# Asymptotics of the total chromatic number for multigraphs*

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**Abstract**

For loopless multigraphs, the total chromatic number is asymptotically its fractional counterpart as the latter invariant tends to infinity. The proof of this is based on a recent theorem of Kahn establishing the analogous asymptotic behaviour of the list-chromatic index for multigraphs.

The total colouring conjecture, proposed independently by Behzad [1] and Vizing [11], asserts that the total chromatic number $\chi_t$ of a simple graph exceeds the maximum degree $\Delta$ by at most two. The most recent increment (better: giant leap) toward a proof of this conjecture was made by Molloy and Reed [8], who established by probabilistic means that the difference between $\chi_t$ and $\Delta$ is at most a constant (say $c$). An immediate consequence of their result is that for *simple* graphs, $\chi_t$ is asymptotically its fractional analogue $\chi_t^*$ as the latter tends to infinity: for this follows from $\Delta + 1 \leq \chi_t^* \leq \chi_t \leq \Delta + c$. This leads naturally to the following question: does $\chi_t$ enjoy the same asymptotic connection with $\chi_t^*$ for loopless multigraphs (henceforth *multigraphs*)? That this question has an affirmative answer was conjectured in [6].

The purpose of this note is to verify that conjecture:

**Theorem 1** *For multigraphs,*

$$\chi_t \sim \chi_t^* \quad \textit{as} \quad \chi_t^* \to \infty. \tag{1}$$

*This is the final version of this note, a revision of [7].

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That is, for each $\varepsilon > 0$ there exists $D = D(\varepsilon)$ such that every multigraph $G$ with $\chi_t^*(G) > D$ satisfies

$$(1+\varepsilon)^{-1} < \frac{\chi_t(G)}{\chi_t^*(G)} < 1+\varepsilon. \tag{2}$$

This adds $\chi_t$ to a growing list of (hyper)graph colouring invariants exhibiting "asymptotically good" behaviour, in the sense elucidated, e.g., in [3] or [6].

Pausing briefly to fix notation, we point the reader to [5, 6] for background and further motivation, and to [2] for omitted definitions. In addition to $\chi_t$, the colouring invariants that come into play here are the chromatic index $\chi'$ and the list-chromatic index $\chi'_\ell$. Regarding these as solutions to integer programming problems leads to their fractional variants $\chi_t^*$, $\chi'^*$, $\chi'^*_\ell$, namely the optimal values of the linear relaxations of the respective IP's (see [10] for omitted LP/IP terminology). We can (and will) restrict our attention to $\chi_t^*$ and $\chi'^*$ since $\chi'^* = \chi'^*_\ell$; see [9].

The key ingredient in the proof of Theorem 1 is the following result of Kahn [4]:

**Theorem 2** *For multigraphs,*

$$\chi'_\ell \sim \chi'^* \quad \textit{as} \quad \chi'^* \to \infty.$$

The convergence here is in the same sense as that in (1), but we again spell out the quantifiers for later reference: for each $\gamma > 0$ there exists $C = C(\gamma)$ such that every multigraph $G$ with $\chi'^*(G) > C$ satisfies $\chi'_\ell(G) < (1+\gamma)\chi'^*(G)$.

Our proof also employs the following elementary inequalities (in (4), $k$ is a positive constant and the multigraph needs to be non-empty):

$$\chi_t^* \leq \chi_t; \tag{3}$$
$$\chi_t^* \leq k\chi'^*; \tag{4}$$
$$\chi_t \leq \chi'_\ell + 2; \tag{5}$$
$$\chi'^* \leq \chi_t^*. \tag{6}$$

*Proof of* (3). The left side is the optimal value of the linear relaxation of the IP defining the right. ■

*Proof of* (4). Kostochka proved (see, e.g., [2, p. 86]) that $\chi_t \leq \lfloor 3\Delta/2 \rfloor$, but, for our needs, this is using a sledge for a finishing nail; greedy colouring yields $\chi_t \leq 2\Delta + 1$. Either of these bounds together with (3) and the obvious $\Delta \leq \chi'^*$ gives (4). ■

*Proof of* (5). See, e.g., [2, p. 87]. ■

*Proof of* (6). Straightforward; see [7]. ■

In light of (3), to complete the proof of Theorem 1 it remains only to establish the right-hand inequality in (2) for arbitrary $\varepsilon > 0$ and sufficiently large $\chi_t^*$. Given $\varepsilon > 0$, let $\gamma = \varepsilon/2$, and choose $C$ so large (according to Theorem 2) that

$$\chi'^* > C \quad \text{implies} \quad \chi'_\ell < (1+\gamma)\chi'^*. \tag{7}$$

Let $k$ be as in (4). If $\chi_t^* > D := \max\{kC, 4k/\varepsilon\}$, then, since $\chi'^* \geq \chi_t^*/k$ (by (4)), we see that $\chi'^*$ exceeds both $C$ and $4/\varepsilon = 2/\gamma$. Thus, provided $\chi_t^* > D$, we have

$$\chi_t \leq \chi_t' + 2 < (1+\gamma)\chi'^* + \gamma\chi'^* = (1+\varepsilon)\chi'^* \leq (1+\varepsilon)\chi_t^*$$

(justifying the inequalities, respectively, by: (5); the preceding sentence and (7); and (6)), as desired. ∎

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