Hadamard 2-groups with arbitrarily large derived length

Jung R. Cho
Department of Mathematics, Pusan National University,
Pusan 609-735, Korea.

Noboru Ito
Department of Mathematics, Meijo University,
Nagoya, Tenpaku 468, Japan.

Pan Soo Kim
Department of Mathematics Education,
Pusan National University of Education,
Pusan 607-736, Korea.

Hyo-Seob Sim
Department of Applied Mathematics, Pukyong National University,
Pusan 608-737, Korea.

Abstract

In this paper we construct a family of Hadamard 2-groups whose derived lengths are arbitrarily large.

1. Introduction

Let G be a group of order $4n$ with a central involution e^*. If there exists a transversal D of G with respect to the subgroup $\langle e^* \rangle$ generated by e^* such that $|D \cap Da| = n$ for every element a of G apart from the identity e and e^*, then we call D and G an *Hadamard subset* and an *Hadamard group* with respect to e^* respectively.

If an Hadamard group G of order $4n$ exists, then we can construct an Hadamard matrix of order $2n$ whose automorphism group contains a regular subgroup isomorphic to G. (See [2].)

A result of Ito[3] shows that every generalized quaternion group is an Hadamard group. This implies that for a given positive integer c, there is an example of an Hadamard group with nilpotency class c. It was previously unknown whether the

The work presented here was supported in part by KOSEF grant 96-0701-03-01-3 .

analogue for the derived length is also true. (An example of an Hadamard 2-group of derived length 3 was given in [1].)

The aim of this paper is to give a family of examples of Hadamard 2-groups with arbitrarily large derived lengths. The main result of this note may be stated as follows:

Theorem 1.1. *Suppose that G_0 is an Hadamard abelian 2-group with respect to a square involution and $G_0 \times \langle e_2^* \rangle$ is Hadamard with respect to e_2^*. For each positive integer n let G_n be the wreath product of G_{n-1} by a cyclic group of order 2. Then G_n is an Hadamard group with derived length $n+1$.*

2. Extensions of Hadamard groups by involutions

The following lemma establishes the Hadamard property of certain kinds of extensions of an Hadamard group.

Lemma 2.1. *Let G be an Hadamard group with an Hadamard subset D with respect to e^*, and let $\overline{G} := G \rtimes \langle t \rangle$ be an extension of G by the cyclic group $\langle t \rangle$ of order 2. Suppose there exists an element x in G with $x^2 = e^*$ such that t commutes with x. Then the extension $\overline{G}$ is also an Hadamard group with the Hadamard subset $\overline{D} := Dx \cup Dt$ with respect to the same involution e^*.*

Proof. Since $|D| = \frac{|\overline{G}|}{4}$, we need to show that $|\overline{D} \cap \overline{D}y| = |D|$ for each element y in $\overline{G}$ other than e and e^*.

(i) Suppose that y belongs to G as embedded in $\overline{G}$. Since y is different from e and e^*, so is y^t by the hypothesis. It easily follows that $\overline{D}y = Dxy \cup Dy^t t$; so $|\overline{D} \cap \overline{D}y| = |Dx \cap Dxy| + |D \cap Dy^t| = |D|$ since Dx and D are Hadamard subsets of G with respect to e^*.

(ii) For the other case, y is an element of the form zt for some z in G. Then $\overline{D}y = Dxzt \cup Dz^t$, and so $|\overline{D} \cap \overline{D}y| = |D \cap Dxz| + |Dx \cap Dz^t|$. Note that t commutes with e^*. It follows that $xz = e$ if and only if $x^{-1}z^t = e^*$, and $xz = e^*$ if and only if $x^{-1}z^t = e$. So if $xz \in \langle e^* \rangle$ then either $|D \cap Dxz| = 0$ and $|Dx \cap Dz^t| = |D|$, or $|D \cap Dxz| = |D|$ and $|Dx \cap Dz^t| = 0$; otherwise, $|D \cap Dxz| = |Dx \cap Dz^t| = \frac{|D|}{2}$. Therefore $|\overline{D} \cap \overline{D}y| = |D|$. This completes the proof of Lemma 2.1.

We then have the following corollary as an immediate consequence of Lemma 2.1.

Corollary 2.2. *Suppose that G is an Hadamard group with respect to e^*. If G contains an element x with $x^2 = e^*$ then $G \times C_2$ is Hadamard.*

We also need the following result:

Lemma 2.3. *Suppose G and $H \times C_2$ are Hadamard with respect to involutions $e_1^* \in G$ and e_2^*, respectively, where $C_2 = \langle e_2^* \rangle$. Then $G \times H$ is Hadamard with respect to e_1^*.*

Proof. By Proposition 3 of [2], $G \times (H \times \langle e_2^* \rangle)/\langle e_1^* e_2^* \rangle$ is Hadamard with respect to $e_1^* \langle e_1^* e_2^* \rangle$. The homomorphism of $G \times H \times \langle e_2^* \rangle$ defined by $(g, h, (e_2^*)^\epsilon) \mapsto (g(e_1^*)^\epsilon, h)$

($\epsilon = 0$ or 1) induces an isomorphism of $G \times (H \times \langle e_2^* \rangle)/\langle e_1^* e_2^* \rangle$ onto $G \times H$. So the result follows.

Corollary 2.4. *Suppose H and $H \times C_2$ are Hadamard abelian groups with respect to a square involution e_1^* and an involution e_2^*, respectively, where $C_2 = \langle e_2^* \rangle$. Then $H \times H \times \cdots \times H$ is Hadamard with respect to a square involution $(x^2, x^2, \ldots, x^2) = (e_1^*, \ldots, e_1^*)$ for some $x \in H$.*

Proof. By Lemma 2.3, $H \times H \times \cdots \times H$ is Hadamard with respect to $(e_1^*, e, e, \ldots, e)$, where e is the identity. Define an automorphism σ of $H \times \cdots \times H$ by

$$(h_1, h_2, \ldots, h_n)^\sigma = (h_1, h_1 h_2, h_1 h_3, \ldots, h_1 h_n).$$

It is easy to check in general that the isomorphic image of an Hadamard subset with respect to e^* is an Hadamard subset with respect to the image of e^*. Since $(e_1^*, e, e, \ldots, e)^\sigma = (e_1^*, e_1^*, \ldots, e_1^*)$, we have the desired result.

Remark There are many Hadamard abelian 2-groups satisfying the hypotheses of the above Corollary and so those of Theorem 1.1. For example, C_4, $C_4 \times C_2, C_4 \times C_2 \times C_2, C_4 \times C_2 \times \cdots \times C_2$ are such groups.

We now return to the proof of Theorem 1.1. Each element g of $G_n = G_{n-1} \wr C_2$ is of the form $(a_{n-1}, b_{n-1}) t_n^\epsilon$ where $a_{n-1}, b_{n-1} \in G_{n-1}$ and $\epsilon = 0$ or 1. We also note that t_n is an automorphism of $G_{n-1} \times G_{n-1}$ exchanging the components, that is $(a_{n-1}, b_{n-1})^{t_n} = (b_{n-1}, a_{n-1})$.

From its construction, G_n is expressed in terms of G_i's ($i < n$) and its automorphisms. For example, G_3 is expressed as follows.

$$\begin{aligned}
G_3 =& G_2 \wr C_2 \\
=& (G_2 \times G_2) \rtimes \langle t_3 \rangle \\
=& [((G_1 \times G_1) \rtimes \langle t_2 \rangle) \times ((G_1 \times G_1) \rtimes \langle t_2 \rangle)] \rtimes \langle t_3 \rangle \\
=& \{\{[((G_0 \times G_0) \rtimes \langle t_1 \rangle) \times ((G_0 \times G_0) \rtimes \langle t_1 \rangle)] \rtimes \langle t_2 \rangle\} \times \\
& \{[((G_0 \times G_0) \rtimes \langle t_1 \rangle) \times ((G_0 \times G_0) \rtimes \langle t_1 \rangle)] \rtimes \langle t_2 \rangle\}\} \rtimes \langle t_3 \rangle \\
=& (G_0)^8 \rtimes (C_2)^4 \rtimes (C_2)^2 \rtimes C_2.
\end{aligned}$$

So we can have that

$$G_n = (G_0)^{2^n} \rtimes (t_1)^{2^{n-1}} \rtimes (t_2)^{2^{n-2}} \rtimes \cdots \rtimes t_n.$$

We also know from Corollary 2.4 that $(G_0)^{2n}$ is Hadamard with respect to a square involution $(x^2, x^2, \ldots, x^2)$. Each t_i acts on $(G_0)^{2n}$, switching the components of elements of this group. So each t_i has order 2 and commutes with $(x, x, \ldots, x)$. By repeated applications of Lemma 2.1, it is now clear that G_n is Hadamard.

The following lemma completes the proof of Theorem 1.1.

Lemma 2.5. *G_n has derived length $n+1$.*

Proof. Since $(g,e)^{t_n}(g,e)^{-1} = (g^{-1}, g)$ for $g \in G_{n-1}$, the derived subgroup G'_n contains all the elements of the type (g, g^{-1}) for $g \in G_{n-1}$. The projection $(g, g^{-1}) \mapsto (g, e)$ implies that G_{n-1} is a subgroup of a quotient of G'_n. So the derived length of G'_n is at least that of G_{n-1}. But G'_n is contained in $G_{n-1} \times G_{n-1}$. Hence G'_n and G_{n-1} have the same derived length. By induction the result follows.

Acknowledgments

We would like to thank the referee for his helpful comments.

References

[1] M. Fukui, *Directly decomposable Hadamard groups and an example of an Hadamard 2-group with derived length 3,* Master's thesis, Meijo University, 1996.
[2] N. Ito, *On Hadamard groups,* J. of Algebra, **168** (1994) 981-987.
[3] N. Ito, *On Hadamard groups, III,* to appear in Kyushu J. of Math.

(Received 2/4/97)