ON CRITICALLY k-EDGE-CONNECTED GRAPHS

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Abstract: Let G be a simple graph on n vertices having edge-connectivity $\kappa'(G) > 0$ and minimum degree $\delta(G)$. We say G is **k-critical** if $\kappa'(G) = k$ and $\kappa'(G - e) < k$ for every edge e of G. In this paper we prove that a k-critical graph has $\kappa'(G) = \delta(G)$. We describe a number of classes of k-critical graphs and consider the problem of determining the edge-maximal ones.

1. INTRODUCTION

For our purposes graphs are undirected, finite, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [2]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ Vertices and $\varepsilon(G)$ edges. However, we denote the complement of G by $\overline{G}$, K_n denotes the complete graph on n vertices, $K_{n,m}$ the complete bipartite graph with bipartioning sets of order n and m, and C_ℓ a cycle of length ℓ. The **join** of disjoint graphs G and H, denoted $G \vee H$, is the graph obtained by joining each vertex of G to each vertex of H.

A good deal of graph theory is concerned with the characterization of graphs having certain specified properties. Graph parameters of particular practical interest include: minimum degree $\delta(G)$, connectivity $\kappa(G)$, edge-connectivity $\kappa'(G)$, diameter $d(G)$, chromatic number $\chi(G)$, and various covering numbers (vertex, edge, clique, etc.). In studying such parameters it is often useful to restrict attention to the so called "critical graphs".

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Let P be a graph parameter. A graph G is said to be **P-edge (vertex)-critical** if $P(G - e) \neq P(G)$ $(P(G - v) \neq P(G))$ for every edge e (vertex v) of G. For a given P, the problem that arises is that of characterizing the class of P-edge-critical and class of P-vertex-critical graphs. In particular those that are edge-minimal or edge-maximal. This problem has been investigated for the edge case when P is: connectivity (Halin [9]); diameter (Caccetta and Haggkvist [3], Fan [8]); chromatic index (Yap [16]); and the vertex covering number (Lovasz and Plummer [14]), and for the vertex case when P is: connectivity (Chartrand [5], Entringer [7], Hamidoune [11], Krol and Veldman [13]); edge-connectivity (Cozzens and Wu [6]). The anologous problem for "edge addition" has been considered for diameter (Caccetta and Smyth [4], Ore [15]).

The object of this paper is to study graphs that are edge-critical with respect to the parameter κ'. For simplicity we say a graph G is **k-critical** if $\kappa'(G) = k$ and $\kappa'(G - e) < k$ for every edge e of G. Observe that: K_n is (n-1)-critical; $K_{n,m}$ is t-critical, where $t = \min\{m,n\}$; every tree is 1-critical; C_n is 2-critical; and $K_1 \vee C_n$ is 3-critical. We prove that a k-critical graph G has $k = \delta(G)$. This is anologous to the corresponding result of Halin [10] for edge-critical graphs with respect to κ. In addition, we shall consider the problem of determining the maximum number of edges in a k-critical graph.

2. RESULTS

Let $\mathscr{C}(n,k)$ denote the class of k-critical graphs on n vertices. We begin our discussion with some constructions.

It is very well known that for any graph G $\kappa'(G) \leq \delta(G)$. Further, given any positive integers a and b with $a \leq b$ there exists a graph G on $n \geq b + 1$ vertices such that $\kappa'(G) = a$ and $\delta(G) = b$. A class of graphs corresponding the case a = b is sometimes referred to as the Harary graphs and are described in standard texts (p.48, [2]). Let H(n,r) denote the class of graphs on n vertices having minimum degree and edge-connectivity r and having $\left\lceil \frac{1}{2}nr \right\rceil$ edges. Observe that for $k \geq 2$, $H(n,k) \subseteq \mathscr{C}(n,k)$. In fact, this class is edge-minimal. The following is an immediate consequence of the definition of criticality.

Lemma 1. Let G be a graph with $\kappa'(G) = \delta(G) = k$ and every edge of G is incident to at least one vertex of minimum degree. Then G is k-critical. □

Thus we have one class of critical graphs. Let $\mathscr{A}(n,k)$ denote the subclass of $\mathscr{C}(n,k)$ consisting of those graphs in which every edge is incident to a vertex of minimum degree. Clearly $K_{k,n-k} \in \mathscr{A}(n,k)$ for $n \geq 2k$. Later we shall show that for $n \geq 3k$, $K_{k,n-k}$ is an edge maximal graph of $\mathscr{A}(n,k)$. We now construct a class of graphs in $\mathscr{A}(n,k)$.

Let $H \in H(n - x, k - x)$ for $1 \leq x \leq n - k$, and define $G = H \vee \overline{K}_x$. If $n - x$ and $k - x$ are both odd and $x \neq n - k$, then G contains an edge $e = uv$ with $u \in H$ and $v \in \overline{K}_x$ such that $G - e$ is k-edge connected; in fact $G - e$ is k-critical. Thus if we let $G' = G - e$ if both $n - x$ and $k - x$ are odd and $G' = G$ otherwise, then $G' \in \mathscr{A}(n,k)$ and has $(n - x)x + \left\lfloor \frac{1}{2}(n - x)(k - x) \right\rfloor$ edges. Figure 1 below illustrates this construction. Note that in our illustration the "=" means all edges

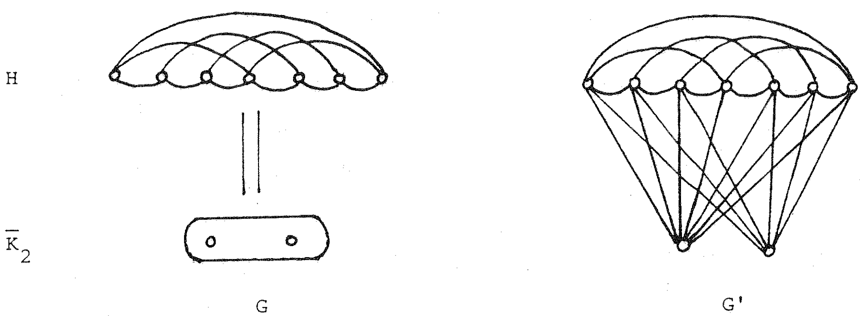
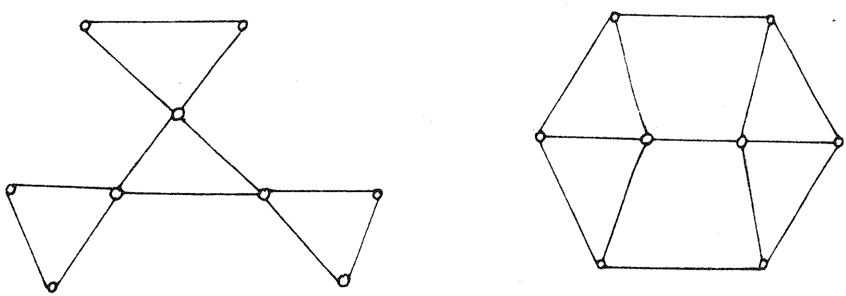


Figure 1

between the vertices of H and the vertices of $\overline{K}_2$. We use this notation in all our diagrams. In Theorem 2 we show that G′ is edge-maximal for $n < 3k$.

The graphs drawn in Figure 2 below show that $\mathcal{A}(n,k) \neq \mathcal{C}(n,k)$. In fact, it is easy to construct graphs in the class $\mathcal{C}(n,k)\backslash\mathcal{A}(n,k)$. One construction is the following. Let $n = 2kt + r$, $0 \le r \le 2k - 1$. The graph G obtained by adding edges to the graph $(2t - 1)\overline{K}_k \cup \overline{K}_{k+r}$ as shown in Figure 3 is in the class $\mathcal{C}(n,k)\backslash\mathcal{A}(n,k)$ for $t \ge 2$. Note that a line joining two graphs means a "perfect matching" between the two graphs. We adopt this convention in all our diagrams.



$G_1 \in \mathcal{C}(9,2)\backslash\mathcal{A}(9,2)$ $G_2 \in \mathcal{C}(8,3)\backslash\mathcal{A}(8,3)$

Figure 2.

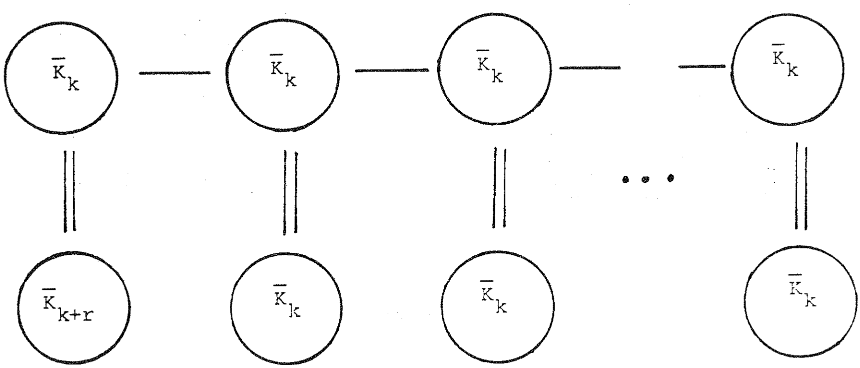


Figure 3.

Let G be a graph with a cut vertex v. That is $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{v\}$. Then $\kappa'(G) = \min\{\kappa'(G_1), \kappa'(G_2)\}$. We thus have the following simple but useful property.

Lemma 2. Let $G_i \in \mathscr{C}(n_i, k)$, $1 \le i \le t$. Then the graph G whose blocks are $G_1, G_2, \ldots, G_t$ is in the class $\mathscr{C}(n_1 + n_2 + \ldots + n_t - t + 1, k)$.

□

This lemma provides a procedure for building larger critical graphs from smaller ones.

Let $\rho(u,v)$ denote the maximum number of edge-disjoint paths between vertices u and v in G. Menger's theorem states that

$$\kappa'(G) = \min_{u,v \in V(G)} \{\rho(u,v)\}.$$

We make use of this result in our next lemma.

Lemma 3. Let G be a k-edge-connected graph. Then $G \in \mathscr{C}(n,k)$ if and only if $\rho(u,v) = k$ for every pair of adjacent vertices u, v in G.

Proof: Suppose $G \in \mathscr{C}(n,k)$ and let $e = uv$ be an edge of G. Consider the graph $G' = G - e$. We have $\kappa'(G') = k - 1$. Let E' be an edge-cut set of G' having $k - 1$ elements. Since $\kappa(G) = k$, the graph $G'' = G' - E'$ consists of exactly two components. Further, $G'' + e$ is connected. Hence the vertices u and v are in different components of G''. Consequently the set $E = E' \cup \{e\}$ is an edge-cut set of G having $k = \kappa(G)$ elements. Thus $\rho(u,v) \le k$. Menger's theorem now implies that $\rho(u,v) = k$ as required.

Conversely, if $\rho(u,v) = k$ for every pair of adjacent vertices u, v in G, then $\kappa'(G - uv) \le k - 1$ and hence $\kappa'(G) \le k$. Now since $\kappa(G) \ge k$ we have $G \in \mathscr{C}(n,k)$. This completes the proof of the lemma. □

Thus we can test whether or not a graph G is k-critical using standard network flow algorithms.

We now prove the main result of this paper.

Theorem 1. If G is a k-critical graph, then $\delta(G) = k$.

Proof: Let $G \in \mathscr{C}(n,k)$. Then $n \geq k + 1$. If $n = k + 1$, then $G = K_{k+1}$ and hence $\delta(G) = k$. So we suppose that $n \geq k + 2$. We prove the theorem by contradiction. Assume $\delta(G) > k$.

Let $\mathscr{E}(k)$ denote the set of edge-cut sets of G having k elements. If $E' \in \mathscr{E}(k)$, then $G - E'$ consists of two components. Let E^* denote an element of $\mathscr{E}(k)$ such that $G - E^*$ has the smallest possible component. Let G_1 and G_2 denote the components of $G - E^*$ and suppose, without loss of generality, that $n_1 = |V(G_1)| \leq n_2 = |V(G_2)|$.

Let A_i denote the set of vertices of G_i, $i = 1,2$, that are incident to an edge of E^*. We prove the theorem by showing that $n_1 = 1$. Suppose that $n_1 \geq 2$. We will show that $n_1 \geq k + 2$. This is certainly the case if $G_1 - A_1 \neq \phi$ as we have assumed that $\delta \geq k + 1$. So suppose that every vertex of G_1 is in A_1. We have

$$\begin{aligned}\sum_{u \in V(G_1)} d_{G_1}(u) &= \sum_{u \in V(G_1)} d_G(u) - k \\ &\geq n_1 (k + 1) - k \\ &= k(n_1 - 1) + n_1, \quad (n_1 \leq k) \\ &\geq n_1(n_1 - 1) + n_1 > n_1(n_1 - 1),\end{aligned}$$

a contradiction. Thus $n_1 \geq k + 2$. Since $|A_1| \leq k$, we must have at least two vertices of G_1 not in A_1. Hence there exists an edge $e = xy$ in G_1 with $x,y \notin A_1$. Since G is k-critical, $\kappa'(G - e) = k - 1$. Now since $n_2 \geq n_1$, G_2 contains vertices which are not in A_2. Let z be one such vertex. Clearly z is joined to the vertices of A_1 by k-edge disjoint paths.

Since $\kappa'(G) = k$ the vertices x and y must each be joined to the vertices of A_1 by at least k-edge disjoint paths. In fact, the choice of E^* ensures that there are at least $k + 1$ such paths. This contradicts Lemma 3. Hence $n_1 = 1$. This completes the proof of our theorem.

□

We mentioned earlier in this section that the Harary graphs were edge-minimal members of $\mathscr{C}(n,k)$. The problem of determining the edge-maximal members of $\mathscr{C}(n,k)$ seems to be difficult. Our next result determines the maximum number of edges for a graph $G \in \mathscr{A}(n,k)$.

Theorem 2. Let G be an edge-maximal graph of $\mathscr{A}(n,k)$. Then

$$\varepsilon(G) = \begin{cases} k(n - k), & \text{if } n \ge 3k \\ \left\lfloor \frac{1}{8}(n + k)^2 \right\rfloor, & \text{otherwise .} \end{cases}$$

Proof: By Theorem 1 $\delta(G) = k$. We denote the set of vertices of G having degree k by X and the remaining vertices by $\overline{X}$. Let $n_1 = |X|$. Since $G \in \mathscr{A}(n,k)$, we must have $n_1 \ge k + 1$. Simple counting gives:

$$\varepsilon(G) \le \begin{cases} n_1 k & , \quad \text{if } n_1 \le n - k \\ n_1(n - n_1) + \left\lfloor \frac{1}{2} n_1(k - n + n_1) \right\rfloor, & \text{otherwise .} \end{cases} \qquad (1)$$

Let $g(n_1)$ denote the right hand side of (1).

Clearly

$$\max_{n_1 \le n - k} \{g(n_1)\} = g(n - k) = k(n - k) .$$

This maximum is attained by the graph $K_{k,n-k}$. For $n_1 \ge n - k$, we have for fixed n and k

$$g(n_1 + 1) - g(n_1) = \left\lfloor \frac{1}{2}(n + k - 1) \right\rfloor - n_1 + \delta(n_1).\delta(n - k - 1), \qquad (2)$$

where $\delta(x) = 0$ or 1 according to whether x is even or odd. There is some algebra involved in establishing (2), but it is fairly elementary.

From (2) we deduce that $g(n_1)$ monotonically increases in n_1 for $n_1 \le \left\lfloor \frac{1}{2}(n + k - 1) \right\rfloor$ and monotonically decreases in $n_1 \ge \left\lfloor \frac{1}{2}(n + k + 1) \right\rfloor$. Now since $n_1 \ge n - k$, $g(n_1)$ is decreasing in n_1 for $n \ge 3k$. Hence

$$\max_{n \ge 3k} \{g(n_1)\} \le g(n - k) = k(n - k) .$$

For $n < 3k$, $g(n_1)$ attains its maximum value at $n_1 = \left\lfloor \frac{1}{2}(n + k + 1) \right\rfloor$. It is a straight forward algebraic exercise to verify that $g(\left\lfloor \frac{1}{2}(n + k + 1) \right\rfloor) = \left\lfloor \frac{1}{8}(n + k)^2 \right\rfloor$. An example of a graph in $\mathcal{A}(n,k)$ having this number of edges is the graph G′ (described following Lemma 1) with $x = n - \left\lfloor \frac{1}{2}(n + k + 1) \right\rfloor$.

Now

$$(n + k)^2 - 8k(n - k) = (n - 3k)^2 \ge 0.$$

Hence

$$\left\lfloor \frac{1}{8}(n + k)^2 \right\rfloor \ge k(n - k)$$

always. This completes the proof of the theorem.

□

It would be interesting to determine whether or not the edge-maximal graphs of $\mathcal{C}(n,k)$ coincide with the edge-maximal graphs of $\mathcal{A}(n,k)$. Krol and Veldman [13] have conjectured that for κ-vertex-critical graphs the analogous question is true for $\kappa \ge 3$.

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