WHEN ARE CHORDAL GRAPHS ALSO PARTITION GRAPHS?

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Abstract

A general partition graph (gpg) is an intersection graph G on a set S so that for every maximal independent set M of vertices in G, the subsets assigned to the vertices in M partition S. These graphs have been characterized by the presence of special clique covers. The Triangle Condition T for a graph G is that for any maximal independent set M and any edge uv in $G - M$, there is a vertex $w \in M$ so that uvw is a triangle in G. Condition T is necessary but not sufficient for a graph to be a gpg and a computer search has found the smallest ten counterexamples, one with nine vertices and nine with ten vertices. Any non-gpg satisfying Condition T is shown to induce a required subgraph on six vertices, and a method of generating an infinite class of such graphs is described. The main result establishes the equivalence of the following conditions in a chordal graph G: (*i*) G is a gpg (*ii*) G satisfies Condition T (*iii*) every edge in G is in an end-clique. The result is extended to a larger class of graphs.

1. INTRODUCTION

All graphs considered will be assumed to be connected and we will follow notation found in [6]. In particular, *cliques* are assumed to be maximal complete subgraphs. A graph G is a *general partition graph* (gpg) on a set S if it is possible to assign to each of its vertices v a subset S_v of S such that:

(1) vertices u and v are adjacent if and only if $S_u \cap S_v \neq \phi$,

(2) $S = \bigcup_{v \in V(G)} S_v$,

(3) for every maximal independent set M of vertices in G, the collection $\{S_m: m \in M\}$ partitions S.

The term *partition graph* has been reserved for a graph G which is a gpg and in addition satisfies the closed neighborhood requirement that $N[u] \neq N[v]$ for all $u \neq v$ in $V(G)$. These graphs (not to be confused with partition intersection graphs introduced in [8]) have been encountered in the geometric setting of triangulations of lattice polygons [4] and their theory developed in [2], [3] and [7]. The following conditions prove to be important in the theory of general partition graphs.

Triangle Condition T. If M is any maximal independent set in G and uv is any edge in $G - M$ then for some $m \in M$, uvm is a triangle in G.

Clique Condition C. If M is any maximal independent set in G, then no complete subgraph of $G - M$ is a clique in G.

Incidence Condition I for a Clique Cover. There is a collection $\mathcal{C}$ of cliques that contains all edges of G with the property that every maximal independent set in G has a vertex from each clique in $\mathcal{C}$.

Condition T is necessary but not sufficient for a gpg [2], Condition C is sufficient but not necessary for a gpg [2]; clearly Condition C implies Condition T. Condition I is a characterization for a gpg [7].

We add a fourth condition which, in a special form, has already been used implicitly in [7] and occurs again in the last section of this paper. An *end-clique* in a graph G is a clique that contains a vertex that lies in no other clique of G.

End-clique Condition E. Every edge of G lies in an end-clique of G.

Condition E is not necessary for a gpg (for example, the cycle on 4 vertices) but it is sufficient.

Lemma 1 Condition E implies Condition I.

Proof: Let $\mathcal{C}$ be the collection of all end-cliques of G. //

Conditions C and E are independent. The graph G^* in Figure 2 satisfies E but not C. The cycle on 4 vertices satisfies C but not E. The path on 4 vertices satisfies neither condition while the path on 3 vertices satisfies both.

One can ask whether there are settings in which Condition T is sufficient for a graph to be a gpg. In the next section we examine the situation where the triangle condition is not sufficient. The concluding section derives our main result, that the triangle condition is sufficient in chordal graphs.

2. A NECESSARY SUBGRAPH FOR GRAPHS WHICH SATISFY CONDITION T BUT ARE NOT A GENERAL PARTITION GRAPH

A computer search, in which Condition T is checked against Condition I, has found all of the connected graphs on ten or fewer vertices which satisfy the triangle condition but are not gpg's [1]. The smallest example, denoted by G_T, has nine vertices and is shown in Figure 1(a). There are nine more such graphs on ten vertices, shown in Figure 1(b)-(j).

Several of the 10-vertex graphs in Figure 1 have a simple relation to the 9-vertex graph G_T at the top of the figure. For example, introducing the new vertex 0 with the same open neighborhood as vertex 7 of G_T yields graph (d). Graph (e) is obtained similarly, but with closed neighborhoods, $N[0] = N[7]$. Graphs (f) and (g) are obtained from G_T by using vertex 1 instead of 7. We also note that $N(0) = V(G_T)$ in graph (j). These examples suggest methods to generate an infinite class of non-gpg's which satisfy Condition T. If G is such a graph, take any vertex $u \in V(G)$, introduce a new vertex $v \notin V(G)$ and join edges so that $N(u) = N(v)$ for the open neighborhoods, or $N[u] = N[v]$ for the closed neighborhoods. Alternatively, introduce a new vertex u that is joined to every vertex of $V(G)$. The resulting graphs are still a non-gpg satisfying Condition T, as follows from parts (a) and (b) of

Lemma 2 Let G be a graph and u and v be vertices so that either $N(u) = N(v)$, $N[u] = N[v]$, or $N[u] = V(G)$. Then

(a) G satisfies Condition I if and only if $G - u$ satisfies Condition I.
(b) G satisfies Condition T if and only if $G - u$ satisfies Condition T.
(c) G satisfies Condition C if and only if $G - u$ satisfies Condition C.
(d) G satisfies Condition E if and only if $G - u$ satisfies Condition E.

Proof: Statement (a) is Theorem 4.3 in [7]. Statements (b), (c) and (d) are routinely justified by considering cases depending on how the particular maximal independent set intersects the appropriate vertex neighborhood. //

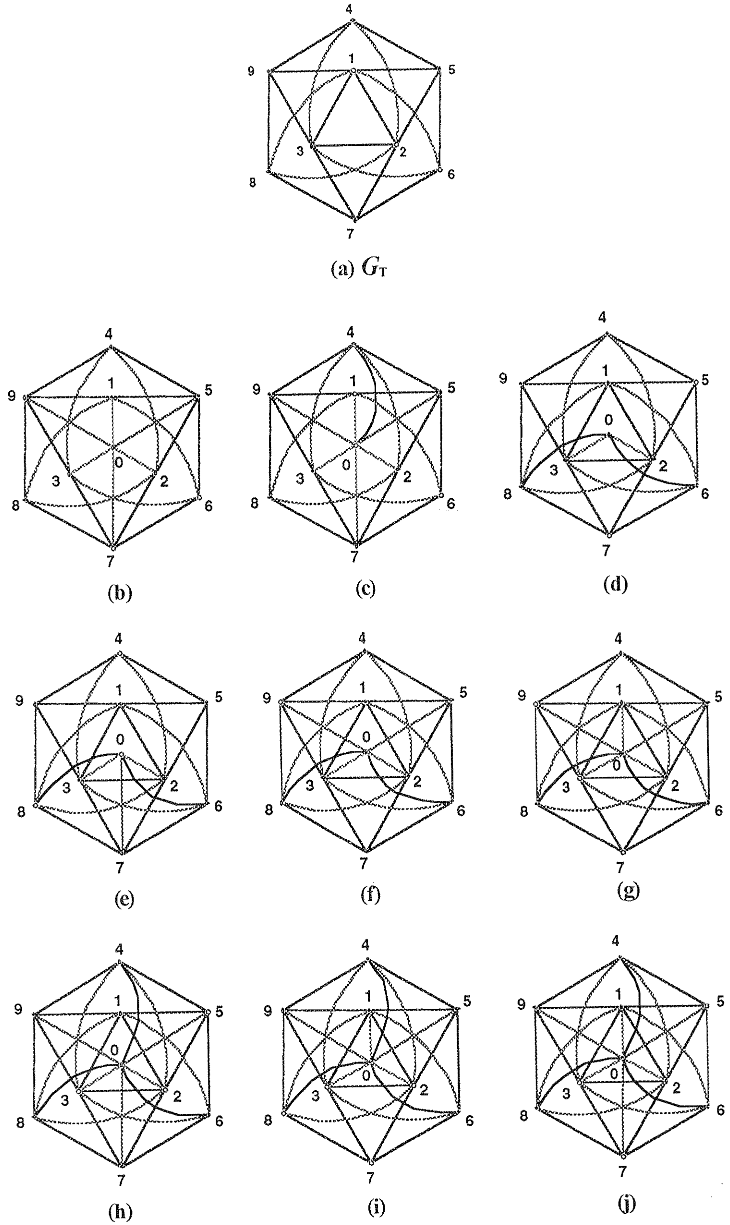

Figure 1. The ten graphs on ten or fewer vertices which satisfy Condition T but are not general partition graphs.

Every graph in Figure 1 has the graph G^* shown in Figure 2 as an induced subgraph. (Note that G^* is a gpg satisfying Condition T but not Condition C.).

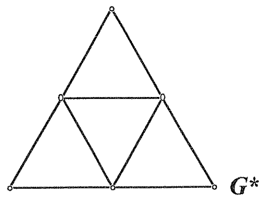

Figure 2. G^*, a required induced subgraph for graphs that satisfy Condition T but are not general partition graphs

Theorem 1 If G satisfies Condition T but is not a gpg then G^* is an induced subgraph of G.

Proof: Since Condition I characterizes a gpg, for any clique cover $\mathcal{C}$ of the edges of G, there is a maximal independent set M and clique $C \in \mathcal{C}$ with no member of M in C. Thus C lies in $G - M$. Clique C is not K_2 because of Condition T. Choose $m_1 \in M$ so that $|N(m_1) \cap V(C)|$ is maximal. By Condition T, this maximum is at least two. Since C is maximal, there is a vertex in C which is not adjacent to m_1. For any edge xy where $x \in V(C) \setminus N(m_1)$ and $y \in V(C) \cap N(m_1)$, there is a vertex $m_2 \in M$ adjacent to both x and y. Choose m_2 so that $|N(m_1) \cap N(m_2) \cap V(C)|$ is maximal. Since $|N(m_1) \cap V(C)|$ is maximal, there is a vertex $a \in V(C) \cap (N(m_1) \setminus N(m_2))$. Let $b \in V(C) \cap (N(m_2) \setminus N(m_1))$. There is a vertex $m_3 \in M$ adjacent to both a and b and since $|N(m_1) \cap N(m_2) \cap V(C)|$ is maximal, there must be a vertex $c \in N(m_1) \cap N(m_2) \cap V(C)$ which is not adjacent to m_3. The vertices a, b, c, m_1, m_2, and m_3 induce G^* in G. //

3. A Chordal Graph Satisfying Condition T is a GPG.

A connected *chordal* graph can be defined recursively using the notion of simplicial vertices [5]. Equivalently, a *chordal* graph is a connected graph in which every cycle on more than three vertices has a chord.

Theorem 2 For a chordal graph G, Conditions I, T and E are equivalent.

Proof: It follows directly from the definitions that all gpg's satisfy Condition T [2], and from Lemma 1 that Condition E implies Condition I. It only remains to show Condition T implies Condition E.

We shall use the following notation for edge uv in a connected chordal graph G.

$\mathcal{C}_{uv}$ = the set of cliques in G that contain edge uv.

T_{uv} = the union of vertex sets of all cliques in $\mathcal{C}_{uv}$.

$\mathcal{B}_{uv}$ = the set of cliques in G that contain vertices in both T_{uv} and its complement.

E_m = the set of edges uv in G that lie in no end-clique of G and for which $\mathcal{C}_{uv}$ is minimal.

F_C = the set of vertices from T_{uv} that lie in clique C from $\mathcal{B}_{uv}$.
We call F_C the *foot* of C in T_{uv}.

Let uv be an edge in E_m and x any vertex in T_{uv}. We show that x lies in a clique from $\mathcal{B}_{uv}$. If not, then x belongs to two cliques C_1 and C_2 from $\mathcal{C}_{uv}$. Let y be a vertex in $V(C_1)\backslash V(C_2)$. All vertices of any clique C containing edge xy must lie in T_{uv} otherwise C belongs to $\mathcal{B}_{uv}$. Moreover C contains uv. Hence $\mathcal{C}_{xy}$ is a subset of $\mathcal{C}_{uv}$. Then xy lies in no end-clique of G but also lies in fewer cliques than uv since xy is not in C_2. This contradicts our definition of uv.

Thus we can choose cliques C_1, C_2, ... from $\mathcal{B}_{uv}$ with distinct feet F_{C1}, F_{C2}, ... whose union equals T_{uv} and we can assume that each $F_{C\,k}$ is maximal with respect to set inclusion over all feet generated by cliques in $\mathcal{B}_{uv}$. For distinct i and j let $x \in V(C_i)\backslash T_{uv}$ and $y \in V(C_j)\backslash T_{uv}$. We show that there is a vertex z in $F_{C\,j}\backslash F_{C\,i}$ not adjacent to x. Suppose not, then there is a vertex w' in $F_{C\,i}\backslash F_{C\,j}$ that is not adjacent to some vertex z' in $F_{C\,j}\backslash F_{C\,i}$ otherwise x, w', and $F_{C\,j}$ lie in a clique from $\mathcal{B}_{uv}$ whose foot properly contains $F_{C\,j}$. This means one of the 4-cycles $x\,w'u\,z'$ or $x\,w'v\,z'$ is chordless contradicting the definition of G. Similarly we have a vertex w in $F_{C\,i}\backslash F_{C\,j}$ that is not adjacent to y. Suppose now that x and y are adjacent. Then w and z are not adjacent otherwise we have the chordless 4-cycle $xwzy$. By considering the 5-cycle $xwuzy$, we see that u is adjacent to both x and y. Hence v is adjacent to neither x nor y and cycle $xwvzy$ is chordless. We conclude that x and y are not adjacent.

Choose $x_i \in V(C_i)\backslash T_{uv}$ and extend $\{x_1, x_2, ...\}$ to a maximal independent set M in G. Edge uv lies in $G - M$ yet forms no triangle with a vertex in M. Thus condition T fails. //

Corollary 1 The only tree which is a gpg is the star $K_{1,n}$.

The conditions given in Theorem 2 are equivalent in a more general class of graphs.

Theorem 3 Let G be any connected triangle-free graph with edges $e_1, e_2, \ldots, e_q$. On each edge e_i construct any connected chordal graph G_i containing edge e_i so that for $i \neq j$, G_i and G_j have no vertices in common other than the vertex which may be common to e_i and e_j. Let H denote the graph so constructed. If $H \neq K_{m,n}$ for $m, n \geq 2$, then conditions I, T, and E are equivalent for H.

Comment: Notice that by construction, each edge of H lies in exactly one subgraph G_i for some i and a G_i may consist only of e_i. Also notice that the graph $K_{m,n}$, $m, n \geq 2$, is a gpg which satisfies Condition T but has no edge in an end-clique.

Proof: Only T implies E needs to be checked; as before, we show the contrapositive. In all that follows we let uv be an edge in H that lies in no end-clique of H, and if uv lies in the chordal graph G_i then the edge e_i is denoted by xy. We consider three cases: (1) e_i is all of G, (2) e_i is a pendant edge in G, but not all of G; or, (3) the degrees of both x and y are at least two in G.

Case 1. If $G_i = H$ then Theorem 2 applies directly to give the result.

Case 2. Let $\deg(x) = 1$ and $\deg(y) \geq 2$ in G, and suppose that yw is the edge e_j in G with $w \neq x$. Let H_i be the subgraph of H consisting of G_i along with edge e_j. Then H_i is chordal and uv belongs to no end-clique in H_i. (It could be in an end-clique in subgraph G_i.) From Theorem 2 we know that H_i contains an edge e and a maximal independent set M_i which lead to a violation of Condition T in H_i. If M_i is extended to a maximal independent set in H, the violation remains in H.

Case 3. Assume $\deg(x) \geq 2$ and $\deg(y) \geq 2$ in G. Choose vertices w and z, neither of which is x or y, so that wx is edge e_j and yz is edge e_k in G. Let H_i be the subgraph of H consisting of G_i along with e_j and e_k. Again H_i is chordal and uv is not in an end-clique in H_i. Applying Theorem 2, let M_i be a maximal independent set in H_i creating a violation of Condition T for some edge of H_i.

Case 3.1. w and z are not adjacent.

If w and z are not adjacent we may extend M_i to a maximal independent set M for H which leads to a violation of Condition T in H for that same edge.

Case 3.2. w and z are adjacent.

In each of the following three subcases we will be able to replace e_j and/or e_k by other edges xg and yf where neither f nor g is in $V(G_i)$ and they are non-adjacent in H. Then we can simply repeat the argument given in Case 3.1.

Since G is triangle-free, the subgraph of G induced by $\{x, y, z, w\}$ is isomorphic to $K_{2,2}$. Let $G' = K_{m,n} = \overline{K_m} + \overline{K_n}$, $m, n \geq 2$, be a maximal complete bipartite induced subgraph of G which contains e_i. Label the vertices of $\overline{K_m}$ as $r_1, \ldots, r_m$ (where one of them is x and $m \geq 2$) and the vertices of $\overline{K_n}$ as $s_1, \ldots, s_n$ (where one of them is y and $n \geq 2$). Since $H \neq K_{m,n}$, there is a vertex $h \in V(H)$ not in $K_{m,n}$ which is adjacent (wlog) to r_1. If $h \in V(G)$, h is not adjacent to some r_j by the maximality of $K_{m,n}$ in G. If $h \notin V(G)$, h can be adjacent only to r_1 in $\overline{K_m}$ because h is a vertex in a chordal graph built on an edge of G.

So we can assume that h is not adjacent to $r_j, r_{j+1}, \ldots, r_m$, for some $j > 1$ and find a maximal independent set M in H which contains $h, r_j, \ldots, r_m$. Then edge r_1s_1 lies in $H - M$ and in order not to violate Condition T, there must be a vertex h' in M (necessarily not in $V(G)$) so that $r_1h's_1$ is a triangle in H. Arguing as above, h' is not adjacent to any of the vertices $r_2, \ldots, r_m$ or $s_2, \ldots, s_n$. We consider three subcases.

Subcase (a). $r_1 \neq x$ and $s_1 \neq y$.

Construct a maximal independent set M which includes $h', r_2, \ldots, r_m$. Edge r_1y lies in $H - M$ so Condition T requires a vertex $f \in V(H)$ such that r_1yf is a triangle in H. Similarly extending $h', s_2, \ldots, s_n$ to a maximal independent set generates another vertex g with s_1xg a triangle in H. Furthermore, f and g are not adjacent in H because they belong to chordal graphs built on different edges of G.

Now let H_i' be the subgraph consisting of G_i along with edges xg and yf. Since f and g are not adjacent we are back to Case 3.1.

Subcase (b). $r_1 = x$ and $s_1 \neq y$.

Let H_i' be G_i along with edges xh' and yr_2. Now h' lies in the chordal graph built on edge r_1s_1. Hence h' is not adjacent to r_2 and we are again back to Case 3.1.

Subcase (c). $r_1 = x$ and $s_1 = y$.

Construct a maximal independent set M containing $h', r_2, \ldots, r_m$. Edge xs_2 lies in $H - M$ so by Condition T there is a vertex f for which fxs_2 is a triangle in H. Similarly, extending $h', s_2, \ldots, s_n$ to a maximal independent set generates a triangle gyr_2. Letting H_i' be G_i along with edges xf and yg we again return to Case 3.1. //

Corollary 2 For any triangle-free graph G other than $K_{m,n}$ for $m, n \geq 2$, conditions I, T, and E are equivalent for G.

All graphs in Figure 1 are non-planar, and Condition T is sufficient for any planar graph H in Theorem 3 to be a gpg. This suggests the following.

Open Question: Is every planar graph which satisfies Condition T a gpg?

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