Characterization of graphs with Hall index 2

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To the memory of M.L. Mehrabadi

Abstract

A proper edge coloring of a simple graph G from some lists assigned to the edges of G is of interest. A. Hilton and P. Johnson (1990) considered a necessary condition for the list coloring of a graph and called it Hall's condition. They introduced the Hall index of a graph G, $h'(G)$, as the smallest positive integer m such that there exists a list coloring whenever the lists are of length at least m and Hall's condition is satisfied. They characterized all graphs G with $h'(G) = 1$. In this paper we characterize the graphs with Hall index 2.

1 *Introduction and Preliminaries*

We consider finite simple graphs and follow the notations in [1]. An *L-list coloring*, or L-coloring for short, of a graph G is an assignment of colors to the vertices such that each vertex v receives a color from a prescribed list $L(v)$ of colors and the adjacent vertices receive distinct colors. If an L-coloring exists then the following inequality, called *Hall's condition*, holds:

$$\sum_{\sigma} t_H(\sigma, L) \geq |V(H)|;$$

here H is an arbitrary subgraph of G and $t_H(\sigma, L)$ denotes the maximum number of independent vertices of H having the color σ in their lists, and σ ranges over $\bigcup_{v \in V(H)} L(v)$. In what follows, we use the same notions and notations for the corresponding edge concepts. Specially we widely use the following definition:

Definition 1. [4] A graph G with a list assignment L to the edges of G satisfies

Hall's condition if for each subgraph H of G, we have:

$$|E(H)| \leq \sum_{\sigma \in \bigcup_e L(e)} t_H(\sigma, L). \qquad (*)$$

We denote $\bigcup_{e \in E(G)} L(e)$ by $\mathcal{L}$ throughout this paper. Although Hall's condition is necessary for the existence of L–coloring, it is not sufficient unless we suppose that the sizes of lists are large enough. The following definitions appeared in [7].

Definition 2. The *Hall number* of a graph G, $h(G)$, is the smallest positive integer m such that, for every list assignment L of G with $|L(v)| \geq m$, $v \in V(G)$, if (G, L) satisfies Hall's condition, then G has an L–coloring.

Definition 3. The *Hall index* of a graph G, $h'(G)$, is defined as $h(L(G))$, where $L(G)$ is the line graph of G and is defined to be the graph whose vertex set is in one to one correspondence with $E(G)$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are incident.

We use the following lemmas frequently:

Lemma A. [4, 5] *For a graph G we have $h(G) = 1$ iff every block of G is a complete graph.*

Lemma B. [7] *For a connected graph G we have $h'(G) = 1$ iff G is a nontrivial tree or K_3.*

Lemma C. [7] *If H is a subgraph of G then $h'(H) \leq h'(G)$.*

In this paper we characterize the graphs with Hall index 2. Our main theorem is:

Theorem (Main Theorem). *A graph G has Hall index at most 2, if and only if every component of G is one of the following graphs.*

a) *A tree.*

b) *A cycle.*

c) *A graph with maximum degree 3 which contains exactly one cycle and only one vertex of degree 3, that lies on the cycle also; see Figure 15.*

d) *A bipartite graph with maximum degree 3 having exactly one cycle and at most two vertices of degree 3 which lie on the cycle and are non–adjacent; see Figure 16.*

e) *A graph whose set of cycles consist of some vertex disjoint triangles such that every triangle has exactly two vertices of degree 2 and one vertex of degree 3; see Figure 18.*

f) *A graph whose set of cycles consist of one triangle such that it has exactly one vertex of degree* 4 *and the other vertices of graph have degree at most* 2*; see Figure* 17.

2 *Some Forbidden Subgraphs*

The following obvious lemma provides a useful technique for verifying Hall's condition.

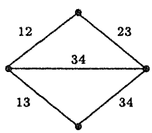
Lemma 1. *A graph G with a list assignment L satisfies Hall's condition if G satisfies* $(*)$ *and for every edge e of G, $G - e$ has an L–coloring.*

To characterize the graphs with Hall index at most two, we start by introducing some forbidden subgraphs. First we repeat the following definition.

Definition 4. A θ–graph consists of two distinguished vertices x and y together with three internally disjoint paths from x to y. Thus a θ–graph $\theta_{k,l,m}$ can be specified by giving k, l, and m, the lengths of three paths.

Proposition 1. *Let G be a θ–graph, $\theta_{k,l,m}$; then $h'(G) > 2$.*

Proof. Obviously every θ–graph is isomorphic to one of the graphs shown in the following figures.



$m = k = 2, \quad l = 1$

Figure 1.

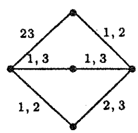
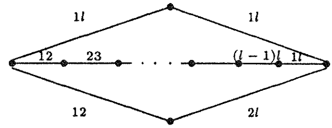
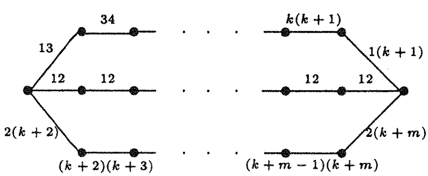


Figure 2



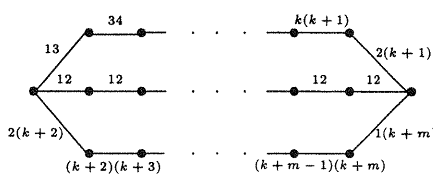
$m = k = 2, \quad l > 1$

Figure 3.



$k \geq 2, \quad m > 2, \quad l$ is odd

Figure 4.



$k \geq 2, \quad m > 2, \quad l$ is even

Figure 5.

The list assignments indicated in the above figures show that $h'(G) > 2$. We show it for Figure 4 and Figure 5.
In both cases it is obvious that the graph G with the indicated list assignments, L, does not have an L–coloring, and for each edge e of G, $G - e$ has an L–coloring. By Lemma 1, it is enough to check that (G, L) satisfies $(*)$. If l is odd, we have $t_G(1, L) + t_G(2, L) \geq l + 2$ and $t_G(3, L) = \cdots = t_G(k + m, L) = 1$. If l is even we have; $t_G(1, L) = t_G(2, L) = l + 2$ and $t_G(3, L) = \cdots = t_G(k + m, L) = 1$. Therefore, $\sum_{\sigma \in \mathcal{L}} t_G(\sigma, L) \geq l + k + m = |E(G)|$, and hence (G, L) satisfies Hall's condition. So, $h'(G) > 2$. □

Proposition 2. *Each of the graphs shown in Figures* $6 - 14$, *in which the lengths of the cycles in Figures* $6, 8$ *and* 9 *are at least* 4 *and the length of the cycle in Figure* 7 *is at least* 3, *has Hall index at least* 3.

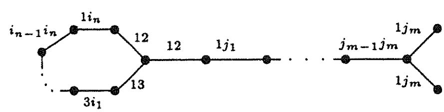


Figure 6.

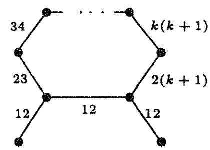


Figure 7.

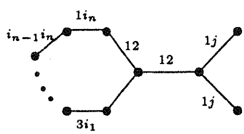


Figure 8.

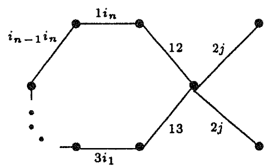


Figure 9.

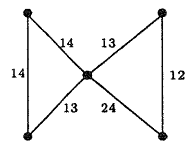


Figure 10.

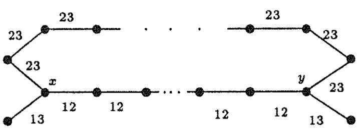


Figure 11. The cycle is odd

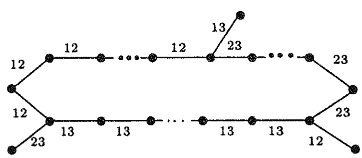


Figure 12. The cycle is even

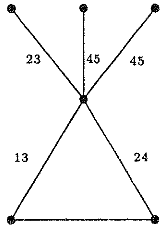


Figure 13.

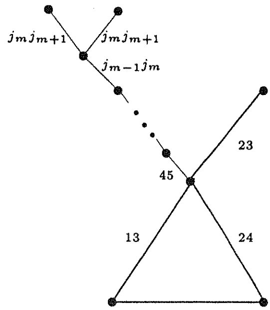


Figure 14.

Proof. The list assignments indicated in the above figures show that their Hall indexes are greater than 2. We show this only for the graph G which is shown in Figure 11. The other cases are easy to check. Let L be the list assignment shown in Figure 11. It is obvious that G does not have an L–coloring, but for each edge e of G, $G - e$ has an L–coloring. Now by Lemma 1, it is enough to show that (G, L) satisfies $(*)$. Suppose $d_G(x, y) = k$ and n is the length of the cycle. We have two cases:

Case 1) k is even : $t_G(1, L) = \frac{k}{2} + 1$, $t_G(2, L) = \frac{n-1}{2}$, and $t_G(3, L) = \frac{n-k-1}{2} + 2$.
Case 2) k is odd : $t_G(1, L) = \frac{k+1}{2} + 1$, $t_G(2, L) = \frac{n-1}{2}$, and $t_G(3, L) = \frac{n-k}{2} + 1$.
In each case we have $\sum_{\sigma \in \mathcal{L}} t_G(\sigma, L) = n + 2 = |E(G)|$. Hence, $h'(G) > 2$. □

3 *The Graphs with Hall Index two*

As we discussed in the last section it is easy, by the Propositions 1, 2, and Lemma C, to see that the only graphs that may have Hall index at most 2 are the subgraphs of the graphs shown in Figures 15, 16, 17, and 18.

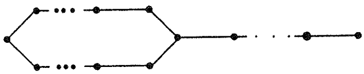


Figure 15.

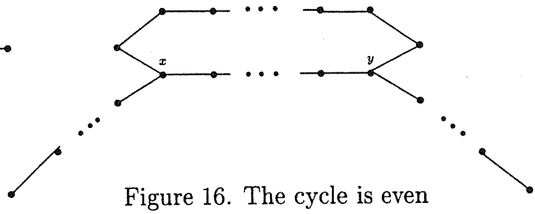


Figure 16. The cycle is even

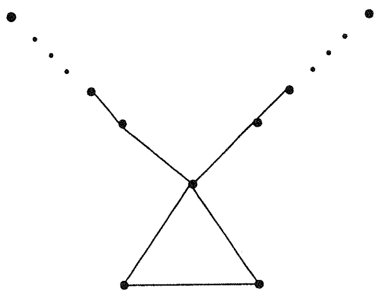


Figure 17.

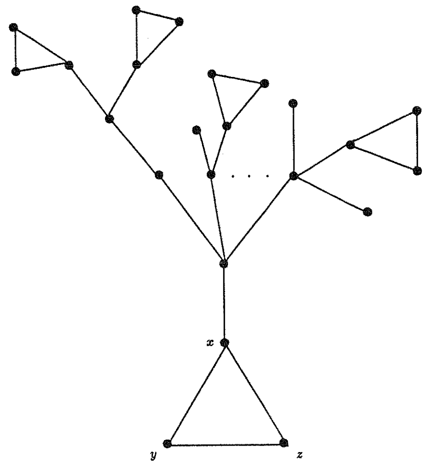


Figure 18.

In the following theorems we prove that these graphs actually have Hall index 2.

Theorem 1. *Let G be a graph which is obtain by joining two leaves of $K_{1,4}$. Then G has Hall index 2.*

Proof: The line graph of G is K_4 with an 'ear' of length 2, and this is proven to have Hall number 2 in [3]. □

Theorem 2. *The Hall index of the graph shown in Figure 15, is equal to 2.*

Proof. Without loss of generality we can consider the graph G shown in Figure 15.a.

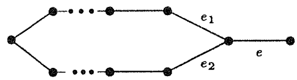


Figure 15.a.

The line graph of G is $\theta_{m,2,1}$, for some $m \geq 2$, and these graphs are shown to have Hall number 2 in [5]. □

Theorem 3. *Let G be an even cycle together with two pendant edges, with no edge of the cycle incident to both. Then G has Hall index 2.*

Proof. The line graph of G is shown to have Hall number 2 in [3]. □

Theorem 4. *Let G be a graph with the vertex set $V(G) = \{x, y, z, w\}$ and edge set $E(G) = \{xy, xz, yz, xw\}$ (see Figure 19). Let L be a list assignment to the edges of G such that $|L(xw)| \geq 1$ and the other lists each have size at least two. If (G, L) satisfies Hall condition then G has an L-coloring.*

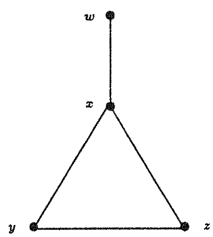


Figure 19.

Proof. The line graph of G is $\theta_{2,2,1}$, shown to have Hall number 2 in [5], so the conclusion follows if $|L(xw)| \geq 2$. suppose $L(xw) = \{1\}$. Since $G - yz$ is a tree, and therefore properly L-colorable, we may as well suppose that $|L(yz)| = 2$. Now, keeping in mind that $G - yz$ is properly L-colorable we see that the only way G can fail to be L-colorable is if, for some colors 2, 3, we have $L(yz) = \{2, 3\}$ and $L(xy)$, $L(xz) \subseteq \{1, 2, 3\}$. But then

$$\sum_{\sigma} t_G(\sigma, L) = 3 < |E(G)|,$$

contradiction the assumption that G and L satisfy Hall's condition. □

Lemma 2. *Suppose that (G, L) has an L-coloring and for some edge e of G, $L(e) =$*

$\{1,2,\cdots,n\}$, such that in every L- coloring of G, e can take the colors only from the set $\{1,2,\cdots,k\}$. Define L' as follows:
$L'(f) = L(f)$ if $f \neq e$, and $L'(e) = \{k+1,\cdots,n\}$. Assume that (G,L') does not satisfy Hall's condition. Then G has a subgraph K which contains e such that, $\sum_{\sigma\in\mathcal{L}} t_K(\sigma,L) \leq |E(K)| + m_{e,K} - 1$, where $m_{e,K} = |\{i : t_K(i,L) = t_K(i,L') + 1, \text{ and } 1 \leq i \leq k\}|$.

Proof. Since (G,L') does not satisfy Hall's condition, there exists a subgraph K of G such that:

$$\sum_{\sigma\in\mathcal{L}} t_K(\sigma,L') < |E(K)|. \tag{1}$$

But K must contain e, otherwise, K is a subgraph of G with the list assignment L and since (G,L) satisfies Hall's condition, (1) can not hold. Now, we have:

$$\begin{array}{rcll} t_K(\sigma,L) & \leq & t_K(\sigma,L')+1 & \sigma = 1,2,\cdots,k \\ t_K(\sigma,L) & = & t_K(\sigma,L') & \sigma \neq 1,2,\cdots,k. \end{array}$$

Therefore

$$\begin{array}{rcl} \sum_{\sigma\in\mathcal{L}} t_K(\sigma,L) & \leq & \sum_{\sigma\in\mathcal{L}'} t_K(\sigma,L') + m_{e,K} \\ & < & |E(K)| + m_{e,K}. \end{array}$$

Thus the lemma holds. □

Theorem 5. *Let G be a graph whose set of cycles consists of some triangles such that every triangle has exactly two vertices of degree 2 and one vertex of degree 3. (See Figure 18). Let L be a list assignment of colors to the edges of G such that $|L(e)| \geq 2$ and (G,L) satisfies Hall's condition. Then G has an L- coloring.*

Proof. Assume G is connected. We prove this theorem by induction on the number of triangles. If G has no triangle, then G is a tree and by Lemma B the theorem holds. Let G be such a graph with $k+1$ triangles. Suppose H_* is a triangle of G with vertices x, y, and z, where $\deg_G x = 3$. Let $e = yz$, $e_1 = xz$, and $e_2 = xy$. By induction, every branch of G with root x has an L–coloring. Let f be the edge other than e_1 and e_2 incident to x and G_1 be the subgraph induced by e, e_1, e_2 and f and let L_1 be a list assignment as follows. $L_1(f)$ be the set of colors which appear in various L–colorings of the branch which begins with f and $L_1(e_i) = L(e_i)$, for $i = 1,2$ and $L_1(e) = L(e)$. Now, we show that (G_1,L_1) satisfies Hall's condition. If not then $L_1(f) \neq L(f)$ and G_1 has a subgraph H such that $f \in E(H)$ and $\sum_{\sigma\in\mathcal{L}} t_H(\sigma,L_1) < |E(H)|$. By Lemma 2, there is a subgraph H_1 of the corresponding branch which contains f such that $\sum_{\sigma\in\mathcal{L}} t_{H_1}(\sigma,L) \leq |E(H_1)| + m_{f,H_1} - 1$. Consider the subgraph K of G induced by the edges of H and the edges of the H_1. Now, we have:

$$\begin{array}{rcl} \sum_{\sigma\in\mathcal{L}} t_K(\sigma,L) & \leq & \sum_{\sigma\in\mathcal{L}} t_H(\sigma,L_1) + \sum_{\sigma\in\mathcal{L}} t_{H_1}(\sigma,L) - |m_{f,H_1}| \\ & < & |E(H)| + (|E(H_1)| - 1) \\ & = & |E(K)|. \end{array}$$

Then K does not satisfy $(*)$, a contradiction. Therefore (G_1,L_1) satisfies Hall's condition and by Theorem 4 it has an L_1– coloring and consequently, G has an L–coloring as desired. □

Now by the previous theorems and propositions the main theorem, stated in the introduction, holds.

Hilton and Johnson conjectured [6] that every graph has Hall index 3, but in [2] the authors prove that from every graph which has Hall index greater than 2, one can construct a graph with Hall index greater than k, for each $k \geq 2$.

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