Stability of i-Connectivity Parameters Under Edge Addition

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Abstract

The inclusive edge (vertex, mixed) connectivity of a vertex v is the minimum number of edges (vertices, graph elements) whose removal yields a subgraph in which v is a cutvertex. Stability under edge addition, in which the value of the parameter remains unchanged with the addition of any edge, is investigated. In particular we examine relationships between stability of the inclusive vertex connectivity parameter and stability of the inclusive mixed connectivity parameter, affirming a conjecture posed in [5].

1. Introduction

The cohesion of a vertex was initially introduced in [7] and was further expanded by [1-3] to the inclusive connectivity parameters of a graph element (vertex or edge). These i-connectivity parameters are local measures of graph vulnerability. In fact, they are shown in [1] to be natural localizations of the widely studied graph connectivity and edge connectivity parameters.

It is natural to inquire about the stability of these parameters with regard to edge addition since knowledge of local network vulnerability changes upon network alteration is desirable. Initial work in stability on the one parameter formerly called cohesion was accomplished in [8-13]. We do not consider λ_i-stability of vertices here since it is largely independent of the other stability types. Initial results concerning stability relationships between inclusive connectivity parameters was begun in [5] and is continued here.

2. Definitions and Fundamental Concepts

Let G be a connected graph without loops or multiple edges with vertex set V(G), edge set E(G), (vertex) connectivity $\kappa(G)$, and edge connectivity $\lambda(G)$. If not defined here, we follow the notation found in [4].

A graph element will be understood to mean either an edge or a vertex of the graph. If S is a set of graph elements, $e \in S$ and $w \in V(G)$ we denote with $S - e + w$ the set consisting of those elements of S less the edge e unioned with $\{w\}$. A cutvertex of G is a vertex whose deletion either increases the number of components or increases the number of isolates in G. Note that this definition permits either end of a K_2-component to be a cutvertex.

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For $v \in V(G)$, the inclusive edge connectivity of v, $\lambda_i(v,G)$, (formerly called cohesion), is the minimum number of edges whose removal yields a subgraph in which v is a cutvertex. Similarly, for $v \in V(G)$, the inclusive vertex (mixed) connectivity of v, $\kappa_i(v,G)$, ($\mu_i(v,G)$) is the minimum number of vertices (graph elements) whose removal yields a subgraph in which v is a cutvertex.

An alternative way of conceptualizing $\kappa_i(v)$ is as the size of the smallest set of vertices whose removal from G - v separates vertices from N(v) into different components or which isolates a neighbor of v. The parameters $\lambda_i(v)$ and $\mu_i(v)$ may be viewed analogously. This view of i-connectivity suggests that Mengers Theorem may be used to aid in computing the parameters. For instance, the smallest set of vertices whose removal from G - v separates vertices from N(v) into different components is the same as the maximum number of internally disjoint paths in G - v, among pairs of vertices from N(v).

Theorem 1 provides a manner of characterizing inclusive mixed connectivity similar to Mengers Theorem for connectivity. The proof and parallel results for other i-connectivity parameters are evident.

Given $v \in V(G)$, let p(u, w) denote the maximum number of internally disjoint u-w paths in the graph G - v.

Theorem 1: For any graph G with $v \in V(G)$ having degree greater than one,
$\mu_i(v, G) = \min \{ p(u, w) ; u, w \in N(v) \}$

Algorithms which compute the i-connectivity parameters (using Theorem 1 and its counterparts for the other parameters) have been implemented [6]. The i-connectivity values for all graphs given here were verified using that program.

The parameters $\lambda_i(e,G)$, $\kappa_i(e,G)$, and $\mu_i(e,G)$ are defined similarly for any edge e of G where 'cutvertex' is replaced by 'bridge' in the preceding definitions. When the underlying graph is apparent reference to that graph may be suppressed, for instance we may use $\lambda_i(v)$ instead of $\lambda_i(v,G)$ when no confusion arises. Inclusive connectivity is also referred to as i-connectivity. If S is a smallest set of vertices (respectively edges, graph elements) whose removal from G makes v a cutvertex, then we call S a κ_i-set (respectively λ_i-set, μ_i-set) for v in G. If a κ_i-set (respectively λ_i-set, μ_i-set) for vertex v in G consists of the set of vertices (respectively edges, graph elements) adjacent to some $u \in N_G(v)$ less the vertex v (or the edge uv), we call that set a neighborhood κ_i-set (respectively λ_i-set, μ_i-set) for v at u.

Figure 1 serves to illustrate the various i-connectivity parameters. Note that in this case $\kappa_i(v) > \lambda_i(v)$ which is in contrast to Whitney's Theorem. This example also serves to illustrate our rather peculiar definition of a cutvertex. Notice that since $\langle N(v) \rangle$ is complete, the only way to make v a cutvertex by removing vertices is to isolate it in a K_2 component. In fact it is now apparent that $\kappa_i(v) = \min \{ \deg(w) : w \in N(v) \} - 1$ whenever N(v) induces a complete subgraph in G.

We are primarily concerned with changes in i-connectivity with respect to the addition of an edge to G; thus we define a vertex $v \in V(G)$ as λ_i-stable if $\lambda_i(v,G) = \lambda_i(v,G + e)$ for every edge $e \notin E(G)$. Similarly, a vertex $v \in V(G)$ is κ_i- (μ_i-) stable if $\kappa_i(v,G) = \kappa_i(v,G + e)$ ($\mu_i(v,G) = \mu_i(v,G + e)$) for every edge $e \notin E(G)$. In Figure 1, if e = va, then $\lambda_i(v,G + e) = \kappa_i(v,G + e) = \mu_i(v,G + e) = 2$, where if e = vb then $\lambda_i(v,G + e) =$

$\kappa_i(v, G + e) = \mu_i(v, G + e) = 1$. Hence vertex v has no stability under edge addition. Figure 2 shows a graph with vertex v having κ_i and μ_i-stability, but not λ_i-stability under edge addition. More examples of graphs exhibiting various types of stability under edge addition may be found in [5].

3. Preliminary Results

Our first four results provide useful tools for examining the stability of i-connectivity parameters under edge addition. The proof of Theorem 2 may be found in [2].

Theorem 2: For any $v \in V(G)$, if $\mu_i(v, G) < \kappa_i(v, G)$ then there exists a μ_i-set for v containing exactly one edge and that edge has as its endpoints neighbors of v.

The next lemma is straightforward to establish.

Lemma 3: If $\mu_i(v, G) < \kappa_i(v, G)$ and S is any μ_i-set for v then G - S - v has exactly two components which contain vertices of $N_G(v)$.

We now examine the effect of edge addition on inclusive mixed connectivity parameter values. The effect on the other i-connectivity parameters is similar and may be found in [13].

Theorem 4: Let u, v, and w be distinct vertices of G and e = uw, where $uw \notin E(G)$. Then

(a) $\mu_i(v, G) \leq \mu_i(v, G + e) \leq \mu_i(v, G) + 1$ and

(b) $\kappa(G - u) \leq \mu_i(u, G + e) \leq \mu_i(u, G)$.

Proof: Let S be any μ_i-set for v in G. Then $S \cup \{e\}$ is a set of graph elements whose removal from G + e makes v a cutvertex showing that $\mu_i(v, G + e) \leq \mu_i(v, G) + 1$. The inequality $\mu_i(v, G) \leq \mu_i(v, G + e)$ holds since $N_G(v) = N_{G+e}(v)$ and between any pair of neighbors of v there are at least as many internally disjoint paths between them in G + e as in G.

Since G - u = (G + e) - u and every pair of neighbors of u in G is a pair of neighbors of u in G + e, it follows that $\mu_i(u, G + e) \leq \mu_i(u, G)$. Finally, $\kappa(G - u) \leq \mu_i(u, G + e)$ holds because any μ_i-set for u in G + e is a set of graph elements whose removal from G - u either results in either a disconnected or trivial graph. ❑

4. Main Results

We now begin investigating stability under edge addition.

Theorem 5: If $v \in V(G)$ satisfies $\mu_i(v, G) < \kappa_i(v, G)$ then v is not κ_i-stable under edge addition in G.

Proof: Let $v \in V(G)$ be such that $\mu_i(v, G) < \kappa_i(v, G)$. By Theorem 2 there exists a μ_i-set , S, for v in G with S containing exactly one edge $e = w_1w_2$ and with the endpoints of e both neighbors of v. Further, by Lemma 3, G - S - v has exactly two components which contain vertices of N(v). Name these components C_1 and C_2. Let $w_1 \in C_1$ and $w_2 \in C_2$. We may assume, without loss of generality, that there is a vertex $x \in V(C_1)$ which is distinct from w_1. For if $|V(C_1)| = |V(C_2)| = 1$, then S is a neighborhood μ_i-set implying that $\mu_i(v, G) = \kappa_i(v, G)$.

If $x \in N(v)$ then $S - e + w_1$ is a set of vertices whose removal from G makes v a cutvertex implying $\kappa_i(v, G) \leq |S - e + w_1| = |S| = \mu_i(v, G)$, a contradiction. Then it must be the case that $x \notin N(v)$. Now consider the graph G + vx. Upon removal from G + vx the set $S - e + w_1$ makes v a cutvertex. This gives $\kappa_i(v, G + vx) \leq |S - e + w_1| = |S| = \mu_i(v, G) < \kappa_i(v, G)$ so that v is not κ_i-stable upon edge addition in G. ❑

Corollary 6: If $v \in V(G)$ is κ_i-stable under edge addition in G then $\mu_i(v, G) = \kappa_i(v, G)$.

The next theorem verifies a conjecture raised in [5].

Theorem 7: If $v \in V(G)$ is κ_i-stable under edge addition in G then v is μ_i-stable under edge addition.

Proof: We establish the contrapositive. Toward that end, suppose that $v \in V(G)$ is not μ_i-stable under edge addition. Let $e \notin E(G)$ with $\mu_i(v, G) \neq \mu_i(v, G + e)$.

Case 1: Suppose $\mu_i(v, G) = \kappa_i(v, G)$.

(a) If $\mu_i(v, G) < \mu_i(v, G + e)$ then $\kappa_i(v, G + e) \geq \mu_i(v, G + e) > \mu_i(v, G) = \kappa_i(v, G)$ implying that v is not κ_i-stable under edge addition.

(b) Suppose $\mu_i(v, G) > \mu_i(v, G + e)$. If $\kappa_i(v, G) \neq \kappa_i(v, G + e)$ then we are done. Assume then that $\kappa_i(v, G) = \kappa_i(v, G + e) > \mu_i(v, G + e)$. By Theorem 2, we let S' be a μ_i-set for v in G + e such that S' contains exactly one edge and that edge has as its endpoints neighbors of v. Since $\mu_i(v, G) > \mu_i(v, G + e)$, Theorem 4 implies that e must be adjacent to v. By Lemma 3, (G + e) - S' - v has exactly two components which contain neighbors of v. Let w_1 be a neighbor of v in component C_1 and w_2 be a neighbor of v in component C_2 of (G + e) - S' - v. Let w_1w_2 be the lone edge in S'.

(i) If w_1 is the only vertex in C_1 then $\mu_i(v, G + e) = \kappa_i(v, G + e)$. To verify this, notice that $S' - w_1w_2 + w_2$ is a set of vertices whose removal from G + e makes v a cutvertex. Hence $\kappa_i(v, G + e) \leq |S' - w_1w_2 + w_2| = |S'| = \mu_i(v, G + e)$. But $\kappa_i(v, G + e) \geq \mu_i(v, G + e)$ by definition resulting in equality. This contradicts our assumption that $\kappa_i(v, G + e) > \mu_i(v, G + e)$.

(ii) There exists another vertex y in C_1 and $y \in N_{G+e}(v)$. Then v is a cutvertex with the removal of $S' - w_1w_2 + w_1$ from G + e so $\kappa_i(v, G + e) = \mu_i(v, G + e)$, a contradiction as above.

(iii) There exists another vertex y and $y \notin N_{G+e}(v)$. Then consider the graph G + vy. Since $S' - w_1w_2 + w_1$ is a set of vertices which makes v a cutvertex upon removal from G + vy, we see that $\kappa_i(v, G + vy) \leq | S' - w_1w_2 + w_1 | = | S' | = \mu_i(v, G + vy) < \mu_i(v, G) = \kappa_i(v, G)$ showing that v is not κ_i-stable under edge addition in G.

Case 2: Suppose $\mu_i(v, G) < \kappa_i(v, G)$. Then by Theorem 5, v is not κ_i-stable under edge addition in G. ❑

We remark that a vertex which is μ_i-stable under edge addition is not necessarily κ_i-stable under edge addition. For example the graph in Figure 3 has vertex v μ_i-stable but not κ_i-stable under edge addition (see [5] for a detailed explanation). We now turn our attention to conditions under which μ_i-stability under edge addition implies κ_i-stability. Theorem 8, which is not difficult to establish, is presented without proof.

Theorem 8: If $v \in V(G)$ has degree one, then v is λ_i, κ_i, and μ_i-stable or v is none of λ_i, κ_i, or μ_i-stable depending on whether G - v is complete or not complete respectively.

Corollary 9: If $v \in V(G)$ is μ_i-stable under edge addition and $\deg_G(v) = 1$, then v is κ_i-stable under edge addition.

We now turn our attention to a more general scenario.

Lemma 10: If $v \in V(G)$ is μ_i-stable under edge addition and every μ_i-set for v in G separates the same pair of neighbors of v, then $\mu_i(v, G) = \kappa_i(v, G)$.

Proof: Let $v \in V(G)$ be μ_i-stable and suppose every μ_i-set for v in G separates the same pair of neighbors. Label the two neighbors of v which get separated by every μ_i-set with u and w. Notice that since the same pair of neighbors get separated by every μ_i-set and v is μ_i-stable under edge addition, $uw \in E(G)$ and uw is in any μ_i-set for v in G.

Assume for the sake of contradiction that $\mu_i(v, G) < \kappa_i(v, G)$. Then no μ_i-set for v in G is a neighborhood μ_i-set for v at u or w. By Theorem 1, there exist exactly $\mu_i(v, G)$ internally disjoint u-w paths in G - v. Let S be any set of $\mu_i(v, G)$ internally disjoint u-w paths each of which is of minimal length. Then there is at least one neighbor for each of u and w in G - v which is not on any path in S (otherwise v has a neighborhood μ_i-set at u or w). Call these neighbors x and y. Notice that x and y are distinct since if not, then u-x-w is a u-w path not in S and internally disjoint from all paths in S, contradicting the maximality of $\mu_i(v, G)$. Similarly, note that $xy \notin E(G)$. Then in G + xy, there are

$\mu_i(v, G) + 1$ internally disjoint u-w paths implying that v is not μ_i-stable under edge addition. Then it must be the case that $\mu_i(v, G) = \kappa_i(v, G)$. ❑

Theorem 11: If $v \in V(G)$ is μ_i-stable under edge addition and every μ_i-set for v in G separates the same pair of neighbors of v, then v is κ_i-stable.

Proof: Let $v \in V(G)$ be μ_i-stable under edge addition and suppose every μ_i-set for v in G separates the same pair of neighbors of v. Label the two neighbors of v which get separated by every μ_i-set with u and w. Then by Lemma 10, $\mu_i(v, G) = \kappa_i(v, G)$.

Case 1: With $deg_G(u) \neq deg_G(w)$ we assume without loss of generality $deg_G(u) > deg_G(w)$. Let P be any set of $\mu_i(v, G)$ internally disjoint u-w paths, each of which is of minimal length. Since $\mu_i(v, G) < deg_G(u)$, then there exists at least one neighbor, x, of u which is not on any path of P. Then $xw \notin E(G)$ by the maximality of $\mu_i(v, G)$. Thus, in G + xw, there are $\mu_i(v, G) + 1$ internally disjoint u-w paths contradicting the fact that v is μ_i-stable under edge addition in G.

Case 2: Suppose $deg_G(u) = deg_G(w) = k$. We now establish $k = \mu_i(v, G) = \kappa_i(v, G)$. Assume $k \neq \mu_i(v, G) = \kappa_i(v, G)$. Then the number of internally disjoint u-w paths in G - v is strictly less than k. Let S be any set of $\mu_i(v, G)$ internally disjoint u-w paths each of which is of minimal length. Then each of u and w has at least one neighbor which is not on any path in S. Then, as in Lemma 10, a new u-w path may be constructed in G + e which is internally disjoint from each path in S for some $e \notin E(G)$. This contradicts the μ_i-stability.

Thus $k = \mu_i(v, G) = \kappa_i(v, G)$ and since $uw \in E(G)$, the degree of at least one of u or w remains the same in G + e for any $e \notin E(G)$. It then follows that $\kappa_i(v, G + e) \leq \kappa_i(v, G)$ for every $e \notin E(G)$. Combining this with $\kappa_i(v, G + e) \geq \mu_i(v, G + e)$ and $\mu_i(v, G + e) = \mu_i(v, G)$ gives $\kappa_i(v, G + e) = \kappa_i(v, G)$ for all $e \notin E(G)$. ❑

We now point out a consequence of Theorem 11 which extends Corollary 9 to the case of a degree two vertex.

Corollary 12: If $v \in V(G)$ is μ_i-stable under edge addition and $deg_G(v) = 2$, then v is κ_i-stable under edge addition.

5. Conclusion

The results presented here compliment initial investigations pertaining to stability under edge addition [5, 8-13] and settle an intriguing conjecture posed in [5]. Stability of the i-connectivity parameters under edge deletion is now under way and is remarkably different from the edge addition stability.

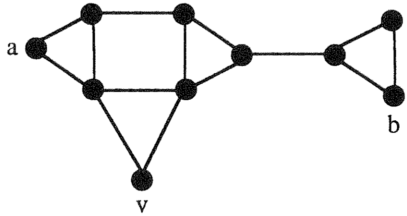

Figure 1: A graph with $\lambda_i(v) = 2$, $\kappa_i(v) = 3$, and $\mu_i(v) = 2$.

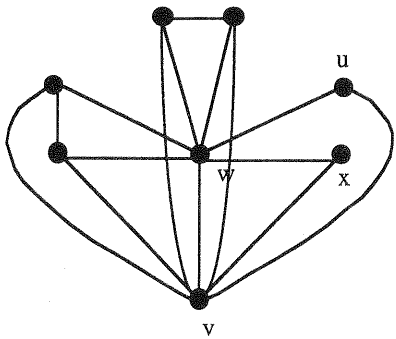

Figure 2: A graph illustrating stability.

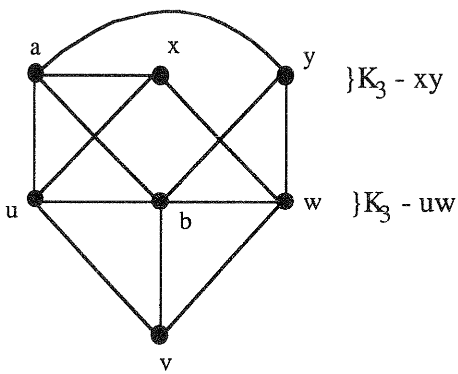

Figure 3: A graph showing that μ_i-stability does not imply κ_i-stability.

References

1. Boland, J.W. and R.D. Ringeisen, *Inclusive Connectivity: An Introduction to a Local Parameter,* Congr. Numer., 1990, **78**, 217-221.

2. Boland, J.W. and R.D. Ringeisen, *Relationships Between i-Connectivity Parameters,* in *Advances in Graph Theory*, 1991, Vishwa International Publications, Gulbarga, 33-40.

3. Boland, J.W. and R.D. Ringeisen, *On Super i-Connected Graphs,* Networks, 1994, **24**, 225-232.

4. Chartrand, G. and L. Lesniak, *Graphs & Digraphs*, 2 ed. 1986, Monterey CA., Wadsworth & Brooks/Cole.

5. Cribb, D.W., J.W. Boland, and R.D. Ringeisen, *Graphs with Stable Inclusive Connectivity Parameters,* Vishwa International J. Graph Theory, 1993, **2**, 93-107.

6. Lee, C., *Computing Inclusive Connectivity*, Masters Paper, 1990, Clemson University.

7. Lipman, M.J. and R.D. Ringeisen, *Cohesion in Alliance Graphs,* Congr. Numer., 1979, **24**, 713-720.

8. Rice, V., *Cohesion Properties in Graphs*, Doctoral Dissertation, 1988, Clemson University.

9. Rice, V. and R.D. Ringeisen, *Cohesion stable edges,* Congr. Numer., 1988, **67**, 205-209.

10. Rice, V. and R.D. Ringeisen, *On Cohesion Stable Graphs,* Discrete Math., 1994, **126**, 257-272.

11. Ringeisen, R. and V. Rice, *Cohesion stability under edge destruction,* Jrnl. Comb. Math. and Comb. Cmpt., 1988, **2**, 153-162.

12. Ringeisen, R. and V. Rice, *When is a graph not stable or are there any stable graphs out there ?* , Jrnl. Comb. Math. and Comb. Cmpt., 1988, **4**, 19-22.

13. Ringeisen, R.D. and M.J. Lipman, *Cohesion and Stability in Graphs.* Discrete Math., 1983, **46**, 253-259.

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