3)$. Hence, the set of triples $\{\{0,r,b_r+n\},\{0,r,d_r+n\} : r = 1,2,\ldots,n\}$ together with the (repeated) triples $\{0,2n+1,4n+2\},\{0,2n+1,4n+2\}$ will form the base blocks for a cyclic two-fold triple system on $6n+3$ points.

For example, the two-fold Rosa sequence of order 2 given by 1102222011 (which is equivalent to the two-fold Skolem sequence 311322223113) yields the pairs $(1,2)$, $(9,10),(4,6),(5,7)$, which in turn give rise to the base blocks $\{0,1,4\},\{0,1,12\}$, $\{0,2,8\},\{0,2,9\}$ (with $\{0,5,10\},\{0,5,10\}$) for a cyclic $TS_2(15)$.

We now move to establish the existence of two-fold Rosa sequences.

Theorem 3.3 *There exists a two-fold Rosa sequence of order n if and only if $n \geq 2$.*

Proof.
If $n \equiv 0$ or $3 \text{mod } 4$ we can take a Rosa sequence $S = (s_1, s_2, \ldots, s_{2n+1})$ of order n (having $s_{n+1} = 0$) and define $2T = (t_1, t_2, \ldots, t_{4n+2})$ as follows. For each $i = 1,2,\ldots,2n+1$ set $t_i = s_i$ and for each $i = 2n+2, 2n+3, \ldots, 4n+2$ set $t_i = s_{4n+2-i+1}$; then $2T$ is a two-fold Rosa sequence of order n (note in particular that $t_{n+1} = s_{n+1} = 0$ and $t_{3n+2} = s_{n+1} = 0$).

If $n \equiv 1$ or $2 \text{mod } 4$, we take a hooked Rosa sequence $S = (s_1, s_2, \ldots, s_{2n+2})$ of order n (having $s_{n+1} = s_{2n+1} = 0$) and define $2T = (t_1, t_2, \ldots, t_{4n+2})$ as follows. For each $i = 1,2,\ldots,2n$ set $t_i = s_i$. Set $t_{2n+1} = t_{2n+2} = s_{2n+2}$, and then for each $i = 2n+3, 2n+4, \ldots, 4n+2$, set $t_i = s_{4n+2-i+1}$. Then $2T$ is a two-fold Rosa sequence of order n (with $t_{n+1} = s_{n+1} = 0$ and $t_{3n+2} = s_{n+1} = 0$).

It is not difficult to see that there is no two-fold Rosa sequence of order 1. □

The two-fold Rosa sequences produced by Theorem 3.3 are obtained by concatenating two (hooked) Rosa sequences and so for $n \equiv 0$ or $3 \text{mod } 4$ would be considered as being decomposable (that is, when $n \equiv 0$ or $3 \text{mod } 4$, $2T$ contains S as a Rosa subsequence). On the other hand, when $n \equiv 1$ or $2 \text{mod } 4$ the two-fold Rosa sequences produced by Theorem 3.3, while being indecomposable, do by construction contain hooked Rosa subsequences. We will say that a two-fold Rosa sequence is *totally indecomposable* if it does not contain a proper Rosa subsequence (in case $n \equiv 0$ or 3mod 4) or a proper hooked Rosa subsequence (in case $n \equiv 1$ or 2mod 4).

Theorem 3.4 *For every $n \geq 2$ there is a totally indecomposable two-fold Rosa sequence of order n.*

Proof.
We use the sequences defined prior to Lemma 2.1.

Case 1: n is even.

A totally indecomposable two-fold Rosa sequence is obtained by writing a copy of A , followed by a hook, followed by a copy of X, followed by a hook, followed by a copy of A.

Case 2: n is odd.

For $n = 3$, a totally indecomposable sequence is:
$(1, 1, 2, 0, 2, 2, 3, 2, 3, 3, 0, 3, 1, 1)$.

For $n = 5$, a totally indecomposable sequence is:
$(3, 1, 1, 3, 5, 0, 2, 3, 2, 5, 3, 4, 1, 1, 5, 4, 0, 4, 2, 5, 2, 4)$.

For $n = 7$, a totally indecomposable sequence is:
$(5, 3, 1, 1, 3, 5, 7, 0, 3, 5, 2, 3, 2, 7, 5, 6, 4, 1, 1, 7, 4, 6, 0, 6, 4, 2, 7, 2, 4, 6)$.

For $n \equiv 3(mod 4), n > 7$, a totally indecomposable sequence is formed by $[O_{n-2}, 1]$, $[1, O_{n-2}]$, $(O_n, 0, O_n - 4, O_n - 2, O_n - 8, O_n - 6, \ldots, (7, 9, 3, 5, 2, 3, 2, 7, 5)^*$, $\ldots, O_n - 4, O_n - 6, O_n, O_n - 2, [E_n, 4], 1, 1, O_n, [4, E_n], 0, [E_n, 2], O_n, [2, E_n])$.

*Replace by (5,7,2,3,2,5,3,9,7) in case of $n \equiv 1(mod 4), n > 5$.

□

Conclusion

There are many interesting constructions for λ-fold triple systems from m-fold Skolem sequences for $m = 1$ and 2, as well as constructions for two-association schemes (with triples) from m-fold Skolem sequences for $m > 2$. We will investigate some of these in a forthcoming article [9], with particular attention to what properties of the Skolem sequences are required in order that the corresponding triple systems be simple and/or indecomposable. (It turns out that indecomposable sequences do *not* necessarily give rise to indecomposable triple systems, nor do decomposable sequences necessarily give rise to decomposable triple systems.) The concept of indecomposable Skolem (Rosa) sequences is closely related to the concept of primitive words. If we consider the set A^* of all m-fold skolem sequences (words) of order n over the alphabet $A = \{1, 2, ...n\}$ then our result can be stated as follows: there exists a primitive word of length $2mn$ in A^*. We hope to find applications for this concept and report that in a future paper.

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