Some Structural Results on Linear Arboricity

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Abstract

A linear forest-factor F of a graph G is a spanning subgraph of G whose components are paths. A linear forest-decomposition of G is a collection $\mathcal{F} = \{F_1, \ldots, F_k\}$ of linear forest-factors of G such that the edge set $E(G)$ of G is the disjoint union of $E(F_1), \ldots, E(F_k)$. The linear arboricity $la(G)$ of G is the minimum cardinality of a linear forest-decomposition of G. In this paper we evolve a method to construct a small linear forest-decomposition of a graph G from given linear forest-decompositions of two subgraphs that are linked by a cut vertex of G. As an application we determine the linear arboricity of block-cactus graphs which extends a result of Zelinka [5] (1986). Our results are connected to the "linear arboricity conjecture" of Akiyama, Exoo and Harary [2] (1980).

1. Introduction

We consider finite, simple, undirected, connected and non-trivial graphs G with vertex set $V(G)$ and edge set $E(G)$. For a vertex x of a graph G the degree $d(x, G)$ of x in G is the cardinality of the neighbourhood of x in G. The maximum degree of a vertex in a graph G is denoted by $\Delta(G)$. The linear arboricity of a graph was defined by F. Harary in [3]. We give here a slightly more formalized version which simplifies some matters of notation.

A linear forest-factor F of a graph G is a spanning subgraph of G whose components are all paths (isolated vertices are allowed). A linear forest-decomposition of G is a collection $\mathcal{F} = \{F_1, \ldots, F_k\}$ of linear forest-factors of G such that $E(G)$ is the disjoint union of $E(F_1), \ldots, E(F_k)$. The linear arboricity $la(G)$ of G is the minimum cardinality of a linear forest-decomposition.

Since the maximum degree in a linear forest-factor is at most 2, the lower bound

$$\left\lceil \frac{\Delta(G)}{2} \right\rceil \leq la(G)$$

is immediate for every graph G.

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To shorten the proofs we now introduce some notation. Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be an linear forest-decomposition of G. For every vertex $x \in V(G)$ we define

$$n_i(x, \mathcal{F}) = |\{F \in \mathcal{F} : d(x, F) = i\}|, \ i = 0, 1, 2$$

and

$$\vec{n}(x, \mathcal{F}) = (n_0(x, \mathcal{F}), n_1(x, \mathcal{F}), n_2(x, \mathcal{F})).$$

A fundamental question in this context is the "linear arboricity conjecture" of Akiyama, Exoo and Harary [2].

Conjecture 1 If G is an r-regular graph, then

$$la(G) = \left\lceil \frac{r+1}{2} \right\rceil.$$

For non regular graphs we state a version of this conjecture formulated by Aït-djafer [1].

Conjecture 2 If G is a graph, then

$$\left\lceil \frac{\Delta(G)}{2} \right\rceil \leq la(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil.$$

2. Structural Results

We start with our main theorem. It examines how to obtain a small linear forest-decomposition of a graph G from given linear forest-decompositions of two subgraphs that are linked by a cut vertex of G.

Theorem 1 Let G be graph with the cut vertex x. Let G_1 and G_2 be two subgraphs of G such that $\{x\} = V(G_1) \cap V(G_2)$, $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be linear forest-decompositions of G_1 and G_2, respectively with $|\mathcal{F}_i| = la(G_i)$ and $\vec{n}_i = (n_0^i, n_1^i, n_2^i) = \vec{n}(x, \mathcal{F}_i)$, $i = 1, 2$. Assume that $n_1^1 \geq n_1^2$.

Then the following relations for $la(G)$ hold:

i. If $n_0^1 \leq n_2^2$ and $n_0^2 \leq n_2^1$, then

$$la(G) = max\left\{\left\lceil \frac{d(x,G)}{2} \right\rceil, la(G_1)\right\}.$$

ii. If $n_0^1 \leq n_2^2$ and $n_2^1 < n_0^2 < n_2^1 + (n_1^1 - n_1^2)$, then

$$la(G) \leq max\left\{la(G_1), \left\lceil \frac{d(x,G) + n_0^2 - n_2^1}{2} \right\rceil\right\}.$$

iii. In all other cases we have

$$la(G) = max\{la(G_1), la(G_2)\}.$$

Proof. Step by step we combine linear forest-factors of $\mathcal{F}_1$ and $\mathcal{F}_2$ to form linear forest-factors of G. After every step we reduce the entries of $\vec{n}_1$ and $\vec{n}_2$ to keep track of the remaining linear forest-factors. If $\vec{n}_1 = \vec{n}_2 = (0,0,0)$, the process is complete.

Since $n_1^2 \leq n_1^1$ we begin in all three cases by forming the union of linear forest-factors $F_1 \in \mathcal{F}_1$ and linear forest-factors $F_2 \in \mathcal{F}_2$ with $d(x, F_i) = 1,\ i = 1,2$ to construct n_1^2 new linear forest-factors of G. This leads to

$$\vec{n}_1 = (n_0^1, n_1^1 - n_1^2, n_2^1) \text{ and } \vec{n}_2 = (n_0^2, 0, n_2^2). \tag{1}$$

In Cases i and ii we form the union of linear forest-factors $F_1 \in \mathcal{F}_1$ and linear forest-factors $F_2 \in \mathcal{F}_2$ with $d(x, F_1) = 0$ and $d(x, F_2) = 2$ to construct n_0^1 new linear forest-factors of G and obtain

$$\vec{n}_1 = (0, n_1^1 - n_1^2, n_2^1) \text{ and } \vec{n}_2 = (n_0^2, 0, n_2^2 - n_0^1). \tag{2}$$

Case i: By forming the union of linear forest-factors $F_1 \in \mathcal{F}_1$ and linear forest-factors $F_2 \in \mathcal{F}_2$ with $d(x, F_1) = 2$ and $d(x, F_2) = 0$ we construct n_0^2 new linear forest-factors of G. Therefore we deduce from (2)

$$\vec{n}_1 = (0, n_1^1 - n_1^2, n_2^1 - n_0^2) \text{ and } \vec{n}_2 = (0, 0, n_2^2 - n_0^1).$$

Now we decompose $m = min\left\{\left\lfloor \frac{n_1^1 - n_1^2}{2} \right\rfloor, n_2^2 - n_0^1\right\}$ of the linear forest-factors $F \in \mathcal{F}_2$ with $d(x, F) = 2$ in $2m$ new linear forest-factors F' of G_2 with $d(x, F') = 1$ and obtain

$$\vec{n}_1 \text{ remains unchanged and } \vec{n}_2 = (0, 2m, n_2^2 - n_0^1 - m).$$

By forming the union of linear forest-factors $F_1 \in \mathcal{F}_1$ and linear forest-factors $F_2 \in \mathcal{F}_2$ with $d(x, F_1) = 1$ and $d(x, F_2) = 1$ we construct $2m$ linear forest-factors of G and obtain

$$\vec{n}_1 = (0, n_1^1 - n_1^2 - 2m, n_2^1 - n_0^2) \text{ and } \vec{n}_2 = (0, 0, n_2^2 - n_0^1 - m).$$

All the remaining linear forest-factors of G_1 and G_2 are adopted unchanged as linear forest-factors of G (just by adding all the missing vertices to get a factor). This yields finally

$$\vec{n}_1 = \vec{n}_2 = (0,0,0).$$

The cardinality of the constructed linear forest-decomposition of G is now

$$\begin{aligned}
&n_1^2 + n_0^1 + n_0^2 + 2m + (n_2^1 - n_0^2) + (n_1^1 - n_1^2 - 2m) + (n_2^2 - n_0^1 - m) \\
&= n_1^1 + n_2^1 + n_2^2 - m \\
&= max\left\{n_1^1 + n_2^1 + n_2^2 - \left\lfloor \frac{n_1^1 - n_1^2}{2} \right\rfloor, n_1^1 + n_2^1 + n_2^2 - n_2^2 + n_0^1\right\} \\
&= max\left\{n_1^1 + n_2^1 + n_2^2 + \left\lceil \frac{n_1^2 - n_1^1}{2} \right\rceil, n_0^1 + n_1^1 + n_2^1\right\}
\end{aligned}$$

$$= max\left\{\left\lceil\frac{n_1^2+n_1^1+2n_2^1+2n_2^2}{2}\right\rceil, n_0^1+n_1^1+n_2^1\right\}$$

$$= max\left\{\left\lceil\frac{d(x,G)}{2}\right\rceil, la(G_1)\right\},$$

and hence it follows that

$$la(G) \leq max\left\{\left\lceil\frac{d(x,G)}{2}\right\rceil, la(G_1)\right\}.$$

Since $la(G) \geq max\left\{\left\lceil\frac{d(x,G)}{2}\right\rceil, la(G_1)\right\}$ the desired equality is proved.

Case ii: By forming the union of linear forest-factors $F_1 \in \mathcal{F}_1$ and linear forest-factors $F_2 \in \mathcal{F}_2$ with $d(x, F_1) = 2$ and $d(x, F_2) = 0$ we construct n_2^1 new linear forest-factors of G. Hence (2) becomes

$$\vec{n}_1 = (0, n_1^1 - n_1^2, 0) \text{ and } \vec{n}_2 = (n_0^2 - n_2^1, 0, n_2^2 - n_0^1).$$

By forming the union of linear forest-factors $F_1 \in \mathcal{F}_1$ and linear forest-factors $F_2 \in \mathcal{F}_2$ with $d(x, F_1) = 1$ and $d(x, F_2) = 0$ we construct $n_0^2 - n_2^1$ new linear forest-factors of G and obtain

$$\vec{n}_1 = (0, n_1^1 - n_1^2 - n_0^2 + n_2^1, 0) \text{ and } \vec{n}_2 = (0, 0, n_2^2 - n_0^1).$$

Now we decompose $m = min\left\{\left\lfloor\frac{n_1^1-n_1^2-n_0^2+n_2^1}{2}\right\rfloor, n_2^2 - n_0^1\right\}$ of the linear forest-factors $F \in \mathcal{F}_2$ with $d(x, F) = 2$ in $2m$ new linear forest-factors F' of G_2 with $d(x, F') = 1$ and obtain

$$\vec{n}_1 \text{ remains unchanged and } \vec{n}_2 = (0, 2m, n_2^2 - n_0^1 - m).$$

By forming the union of linear forest-factors $F_1 \in \mathcal{F}_1$ and linear forest-factors $F_2 \in \mathcal{F}_2$ with $d(x, F_1) = 1$ and $d(x, F_2) = 1$ we construct $2m$ new linear forest-factors of G and obtain

$$\vec{n}_1 = (0, n_1^1 - n_1^2 - n_0^2 + n_2^1 - 2m, 0) \text{ and } \vec{n}_2 = (0, 0, n_2^2 - n_0^1 - m).$$

All the remaining linear forest-factors of G_1 and G_2 are adopted unchanged as linear forest-factors of G. A similar calculation as in Case i leads to the cardinality of the constructed linear forest-decomposition of G, which implies the desired inequality

$$la(G) \leq max\left\{\left\lceil\frac{d(x,G)+n_0^2-n_2^1}{2}\right\rceil, la(G_1)\right\}.$$

Case iii: Starting from (1) we distinguish two subcases.

(a) $n_2^2 < n_0^1$. We construct n_2^2 new linear forest-factors of G by forming the union of linear forest-factors $F_1 \in \mathcal{F}_1$ and linear forest-factors $F_2 \in \mathcal{F}_2$ with $d(x, F_1) = 0$ and $d(x, F_2) = 2$. We obtain

$$\vec{n}_1 = (n_0^1 - n_2^2, n_1^1 - n_1^2, n_2^1) \text{ and } \vec{n}_2 = (n_0^2, 0, 0).$$

We construct $max\{n_0^2, n_2^1 + (n_1^1 - n_1^2) + (n_0^1 - n_2^2)\}$ new linear forest-factors of G by forming the union of $min\{n_0^2, n_2^1 + (n_1^1 - n_1^2) + (n_0^1 - n_2^2)\}$ linear forest-factors $F_1 \in \mathcal{F}_1$ and linear forest-factors $F_2 \in \mathcal{F}_2$ and by adopting all remaining linear forest-factors unchanged. We obtain a linear forest-decomposition of G with cardinality

$$\begin{aligned} & n_1^2 + n_2^2 + max\{n_0^2, n_2^1 + (n_1^1 - n_1^2) + (n_0^1 - n_2^2)\} \\ &= max\{n_0^2 + n_1^2 + n_2^2, n_0^1 + n_1^1 + n_2^1\} \\ &= max\{la(G_1), la(G_2)\}. \end{aligned}$$

Therefore $la(G) \leq max\{la(G_1), la(G_2)\}$. Since clearly $la(G) \geq la(G_i)$ for $i = 1, 2$ we obtain the desired result.

(b) The only remaining case, $n_2^2 \geq n_0^1$ and $n_0^2 \geq n_2^1 + (n_1^1 - n_1^2)$ is similar to Case iii (a) and is therefore omitted. ■

To apply this result we need to know the vectors $\vec{n}_i$. For a vertex $x \in V(G)$ of maximum degree in a graph G satisfying Conjecture 2, the following proposition summarizes all possible values of $\vec{n}(x, \mathcal{F})$.

Proposition 1 Let G be a graph with $\left\lceil \frac{\Delta(G)}{2} \right\rceil \leq la(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$. Let x be a vertex of G with maximum degree $d(x, G) = \Delta(G)$ and $\mathcal{F}$ be a linear forest-decomposition of G with $|\mathcal{F}| = la(G)$.

1.1. If $\Delta(G)$ is odd, then $\vec{n}(x, \mathcal{F}) = (0, 1, la(G) - 1)$.

1.2. If $\Delta(G)$ is even and $la(G) = \frac{\Delta(G)}{2}$, then $\vec{n}(x, \mathcal{F}) = (0, 0, la(G))$.

1.3. If $\Delta(G)$ is even and $la(G) = \frac{\Delta(G)}{2} + 1$, then either
(a) $\vec{n}(x, \mathcal{F}) = (0, 2, la(G) - 2)$ or (b) $\vec{n}(x, \mathcal{F}) = (1, 0, la(G) - 1)$.

The simple proof of Proposition 1 is left to the reader. Now we proceed to the first application of Theorem 1.

Theorem 2 Let G be a graph with at least two blocks $B_1, \ldots, B_r$ such that

2.1. for each $i \in \{1, \ldots, r\}$, $\left\lceil \frac{\Delta(B_i)}{2} \right\rceil \leq la(B_i) \leq \left\lceil \frac{\Delta(B_i)+1}{2} \right\rceil$;

2.2. for each $i \in \{1, \ldots, r\}$ and for every cut vertex x of G in $V(B_i)$ we have $d(x, B_i) = \Delta(B_i)$; and

2.3. for each $i \in \{1, \ldots, r\}$ such that $\Delta(B_i)$ is even and $la(B_i) = \frac{\Delta(B_i)}{2} + 1$, for every cut vertex x of G in $V(B_i)$ there are linear forest-decompositions $\mathcal{F}_1$ and $\mathcal{F}_2$ of B_i such that $\vec{n}(x, \mathcal{F}_1) = (0, 2, la(B_i) - 2)$ and $\vec{n}(x, \mathcal{F}_2) = (1, 0, la(B_i) - 1)$.

Then

$$la(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil . \tag{3}$$

Proof. We use induction on r to show that if the graph G with $r \geq 1$ blocks $B_1, \ldots, B_r$ has properties 2.1, 2.2 and 2.3 then

$$la(G) \leq max\left\{\left\lceil\frac{\Delta(B_i)+1}{2}\right\rceil, i=1,\ldots,r, \left\lceil\frac{\Delta(G)}{2}\right\rceil\right\}. \quad (4)$$

When $r \geq 2$, from 2.2 we have that $\Delta(B_i)+1 \leq \Delta(G)$ for $i=1,\ldots,r$, so (3) follows from (4) and the theorem will be proved. Since (4) is immediate from 2.1 when $r=1$, we assume now than $r \geq 2$.

Without loss of generality we assume that B_1 is an endblock with the cut vertex x. By induction the graph $G' = G - (V(B_1) - \{x\})$ satisfies (4), i.e.

$$la(G') \leq max\left\{\left\lceil\frac{\Delta(B_i)+1}{2}\right\rceil, i=2,\ldots,r, \left\lceil\frac{\Delta(G')}{2}\right\rceil\right\}. \quad (5)$$

Let $\mathcal{F}$ and $\mathcal{F}'$ be linear forest-decompositions of B_1 and G' with $la(B_1) = |\mathcal{F}|$ and $la(G') = |\mathcal{F}'|$. Define $\vec{n} = \vec{n}(x, \mathcal{F})$ and $\vec{n}' = \vec{n}(x, \mathcal{F}')$.

Theorem 1 can be applied to B_1 and G' as G_1 and G_2. Since $d(x, B_1) = \Delta(B_1)$ and B_1 satisfies $\left\lceil\frac{\Delta(B_1)}{2}\right\rceil \leq la(B_1) \leq \left\lceil\frac{\Delta(B_1)+1}{2}\right\rceil$, the only possible values for $\vec{n}$ are all mentioned in Proposition 1. We now show that all these values exclude Case ii of Theorem 1 for $\vec{n}$ and $\vec{n}'$.

1. Let $\Delta(B_1)$ be odd and thus $\vec{n} = (0, 1, la(B_1) - 1)$.
   (a) If $n'_1 = 0$, then we use Theorem 1 with $G_1 = B_1$ and $G_2 = G'$. Since $la(B_1) - 1 < n'_0 = n^2_0 < la(B_1)$ is not possible, Case ii is excluded.
   (b) If $n'_1 \geq 1$, then we use Theorem 1 with $G_1 = G'$ and $G_2 = B_1$. Since $0 = n^2_0 > n^1_2 \geq 0$ is false, Case ii is excluded.
2. Let $\Delta(B_1)$ be even, $la(B_1) = \frac{\Delta(B_1)}{2}$ and thus $\vec{n} = (0, 0, la(B_1))$. We define $G_1 = G'$ and $G_2 = B_1$. Since $0 = n^2_0 > n^1_2 \geq 0$ is false, Case ii is excluded.
3. Let $\Delta(B_1)$ be even and $la(B_1) = \frac{\Delta(B_1)}{2} + 1$.
   (a) If $n'_1 \geq 2$, then choose $\mathcal{F}$ such that $\vec{n} = (0, 2, la(B_1) - 2)$. We define $G_1 = G'$ and $G_2 = B_1$. Since $0 = n^2_0 > n^1_2 \geq 0$ is false, Case ii is excluded.
   (b) If $n'_1 < 2$, then choose $\mathcal{F}$ such that $\vec{n} = (1, 0, la(B_1) - 1)$. We define $G_1 = G'$ and $G_2 = B_1$. Since $n^1_2 = n'_2 < n^2_0 = 1 < n^1_2 + (n^1_1 - n^2_1) = n'_2 + n'_1$ is not possible, Case ii is excluded.

Thus for B_1 and G' only Cases i and iii of Theorem 1 occur which implies

$$la(G) \leq max\left\{la(G'), la(B_1), \left\lceil\frac{d(x,G)}{2}\right\rceil\right\}.$$

Together with (5) and the bounds $la(B_1) \leq \left\lceil\frac{\Delta(B_1)+1}{2}\right\rceil$ and $\Delta(G') \leq \Delta(G)$, we deduce (4) and the proof is complete. ∎

Under weaker conditions the same proof-methods lead to a similar result.

Theorem 3 Let G be a graph with $r \geq 1$ blocks $B_1, \dots, B_r$ such that

3.1. for each $i \in \{1, \dots, r\}$, $\left\lceil \frac{\Delta(B_i)}{2} \right\rceil \leq la(B_i) \leq \left\lceil \frac{\Delta(B_i)+1}{2} \right\rceil$ and

3.2. for each $i \in \{1, \dots, r\}$ and for every cut vertex x of G in $V(B_i)$ we have $d(x, B_i) = \Delta(B_i)$.

Then

$$la(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil,$$

i.e. the graph G satisfies Conjecture 2.

3. Applications

A block-cactus graph is a graph whose blocks are either complete or cycles. In view of Theorem 2 the linear arboricity of block-cactus graphs is now easy to determine.

Corollary 1 If G is a block-cactus graph with at least two blocks, then

$$la(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil.$$

Proof. Since for the blocks which are cycles all conditions in Theorem 2 are evident, we only verify them for the complete blocks.

For the complete graph K_n on n vertices Condition 2.2 of Theorem 2 is immediate, and the linear arboricity was determined by Stanton, Cowan and James [4] and is given by (see also [2])

$$la(K_n) = \left\lceil \frac{\Delta(K_n)+1}{2} \right\rceil.$$

Therefore the complete blocks satisfy Condition 2.1.

For Condition 2.3 let K_n be a complete graph of odd order and let x be an arbitrarily chosen vertex in $V(K_n)$. The linear arboricity is

$$la(K_n) = \left\lceil \frac{\Delta(K_n)+1}{2} \right\rceil = \frac{\Delta(K_n)}{2} + 1 = \frac{n-1}{2} + 1.$$

Now by [4] there exists even a linear forest-decomposition $\mathcal{F}$ of K_n in $la(K_n)$ factors such that each factor in $\mathcal{F}$ contains only one path of length different from 0. Hence it is easily seen that there are at least two vertices x_1 and x_2 in $V(K_n)$ such that $\vec{n}(x_1, \mathcal{F}) = (0, 2, la(K_n) - 2)$ and $\vec{n}(x_2, \mathcal{F}) = (1, 0, la(K_n) - 1)$. Now the symmetry of K_n implies the existence of two linear forest-decompositions of K_n in $la(K_n)$ factors where x takes the positions of x_1 and x_2 respectively. Hence the desired decompositions of Condition 2.3 do exist. This completes the proof. ∎

As an immediate consequence, we obtain the following two results.

Corollary 2 (Zelinka [5], 1986) If G is a cactus graph with at least two blocks, then

$$la(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil.$$

Corollary 3 (Akiyama, Exoo and Harary [2], 1980) If G is a tree, then

$$la(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil.$$

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