Two classes of extreme graphs related to the maximum degree of an interchange graph

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Abstract

Let $\mathcal{U}(R,S)$ denote the class of all $m \times n$ matrices of 0's and 1's having row sum vector R and column sum vector S. The *interchange graph* $G(R,S)$ is the graph where the vertices are the matrices in $\mathcal{U}(R,S)$ and two vertices are adjacent provided they differ by an interchange. Two tight upper-bounds of the maximum degree $\Delta(G(R,S))$ are given. Furthermore, those extreme graphs whose maximum degrees reach the upper-bounds are determined.

1 INTRODUCTION

All graphs discussed in this paper are simple, undirected finite graphs. For notation and terminology not defined in this paper see [15].

Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. The *degree* of a vertex x of G, denoted by $d(x)$, is the number of vertices which are adjacent (are joined by an edge) to x. The *maximum degree* of G, denoted by $\Delta(G)$, is the maximum degree of vertices of G, i.e. $\Delta(G) = \max\{d(x) : x \in V(G)\}$.

Let $R = (r_1, r_2, \cdots, r_m)$, $S = (s_1, s_2, \cdots, s_n)$ be two (positive) integral vectors with $r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n = N$. Denote by $\mathcal{U}(R,S)$ [2,16] the class of all $m \times n$ (0,1)-matrices $x = (x_{ij})_{m\times n}$ having row sum vector R and column sum vector S: $x_{ij} = 0$ or 1 $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$, $\sum_{j=1}^{n} a_{ij} = r_i$ $(i = 1, 2, \cdots, m)$ and $\sum_{i=1}^{m} x_{ij}$ $= s_j$ $(j = 1, 2, \cdots, n)$. An *interchange* of a matrix x of $\mathcal{U}(R,S)$ is a transformation which replaces a 2×2 submatrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ of x with the 2×2 submatrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or vice versa. The *interchange graph* $G(R,S)$ is defined as follows [2]: the vertices are the matrices in $\mathcal{U}(R,S)$ where two matrices x and y are adjacent iff x can be obtained from y by one interchange (also y can be obtained from x by one interchange). Clearly, for $x \in V(G(R,S))$, the number of different interchanges of x equals its degree: $d(x)$.

Many results about the number of vertices, connectivity, diameter, transitivity and hamiltonicity etc. on interchange graphs have been obtained [1-14].

In this paper, we study the maximum degree of $G(R,S)$. Two tight upper-bounds of $\Delta(G(R,S))$ are obtained:

(1) $\Delta(G(R,S)) \leq \binom{N}{2} - \sum_{i=1}^{m}\binom{r_i}{2} - \sum_{j=1}^{n}\binom{s_j}{2}$;

(2) $\Delta(G(R,S)) \leq \binom{N}{2} - \sum_{i=1}^{m}\binom{r_i}{2} - \sum_{j=1}^{n}\binom{s_j}{2} - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=n-r_i+1}^{n}(r_i-1)(s_j-1)$,
$s_1 \geq s_2 \geq \cdots \geq s_n$;

where $\binom{m}{n}$ is the binomial coefficient (with $\binom{m}{n} = 0$ if $m < n$). All extreme graphs which reach these two bounds are also determined.

2 MAIN RESULTS

By the definition of $G(R,S)$, it does not affect the isomorphism type of $G(R,S)$ to rearrange rows and rearrange columns. We now make the assumption that $r_1 \geq r_2 \geq \cdots \geq r_m$ and $s_1 \geq s_2 \geq \cdots \geq s_n$ throughout the following.

Let $x = (x_{ij})_{m\times n} \in V(G(R,S))$. By the assumption: $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j = N$, the total number of 1's in x is N, i.e. $|\{x_{ij} : x_{ij} = 1\}| = N$. Because every interchange of x involves just two 1's (also two 0's) and any pair of 1's which are in the same row or the same column of x does not form an interchange, we have an upper-bound of $\Delta(G(R.S))$ immediately:

$$\Delta(G(R,S)) \leq \binom{N}{2} - \sum_{i=1}^{m}\binom{r_i}{2} - \sum_{j=1}^{n}\binom{s_j}{2}. \tag{1}$$

Denote $\binom{N}{2} - \sum_{i=1}^{m}\binom{r_i}{2} - \sum_{j=1}^{n}\binom{s_j}{2}$ by $\phi(R,S)$. The following result gives us a characterization of the extreme graphs which reach the bound.

Theorem 1. Let $\mathcal{R} = \{i : r_i > 1\}$, $\mathcal{S} = \{j : s_j > 1\}$, $k = |\mathcal{R}|$ and $h = |\mathcal{S}|$. We have

$$\Delta(G(R,S)) = \phi(R,S) \tag{2}$$

if and only if $n - \sum_{i\in\mathcal{R}} r_i \geq h$ and $m - \sum_{j\in\mathcal{S}} s_j \geq k$.

Proof. Note that $r_1 \geq r_2 \geq \cdots \geq r_m$ and $s_1 \geq s_2 \geq \cdots \geq s_n$, so $\mathcal{R} = \{1, 2, \cdots, k\}$ and $\mathcal{S} = \{1, 2, \cdots, h\}$.

Firstly, assume that $n - \sum_{i=1}^{k} r_i \geq h$ and $m - \sum_{j=1}^{h} s_j \geq k$.

Let $\alpha = \sum_{i=1}^{k} r_i$, $\beta = \sum_{j=1}^{h} s_j$, $\gamma = n-\alpha-h$, $\gamma' = m-\beta-k$. Due to $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j = N$, so $N - \alpha = m - k$, $N - \beta = n - h$. Hence $\gamma = n - \alpha - h = m - k - \beta = \gamma'$. Since

$n-\alpha \geq h$, $m-\beta \geq k$, we have $\gamma = \gamma' \geq 0$. Let

$$x_0 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & A \\ \mathbf{0} & I_\gamma & \mathbf{0} \\ B & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

where

$$A = \begin{pmatrix} \overbrace{1\cdots 1}^{r_1} & & & \\ & \overbrace{1\cdots 1}^{r_2} & & \\ & & \ddots & \\ & & & \overbrace{1\cdots 1}^{r_k} \end{pmatrix}_{k\times\alpha} \qquad B = \begin{pmatrix} \left.\begin{matrix}1\\ \vdots \\ 1\end{matrix}\right\} s_1 & & & \\ & \left.\begin{matrix}1\\ \vdots \\ 1\end{matrix}\right\} s_2 & & \\ & & \ddots & \\ & & & \left.\begin{matrix}1\\ \vdots \\ 1\end{matrix}\right\} s_h \end{pmatrix}_{\beta\times h}$$

and I_γ is the $\gamma \times \gamma$ identity matrix, $\mathbf{0}$ is the matrix of all 0's.

It is easy to see that $x_0 \in V(G(R,S))$, and its any pair of 1's which are in different rows and different columns forms an interchange of x_0. So $d(x_0) = \phi(R,S)$.

Conversely, assume that $n - \sum_{i=1}^{k} r_i < h$ or $m - \sum_{j=1}^{h} s_j < k$.

In this case, for any $x = (x_{ij})_{m\times n} \in V(G(R,S))$, the submatrix of x which lies in rows $1,2,\cdots,k$ and columns $1,2,\cdots,h$ contains at least one 1, say x_{pq}. Since $r_i > 1$, $i \in \{1,2,\cdots,k\}$ and $s_j > 1$, $j \in \{1,2,\cdots,h\}$, then $r_p, s_q > 1$. So there are $p' \in \{1,2,\cdots,m\}$, $q' \in \{1,2,\cdots,n\}$, $p' \neq p$, $q' \neq q$ such that $x_{pq'} = x_{p'q} = 1$. Clearly, $x_{pq'}$ and $x_{p'q}$ are in different rows and different columns but do not form an interchange of x. So we have $d(x) < \phi(R,S)$. The theorem follows.

Let x be a matrix. A 1-*type* submatrix, or 1-T for short, of x is one of the 2×2 submatrices: $\begin{pmatrix} * & 1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & * \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ * & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & * \end{pmatrix}$, where $*$ equals 0 or 1. Let T be a 1-T of x. The pair consisting of two 1's which lie in different rows and different columns of T is called the *acute*-1 of T, while the other 1 of T is called its *right*-1. If T lies in rows i,j and columns s,t while its right-1 is at the (i,s)-position, then we denote it by $T_{1x}(i,s:j,t)$. Two 1-T's, say $T_{1x}(i,s:j,t)$ and $T_{1x}(i',s':j',t')$, are considered to be the same iff $i=i', j=j', s=s'$ and $t=t'$. A 2-*type* submatrix of x, or 2-T for short, is a 2×2 submatrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ of x. The 2-T of x which lies in rows i,j and columns s,t is denoted by $T_{2x}(i,s:j,t)$. Similarly,two 2-T's, say $T_{2x}(i,s:j,t)$ and $T_{2x}(i',s':j',t')$, are said to be the same iff $\{i,j\} = \{i',j'\}$ and $\{s,t\} = \{s',t'\}$. The total number of 1-T's and 2-T's of x are denoted by $t_1(x)$ and $t_2(x)$ respectively.

Example. Let

$$x = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

We have $t_1(x) = 8$ and $t_2(x) = 1$. Its eight 1-T's are $T_{1x}(1,1:3,3); T_{1x}(1,3:2,1)$; $T_{1x}(1,3:3,1)$; $T_{1x}(2,3:1,2)$; $T_{1x}(2,3:3,2)$; $T_{1x}(3,1:1,3)$; $T_{1x}(3,3:1,1)$ and $T_{1x}(3,3:2,1)$. Its unique 2-T is $T_{2x}(1,1:3,3)$.

Lemma 1. Let $x = (x_{ij})_{m\times n} \in V(G(R,S))$. We have
(1) $t_1(x) = \sum_{x_{ij}=1} (r_i - 1)(s_j - 1)$
(2) $t_1(x) \geq 4t_2(x)$
(3) $d(x) = \phi(R,S) - t_1(x) + 2t_2(x)$.

Proof. (1) If $x_{ij} = 1$, then the number of different 1-T's with x_{ij} being its right-1 is $(r_i - 1)(s_j - 1)$. Since every 1-T of x has just one right-1, and any two 1-T's with different right-1 are different, then (1) follows.
(2) Because every 2-T contains four 1-T's, (2) is immediate.
(3) We know that every interchange of x involves just two 1's (also two 0's) of x, while the number of ways of choosing two 1's from x is $\binom{N}{2}$. Thus, to determine the number of interchanges of x, we need only to calculate the number of such pairs of 1's which can not form interchanges.

Two 1's of x which cannot form an interchange must come from one of the following three cases.
Case 1. They lie in the same row of x.

In this case, the number of pairs is $\sum_{i=1}^{m} \binom{r_i}{2}$.
Case 2. They lie in the same column of x.

The number of pairs is $\sum_{j=1}^{n} \binom{s_j}{2}$.
Case 3. The pair consisting of these two 1's is the acute-1 of a 1-T.

In this case, we prove that the number of pairs is $t_1(x) - 2t_2(x)$.

In fact, assume that these two 1's are at the (i,s)-position and (j,t)-position respectively, $i \neq j, s \neq t$. Clearly, the pair consisting of these two 1's is the acute-1 of at most two 1-T's. If it is the acute-1 of exactly one 1-T, then it contributes to $t_1(x)$ just once. If it is the acute-1 of two 1-T's, then it contributes to $t_1(x)$ twice and the entries at the (i,s)-,(j,s)-,(i,t)- and (j,t)-positions are all 1's. That is, these four 1's form a 2-T. Hence, the pair consisting of the other two 1's (i.e. at the (i,t)- and (j,s)-positions) is also the acute-1 of two 1-T's. So the pair contributes to $t_1(x)$ twice too. Therefore, the number of pairs in this case is no less than $t_1(x) - 2t_2(x)$.

Conversely, every 2-T contains just four 1-T's and two pairs of 1's which lie in different rows and different columns. Among these, each pair is the acute-1 of two 1-T's. Hence, the number of pairs is no more than $t_1(x) - 2t_2(x)$.

From Case 1, Case 2 and Case 3, (3) is immediate. The proof of the lemma is completed.

By Lemma 1, for any $x \in V(G(R,S))$, we have

$$d(x) \leq \phi(R,S) - \frac{1}{2}t_1(x). \tag{3}$$

The equality holds iff $t_1(x) = 4t_2(x)$. On the other hand, $t_1(x) = 4t_2(x)$ iff every 1-T of x is contained in a 2-T. Thus, (3) holds iff every 1-T of x is contained in a 2-T.

Theorem 2.

$$\Delta(G(R,S)) \leq \phi(R,S) - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=n-r_i+1}^{n}(r_i-1)(s_j-1). \tag{4}$$

Equality holds if and only if $s_1 = s_2 = \cdots = s_n = s$, and $r_1 = \cdots = r_s \geq r_{s+1} = \cdots = r_{2s} \geq \cdots \geq r_{(k-1)s+1} = \cdots = r_{ks}$.

Proof. Since $s_1 \geq s_2 \geq \cdots \geq s_n$, then

$$\sum_{i=1}^{m}\sum_{j=n-r_i+1}^{n}(r_i-1)(s_j-1) \leq \sum_{x_{ij}=1}(r_i-1)(s_j-1) = t_1(x) \tag{5}$$

for any $x \in V(G(R,S))$. By (3) and Lemma 1.1, (4) is immediate.

Now assume that

$$\Delta(G(R,S)) = \phi(R,S) - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=n-r_i+1}^{n}(r_i-1)(s_j-1). \tag{6}$$

Let $x_0 = (x_{ij})_{m\times n}$ be a vertex of $G(R,S)$ with

$$d(x_0) = \phi(R,S) - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=n-r_i+1}^{n}(r_i-1)(s_j-1). \tag{7}$$

We first prove that $s_1 = s_2 = \cdots = s_n = s$.

By (3), (5) and (7), we have $\phi(R,S) - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=n-r_i+1}^{n}(r_i-1)(s_j-1) = d(x_0) \leq \phi(R,S) - \frac{1}{2}\sum_{x_{ij}=1}(r_i-1)(s_j-1) \leq \phi(G,S) - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=n-r_i+1}^{n}(r_i-1)(s_j-1)$. So $\sum_{i=1}^{m}\sum_{j=n-r_i+1}^{n}(r_i-1)(s_j-1) = \sum_{x_{ij}}(r_i-1)(s_j-1)$. If $s_1 > s_n$, then there exists $p \in \{1,2,\cdots,m\}$ such that the entry at the (p,n)-position is 0 while the entry at the $(p,1)$-position is 1. Since $s_1 \geq s_2 \geq \cdots \geq s_n$, we have

$$\begin{aligned}
&\sum_{x_{ij}=1}(r_i-1)(s_j-1)\\
&= \Big(\sum_{x_{ij}=1,(i,j)\neq(p,1)}(r_i-1)(s_j-1) + (r_p-1)(s_n-1)\Big) + (r_p-1)(s_1-1) - (r_p-1)(s_n-1)
\end{aligned}$$

$$\geq \sum_{i=1}^{m} \sum_{j=n-r_i+1}^{n} (r_i - 1)(s_j - 1) + (r_p - 1)(s_1 - 1) - (r_p - 1)(s_n - 1)$$
$$> \sum_{i=1}^{m} \sum_{j=n-r_i+1}^{n} (r_i - 1)(s_j - 1).$$

This is a contradiction. Hence $s_1 = s_2 = \cdots = s_n$.

We know that $\sum_{i=1}^{m} \sum_{j=n-r_i+1}^{n} (r_i - 1)(s_j - 1) = \sum_{x_{ij}=1} (r_i - 1)(s_j - 1) = t_1(x_0)$, i.e. $d(x_0) = \phi(R, S) - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=n-r_i+1}^{n} (r_i - 1)(s_j - 1) = \phi(G, S) - \frac{1}{2} t_1(x_0)$. So every 1-$T$ must be contained in a 2-T of x_0. Thus, there is no 2×2 submatrix in x_0 with the form: $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Suppose that all the 1's which lie in row 1 (the 1^{st} row) are at the positions: $(1, j_1), (1, j_2), \cdots, (1, j_{r_1})$ respectively, and all the 1's which lie in column j_1 are at the positions: $(1, j_1), (i_2, j_1), \cdots, (i_s, j_1)$ respectively. Since every 1-T of x_0 is contained in a 2-T, then the entry at the (i, j)-position, $i \in \{1, i_2, \cdots, i_s\}$, is 1 iff $j \in \{j_1, j_2, \cdots, j_{r_1}\}$. Similarly, the entry at the (s, t)-position, $t \in \{j_1, j_2, \cdots, j_{r_1}\}$, is 1 iff $s \in \{1, i_2, \cdots, i_s\}$. This shows that the submatrix of x_0 which lies in rows $1, i_2, \cdots, i_s$, and columns $j_1, j_2, \cdots, j_{r_1}$ is an $s \times r_1$ matrix of all 1's. Furthermore, this submatrix contains all the 1's of x_0 which lie in rows $1, i_2, \cdots, i_s$ and all the 1's which lie in columns $j_1, j_2, \cdots, j_{r_1}$. Denote this submatrix by x_0'.

Delete rows $1, i_2, \cdots, i_s$ and columns $j_1, j_2, \cdots, j_{r_1}$ from x_0. The remaining submatrix, denoted by x_1, also has the property: every 1-T of x_1 is contained in a 2-T, and each of its column sums is s. We can obtain a submatrix x_1' from x_1 in the same way as that from x_0. For the same reason, x_1' is a matrix of all 1's with s rows.

In this way, we obtain a sequence of submatrices: $x_0', x_1', \cdots, x_{k-1}'$. Each of them is a matrix of all 1's with s rows. Clearly, every 1 of x_0 is contained and only contained in one of them. By $r_1 \geq r_2 \geq \cdots \geq r_m$ and the property of $x_0', x_1', \cdots, x_{k-1}'$, we have the following immediately: $r_1 = \cdots = r_s \geq r_{s+1} = \cdots = r_{2s} \geq \cdots \geq r_{(k-1)s} = \cdots = r_{ks}$.

Conversely, assume $s_1 = s_2 = \cdots = s_n = s$, and $r_1 = \cdots = r_s \geq r_{s+1} = \cdots = r_{2s} \geq \cdots \geq r_{(k-1)s} = \cdots = r_{ks}$. Let

$$x_0 = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}$$

where A_i is the $s \times r_{is}$ matrix of all 1's, $i = 1, 2, \cdots, k$, and all other entries are 0. It is easy to check that $x_0 \in V(G(R, S))$ and

$$d(x_0) = \phi(R, S) - t_1(x_0) + 2t_2(x_0) = \phi(R, S) - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=n-r_i+1}^{n} (r_i - 1)(s_j - 1).$$

This completes the proof of the theorem.

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