A Sufficient Condition for a Semicomplete Multipartite Digraph to be Hamiltonian*

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Abstract

A digraph obtained by replacing each edge of a complete n-partite ($n \geq 2$) graph by an arc or a pair of mutually opposite arcs with the same end vertices is called a *semicomplete n-partite digraph* or *semicomplete multipartite digraph* (abbreviated to SMD). In this paper we show the following result for a semicomplete multipartite digraph of order p with the partite sets $V_1, V_2, \ldots, V_n$. Let $r = \min_{1 \leq i \leq n}\{|V_i|\}$. If for each pair of dominated nonadjacent vertices $\{x, y\}$, $d(x)+d(y) \geq \min\{2(p-r)+3, 2p-1\}$, then T is Hamiltonian. This result is best possible in a sense.

1. Introduction

For the convenience of the reader we provide all necessary terminology and notation in section 2.

There are some degree conditions that guarantee Hamiltonicity in strong digraphs of order p:

Theorem 1.1 ([4]) *If $d(x) \geq p$ for each vertex $x \in V(D)$, then D is Hamiltonian.*

Theorem 1.2 ([8]) *If $d^+(x) + d^-(y) \geq p$ for all pair of vertices x and y such that there is no arc from x to y, then D is Hamiltonian.*

Theorem 1.3 ([6]) *If $d(x) + d(y) \geq 2p - 1$ for each pair of nonadjacent vertices in D, then D is Hamiltonian.*

Theorem 1.4 ([1]) *If $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq p$ for every pair of dominating nonadjacent and dominated nonadjacent vertices $\{x, y\}$. Then D is Hamiltonian.*

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Furthermore, in [1], Bang-Jensen, Gutin and Li conjectured that if $d(x)+d(y) \geq 2p-1$ for every pair of dominating nonadjacent vertices $\{x, y\}$, then D is Hamiltonian. In this paper, we give a sufficient degree condition to guarantee Hamiltonicity in SMDs. This result implies that the above conjecture is valid for semicomplete multipartite tournaments. For surveys on SMDs, see [5] and [7].

2. Terminology and notation

We shall assume that the reader is familiar with the standard terminology on digraphs and refer to [2] for terminology not provided in this paper.

Let D denote a digraph of order p with vertex set V. D is *strict* if it has no loops and no two arcs with the same ends having the same orientation, and *strong* if, for any two vertices u and v, there is a directed path from u to v. If xy is an arc of D, then we say that x dominates y, denoted by $x \to y$. More generally, if A and B are two disjoint vertex set of D such that every vertex of A dominates every vertex of B, then we say that A dominates B, denoted by $A \Rightarrow B$. Let $x \in V(D)$, we define $d^+(x)$ ($d^-(x)$) to be the number of vertices dominated by (dominating) x, and $d(x) = d^+(x) + d^-(x)$. If there is $u \in V$ such that $u \Rightarrow \{x, y\}$, we call the pair $\{x, y\}$ *dominated*. If $v \in V$ and $S \subseteq V$, we denote the set of arcs between v and S by $E(v, S)$. An S-path is a directed path of length at least two having exactly its origin and terminus in common with S. An (x, y)-path is a directed path from x to y.

A digraph obtained by replacing each edge of a complete n-partite ($n \geq 2$) graph by an arc or a pair of mutually opposite arcs with the same end vertices is called a *semicomplete n-partite digraph* or *semicomplete multipartite digraph* (abbreviated to SMD). Let T be an SMD and $x \in V(T)$, we denote by $V(x)$ the partite set of T containing x.

3. Main result

The following lemma is known.

Lemma 3.1. ([3]) *Let $P = v_1v_2 \ldots v_k$ be a directed path in a strict digraph D, and let $v \in V(D) \backslash V(P)$. If D has no (v_1, v_k)-path with vertex set $V(P) \cup \{v\}$, then $|E(v, V(P))| \leq k+1$.*

Theorem 3.2. *Let T be a strong semicomplete n-partite digraph of order p with the partite sets $V_1, V_2, \ldots, V_n$. Let $r = \min_{1 \leq i \leq n}\{|V_i|\}$. If for each pair of dominated nonadjacent vertices $\{x, y\}$, $d(x) + d(y) \geq \min\{2(p-r)+3, 2p-1\}$, then T is Hamiltonian.*

Proof. Assume that T is non-Hamiltonian and $C = x_1x_2 \ldots x_mx_1$ is a longest cycle in T.

Suppose that there is no $V(C)$-path in T. Since T is strong and C is a longest cycle in T, T contains a directed cycle C' having precisely one vertex, say x_1, in common with C. Let v denote the successor of x_1 in C'. If T contains a path of the form $x_2 \to y \to v$ or $v \to y \to x_2$, where $y \in V(T)\backslash V(C)$, then obviously we get a contradiction to the assumption that T has no $V(C)$-path. So we can assume that no such path exists. If $m \geq 3$, then $E(\{x_2, x_3\}, v) \neq \emptyset$ since T is an SMD, which contradicts that T has no $V(C)$-path. Hence we have $|V(C)| = 2$. Note that $V(x_2) = V(v)$ and $x_1 \Rightarrow \{x_2, v\}$, thus, we have

$$d(v) + d(x_2) \leq 2 + 2 + 2(p - 2 - (r - 1)) = 2p - 2r + 2,$$

which also cotradicts the initial assumption.

Hence T contains a $V(C)$-path $P = x_\alpha y_1 y_2 \dots y_s x_{\alpha+\gamma}$. Let the path be chosen so that γ is minimum. Then it is easy to verify that y_1 is not adjacent to any vertex of $\{x_{\alpha+1}, x_{\alpha+2}, \dots, x_{\alpha+\gamma-1}\}$. Hence $\gamma = 2$ since T is an SMD. Thus, $s = 1$ since C is a longest cycle of T. Let $A = V(y_1) \cap V(C)$, $B = V(y_1) \cap (V(T)\backslash V(C))$, then $|A| + |B| \geq r$. Now, by the maximality property of C, T has no $(x_{\alpha+2}, x_\alpha)$-path with vertex set $V(C) \cup \{y_1\}\backslash\{x_{\alpha+1}\}$. Hence by Lemma 3.1, we get that:

(1) y_1 is adjacent to the path $x_{\alpha+2}x_{\alpha+3}\dots x_\alpha$ by at most $m - 1 - (|A| - 1) + 1 = m - |A| + 1$ edges.

Because of the minimality of γ, we get:

(2) T contains no path of the form $x_{\alpha+1} \to y \to y_1$, or $y_1 \to y \to x_{\alpha+1}$ with $y \in V(T)\backslash V(C)$.

Also, by the maximality of C, there are no $(x_{\alpha+2}, x_\alpha)$-paths with the vertex set $V(C)$. By Lemma 3.1, we have $|E(x_{\alpha+1}, V(C)\backslash\{x_{\alpha+1}\})| \leq m - |A| + 1$. Combining this with (1) and (2), we get:

$$d(x_{\alpha+1}) + d(y_1) \leq 2(m - |A| + 1) + \sum_{y \in V(T)\backslash V(C)} |E(y, \{y_1, x_{\alpha+1}\})|$$
$$\leq 2(m - |A| + 1) + 2(p - m - |B|) = 2p - 2(|A| + |B|) + 2 \leq 2p - 2r + 2.$$

This contradicts the initial assumption.

This completes the proof of the theorem. □

4. Remark

Let $t = \min\{2p - 2r + 3, 2p - 1\}$. Consider the following digraph D: $V(D) = \{v_1, v_2, v_3\}$, $A(D) = \{v_1 \to v_i | 2 \leq i \leq 3\} \cup \{v_i \to v_1 | 2 \leq i \leq 3\}$. This is a semicomplete bipartite digraph with $r = 1$, it satisfies the condiction that for any pair of dominated nonadjacent vertices $\{x, y\}$, $d(x) + d(y) \geq t - 1$, but obviously it is not Hamiltonian. So Theorem 3.2 is best possible in a sense.

References

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